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The Beta-Modified Weighted Rayleigh Distribution: Application to Virulent Tubercle Disease

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Abstract

Flexible parametric models are useful for modeling survival data and this has become an important field in statistics; and concern of statisticians in data analysis. Therefore, this paper presents a univariate model called Beta modified weighted Rayleigh distribution constructed from modified weighted Rayleigh distribution. The new distribution is achieved by introducing two shape parameters to the existing modified Weighted Rayleigh distribution using logit of beta function. The idea is to verify if the Beta modified Weighted Rayleigh distribution would perform better than modified Weighted Rayleigh distribution in modeling survival data. The statistical properties such as; survival rate, hazard rate, moment generating functions, skewness and kurtosis are determined for the new distribution. We also performed the expected estimation of model parameters by maximum likelihood. The Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC) are employed to select the best model. The superiority and flexibility of the proposed distribution is illustrated and applied to survival times of guinea pigs with virulent tubercle data sets. The results with the baseline distribution are also compared. Likewise, results from the model selection criteria: the AIC, BIC and CAIC favoured BMWR; indicating that the proposed distribution performs and has better representation of the data than the MWR distribution.

Keywords: Akaike Information Criterion, Beta function, Beta modified Weighted Rayleigh, shape parameters, survival data.
MSC2010: 60E05

1 Introduction

Rayleigh distribution is a special case of Weibull distribution which has been widely used for modeling lifetime data in a wide variety of areas including reliability, engineering, and survival analysis to mention but a few; while weighted distributions are used to adjust the probabilities of the events as observed and recorded [27]. In this paper, we introduce and study the properties of the Beta-modified weighted Rayleigh distribution. This class of distributions is flexible in accommodating skewed data and contains new sub-models such as Lehmann Type II modified weighted Rayleigh, Exponentiated modified weighted Rayleigh distribution.

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Beta generated (Beta-G) family [20] have been studied by numerous authors such as [24], [25] Beta Frechet distribution, [24], [1], [30], Beta generalized exponential [8] the Beta extended-G family [11], Beta generalized Weibull distribution [29], Beta weighted weibull distribution [4], Beta generalized Rayleigh distribution [13], Beta extended half normal distribution [12], [18], Beta log-logistic distribution [22]. Life length of components estimates with Beta weighted Weibull distribution [5], Beta Marshall-Olkin family of distribution [3], the Beta weighted exponential distribution [6], Beta exponential Frechet distribution [23], Beta-Dagum distribution and Beta-Singh-Maddala distribution [15] among others.

Therefore, this paper is arranged as follows. In section 2, we present the generalized distribution including the corresponding probability density function (pdf), cumulative distribution function (cdf), reliability, hazard rate and reverse hazard functions and various sub-models. Section 3 contains the derivation of the moments and moment generating functions, skewness and kurtosis. Estimation of model parameters using method of maximum likelihood estimation is presented in section 4. Applications of the proposed model to real data are given in section 5, followed by conclusion in section 6.

2 Material and Method

2.1 The Beta-modified Weighted Rayleigh (BMWR) Distribution

2.1.1 The Density Function of BMWR Distribution

We introduce two shape parameters into the work of [2], to generate more skewed distribution called BMWR distribution (where they mentioned and gave only the pdf of modified Weighted Rayleigh distribution). The pdf of the MWR is given as

$$f_{MWR}(g) = 2\alpha(\beta\gamma^2+1)ge^{-\alpha(\beta\gamma^2+1)g^2} \tag{2.1}$$

where, α is a scale parameter while, β and γ are shape parameters; and the corresponding cdf is

$$F_{MWR}(g) = 1 - e^{-\alpha(\beta\gamma^2+1)g^2} \tag{2.2}$$

But we explore a new Beta-modified weighted Rayleigh distribution using generalized Beta function by [20] which of course several authors have used in literature. Some more recent extensions on generalizations are: exponentiated Weibull (EW) [17], the modified Weibull (MW) [21], and the Beta exponential (BE) [26], the generalized modified Weibull (GMW) [9], the Beta modified Weibull (BMW) [28], Beta generalized exponential ([8], [10]), Beta generalized Weibull distribution [29], Beta generalized Rayleigh distribution [13], Beta generalized inverse Weibull distribution [7].

The Beta link function is given as:

$$f(g) = \frac{(f(g)[F(g)]^{(m-1)}[1-F(g)]^{(n-1)})}{B(m,n)}, m,n>0 \tag{2.3}$$

where, $g>0,\ f(g) = \dfrac{d}{dg}F(g)$ and $m>0,\ n>0$ are two shape parameters added to the existing parent (MWR) distribution. The primary motivations for developing this model are the advantages presented by this generalized distribution due to its flexibility. It has bathtub and a unimodal shape hazard function, as well as the versatility and flexibility of the Beta-modified weighted Rayleigh distributions in modeling lifetime data. $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ is the Beta function and $f(g),\ F(g)$ are pdf and cdf respectively of base distribution (MWR).

Substituting equations in (1) and (2) into (3) to obtain the density function of BMWR, we have the following:

$$f_{(BMWR)}(g) = \frac{(f_{MWR}(g)[F_{MWR}(g)]^{(m-1)}[1 - F_{MWR}(g)]^{(n-1)})}{B(m,n)},$$

$$\alpha, \beta, \gamma, m, n \simeq BMWR(\alpha, \beta, \gamma, m, n, g > 0) \tag{2.4}$$

$$f_{(BMWR|\alpha,\beta,\gamma,m,n)}(g) = \frac{1}{B(m,n)}[1 - e^{-\alpha(\beta\gamma^2+1)g^2}]^{(m-1)}[e^{-\alpha(\beta\gamma^2+1)g^2}]^{(n-1)}2\alpha(\beta\gamma^2+1)ge^{(-\alpha(\beta\gamma^2+1)g^2} \tag{2.5}$$

where, $f_{MWR}(g)$ and $F_{MWR}(g)$ are the pdf and cdf of base distribution; and expression (4) becomes the pdf of the new BMWR distribution with five parameters.

The plots (a - d) below are the pdf plots of the BMWR, LMWR, EMWR and MWR distributions at different values of $m = (100, 1, 50, 2.5, 1.5)$ and $n = (3.5, 1, 2.5, 5, 1.5)$ when $c = \alpha$, $d = \beta$ and $e = \gamma$ are fixed at (0.3,5,0.3). These are initial values given to each parameter in the distribution (R code) to achieve different shapes from each distribution. For instance, if $m = 1$, the distribution becomes Lehmann Type II modified weighted Rayleigh (LMWR) (the broken line closed to the left side in each plot below), when $n = 1$, then the distribution yields Exponentiatated modified weighted Rayleigh (EMWR) (the fainted line in each graph below) (the two distributions mentioned are sub-model of BMWR). Also the distribution can accomodates the four risk functions, e. g (increasing, decreasing, bathtub and unimodal shape hazard function). Therefore, as the values of m and n increases, the skewness of the BMWR (the bold line in each graph below) decreases and as values reduces the graphs of the BMWR skewed to the right and so on.

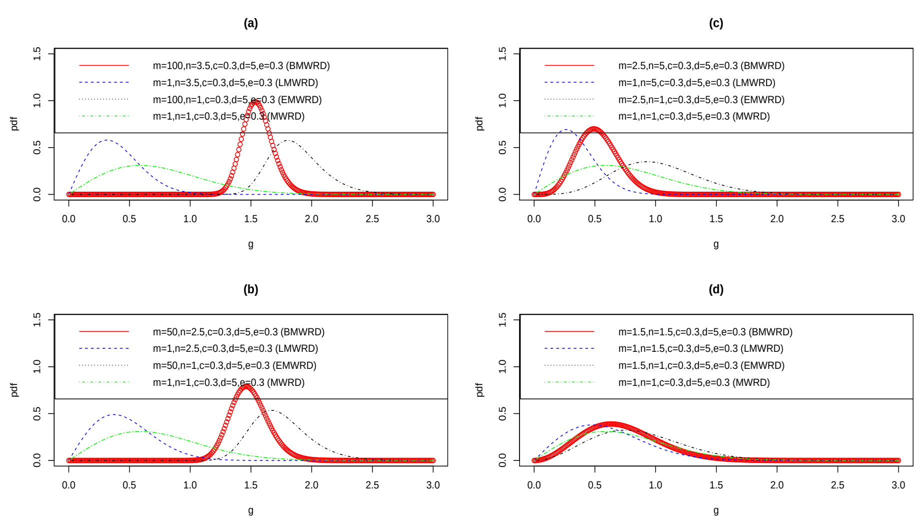

Figure 1: The pdf plots of the BMWR, LMWR, EMWR and MWR distributions.

Letting

$$T(g) = F_{MWR}(g) = 1 - e^{-\alpha(\beta\gamma^2+1)g^2}$$

where, $T(g)$ is the cdf of the baseline gives

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$$\frac{dT}{dg} = [f_{MWR}(g)][F_{MWR}(g)] \tag{2.6}$$

and expression (4) becomes

$$f_{BMWR}(g) = \frac{1}{B(m,n)}[T(g)]^{(m-1)}[1-T(g)]^{(n-1)}d(g) \tag{2.7}$$

2.1.2 Cumulative Distribution Function (CDF)

Here, we used equation (7) to obtain cumulative distribution function of BMWR where variable G is given by

$$F_{BMWR}(g) = \int_0^g f_{BMWR}(g) = \frac{(\int_0^g [T(g)]^{(m-1)}[1-T(g)]^{(n-1)}d(g)])}{B(m,n)} \tag{2.8}$$

$$F_{BMWR}(g) = \frac{B(g;m,n)}{B(m,n)} \tag{2.9}$$

where, $l_g = (m.n) = B_g(m,n)/B(m,n)$ is called incomplete Beta function ratio and $B_g(m,n) = \int_0^g W^{m-1}[1-W]^{n-1}d(g)$ is the Beta function.

2.1.3 The Reliability Function

The reliability function $R_{(BMWR|\alpha,\beta,\gamma,m,n)}$ of a random $BMWR\alpha,\beta,\gamma,m,n$ variable G with cdf F (g) is defined as

$$R_{(BMWR)}[\alpha,\beta,\gamma,m,n](g) = 1 - F_{BMWR}(g) \tag{2.10}$$

$$R_{(BMWR)}[\alpha,\beta,\gamma,m,n](g) = \frac{B(m,nb) - B(g;m,n)}{B(m,n)} \tag{2.11}$$

2.1.4 The Hazard Rate Function

The hazard rate function $H_{(BMWR|\alpha,\beta,\gamma,m,n)}$ of a random $BMWR(\alpha,\beta,\gamma,m,n)$ variable G with cdf F (g) is given as

$$h_{BMWR|(\alpha,\beta,\gamma,m,n)}(g) = \frac{(f_{BMWR(\alpha,\beta,\gamma,m,n)}(g))}{(1 - F_{BMWR(\alpha,\beta,\gamma,m,n)}(g))} \tag{2.12}$$

Substituting equations (7) and (11) yields

$$h_{BMWR(\alpha,\beta,\gamma,m,n)}(g) = \frac{([T(g)]^{(m-1)}[1-T(g)]^{(n-1)}d(g))}{B(m,n)-B(g;m,n)} \tag{2.13}$$

Technically, the hazard function is a probability of failure in a very small time interval.

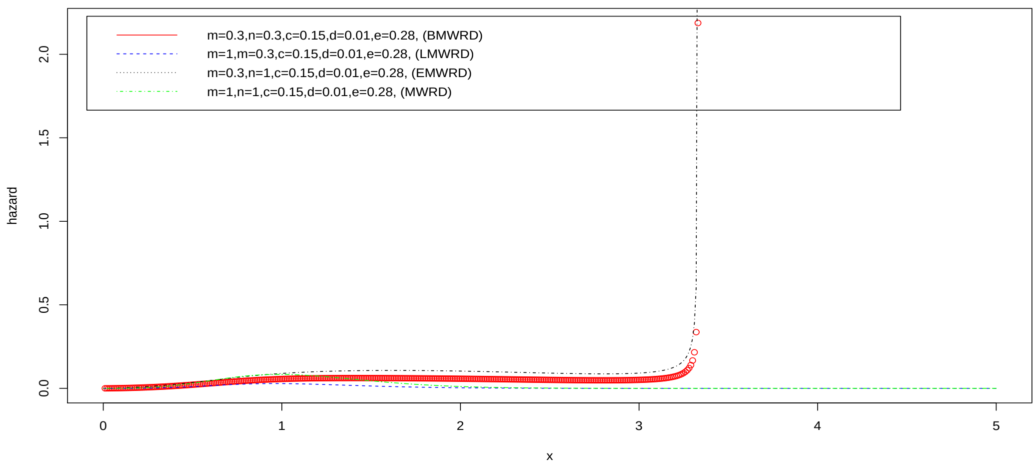

Figure 2: The graph of hazard rates of the $BMWR = (m,n,\alpha,\beta,\gamma)$, $LMWR = (1,n,\alpha,\beta,\gamma)$, $EMWR = (m,1,\alpha,\beta,\gamma)$ and $MWR = (1,1,\alpha,\beta,\gamma)$ at $((0.3,0.3,0.15,0.01,0.28),(1,0.3,0.15,0.01,0.28),(0.3,1,0.15,0.01,0.28),(1,1,0.15,0.01,0.28))$, respectively.

2.1.5 The Reversed Hazard Function

The reserved hazard function $r_{(BMWR|\alpha,\beta,\gamma,m,n)}$of a random $BMWR(\alpha,\beta,\gamma,m,n)$variable G with cdf F (g) is given by

$$r_{BMWR(\alpha,\beta,\gamma,m,n}(g) = \frac{(f_{BMWR(\alpha,\beta,\gamma,m,n)}(g))}{(F_{BMWR(\alpha,\beta,\gamma,m,n)}(g))} \tag{2.14}$$

Also, putting equations (7) and (9) in expression (14), we obtain

$$r_{BMWR(\alpha,\beta,\gamma,m,n)}(g) = \frac{([T(g)]^{(m-1)}[1-T(g)]^{(n-1)}d(g))}{B(g;m,n)} \tag{2.15}$$

2.2 Some Special Cases of the BMWR Distribution

2.2.1 Some new distributions emanate from the BMWR distribution depending on the values of the parameter. These include:

(a) If m=1 in (3) we get Lehmann Type II Modified Weighted Rayleigh (LMWR) distribution (New)

$$f_{(LMWR|\alpha,\beta,\gamma,1,n)}(g) = n[e^{(-\alpha(\beta\gamma^2+1)g^2)}]^{(n-1)}2\alpha(\beta\gamma^2+1)ge^{(-\alpha(\beta\gamma^2+1)g^2)} \tag{2.16}$$

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(b) Also, by letting n=1 in (3) we obtain Exponentiated Modified Weighted Rayleigh (EMWR) distribution (New)

$$f_{(EMWR|(\alpha,\beta,\gamma,m,1)}(g) = m[e^{(-\alpha(\beta\gamma^2+1))}g^2]^{(m-1)}2\alpha(\beta\gamma^2+1)ge^{(-\alpha(\beta\gamma^2+1))}g^2 \tag{2.17}$$

Table 1: Summary of Some New Distributions Emanating from BMWR Distribution

Distribution/Parameter	m	n	α	β	γ
LMWR	1	–	–	–	–
EMWR	–	1	–	–	–

3 Moments and Generating Function

3.1 Generating Function

The moment generating function (MGF) of the BMWR distribution is obtained following the idea of [19] and used in [6].

The MGF $M(g) = E(e^{tg})$is given as

$$M(g) = \frac{1}{B(m,n)}\sum_{i=0}^{\infty}(-1)^i\binom{n-1}{i}\int_{-\infty}^{\infty}e^{tg}f_{MWR}(g)[F_{MWR}(g)]^{(m(i+1)-1)}dg. \tag{3.1}$$

Putting the pdf $f_{MWR}(g)$ and cdf $F_{MWR},(g)$ as defined in equations (1) and (2) above, into MGF $M(g)$ in equation (18) gives

$$M_{BMWR}(t) = \frac{1}{B(m,n)}\sum_{i=0}^{\infty}(-1)^i\binom{n-1}{i}\int_{-\infty}^{\infty}e^{tg}[e^{(-\alpha(\beta\gamma^2+1)g^2)}]^{(m(i+1)-1)}2\alpha(\beta\gamma^2+1)ge^{(-\alpha(\beta\gamma^2+1)g^2)}]dg \tag{3.2}$$

3.2 Moments

Following [2]; the rth non-central moment of the class of Modified Weighted Rayleigh distribution $MWR(\alpha,\beta,\gamma)$ is given as:

$$\mu'_{MWRr} = E(G^r) = 2(\beta\gamma^2+1)\alpha^{-\frac{r}{2}}\Gamma(\frac{r}{2}+1)(\beta\gamma^2+1)^{-\frac{r}{2}-1} \tag{3.3}$$

The rth non-central moment of the Beta Modified Weighted Rayleigh distribution would be given as

$$\mu'_{(BMWR(r))} = \int_0^{\infty} g^r f_{BMWR}(g)dg$$

that is

$$\mu'_{BMWR(r)} = \int_0^{\infty} g^r(\frac{1}{B(m,n)}[T(g)]^{(m-1)}[1-T(g)]^{(n-1)}dT(g))$$

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where

$$T(g) = e^{-\alpha(\beta\gamma^2+1)g^2}, k(g) = e^{-\alpha g^2}, \theta = (\beta\gamma^2+1)$$

Then,

$$\mu'_{BMWW(r)} = [\frac{(2(\theta)\alpha^{-\frac{r}{2}}\Gamma(\frac{r}{2}+1)(\theta)^{(-\frac{r}{2}-1)})}{B(m,n)}]\sum_{i=0}^{\infty}(-1)^i\binom{n-1}{i} \times \int_0^{\infty}[k(g)(\theta)]^{(m(i+1)-1)}dg$$

$$\mu'_{BMWW(r)} = Q\{2(\theta)\alpha^{(-\frac{r}{2})}\Gamma(\frac{r}{2}+1)(\theta)^{-\frac{r}{2}+1}\} \tag{3.4}$$

where

$$Q = \sum_{i=0}^{\infty}\binom{n-1}{i}\frac{\int_0^{\infty}[k(g)(\theta)]^{(m(i+1)-1)}dg}{B(m,n)}$$

see [27] and [6].

We obtained the first four non-central moments $\mu_r^{'}$,by letting r=1,2,3 and 4 respectively in equation (21); i.e.$\mu_1^{'}$is given as

$$\mu_1^{'} = E_{BMWR}(g) = [\frac{(2(\theta)\alpha^{-(\frac{1}{2})}\Gamma(\frac{3}{2})(\theta)^{-(\frac{3}{2})})}{B(m,n)}][\sum_{i=0}^{\infty}(-1)^i\binom{n-1}{i}].$$

Also, central moments $\mu_r, r = 1, 2, 3, 4, \ldots$ are related to non-central moments μ'_r as

$$\mu_r = \sum_{w=0}^{r}\binom{r}{w}\mu'_{r-w}\mu_w'^{w} \tag{3.5}$$

where $\mu'_1 = \mu$ and $\mu'_0 = 1$.

Conversely, the mean and variance, 3rd and 4th moments of the BMWR distribution are given by

$$\mu = \mu_1^{'}$$

$$\mu_2 = \mu_2^{'} - \mu^2$$

$$\mu_3 = \mu_3^{'} - 3\mu\mu_2^{'} + 2\mu^3$$

and

$$\mu_4 = \mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 3\mu^4$$

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where,

$$\mu_1^{'} = Q[2(\theta)\alpha^{-(\frac{1}{2})}\Gamma(\frac{3}{3})(\theta)^{-(\frac{3}{2})}] \quad (3.6)$$

$$\mu_2^{'} = Q[2(\theta)\alpha^{-(\frac{2}{2})}\Gamma(\frac{(2+2)}{2})(\theta)^{-(\frac{(2+2)}{2})}] = 2Q[2(\theta)\alpha^{-1}(\theta)^{-2}] \quad (3.7)$$

$$\mu_3^{'} = Q[2(\theta)\alpha^{-(\frac{3}{2})}\Gamma\frac{(3+2)}{2}(\theta)^{-(\frac{(3+2)}{2})}] = 6Q[2(\theta)\alpha^{-(\frac{3}{2})}\Gamma(\frac{5}{2})(\theta)^{-(\frac{5}{2})}] \quad (3.8)$$

$$\mu_4^{'} = Q[2(\theta)\alpha^{-(\frac{4}{2})}\Gamma\frac{(4+2)}{2})(\theta)^{-(\frac{(4+2)}{2})}] = 24Q[2(\theta)\alpha^{-2}2(\theta)^{-3}]. \quad (3.9)$$

Moments measures of Skewness, ϑ_1 and of excess kurtosis, ϑ_2, are respectively given as

$$\vartheta_1 = \frac{\mu_3}{2\&\sqrt{\mu_2^3}} \quad (3.10)$$

$$\vartheta_2 = \frac{\mu_4}{\mu_2^2} - 3. \quad (3.11)$$

We compute the the third and fourth noncentral moments of the BMWR distribution using equations (27 and 28) above. The first-four ordinary moments, variance, skewness and kurtosis are listed in Table 2 for selected parameter values of the BMWR distribution $(m, n, \beta, \gamma, \alpha); (\beta, \gamma, \alpha)$ fixed at (0.3, 5, 0.3).

Table 2: Moments of the BMWR distribution for some parameter values; $\beta = 0.3, \gamma = 5$ and $\alpha = 0.3$

μ_r'	$m = n = 0.5$	$m = 1.5, n = 0.5$	$m = 1, n = 0.5$
μ_1'	0.35331	0.70662	0.55498
μ_2'	0.24965	0.49931	0.39216
μ_3'	0.24965	0.41566	0.32646
μ_4'	0.58742	1.17485	0.92272
Variance	0.12482	0.00000	0.08416
Skewness	0.71259	0.00000	0.63108
Kurtosis	3.96253	-1.87e-05	1.14935

4 Parameter Estimation

In this section, we employ the method used in ([6] and [10]) to derive the maximum likelihood estimation (MLEs) of the parameter of the $BMWR(\alpha, \beta, \gamma, m, n)$ distribution and from [6]; setting $\tau = (m, n, \rho, \omega)$,

where $\omega = (\beta, \gamma, \theta)$ and is a vector of parameters. We had the likelihood

$$L_{BMWR}(\tau) = vlog\rho - nlog[B(m,n)] + \sum_{(i=1)}^{v} log[f(g;\tau)] + (m-1)\sum_{(i=1)}^{v} log[F(g;\tau)]$$

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$$(n-1)\sum_{(i=1)}^{v} log[1-F(g;\tau)] \tag{4.1}$$

$$L_{BMWR}(\tau) = Const - vlog[B(m,n)] + \sum_{(i=1)}^{v} log[f(g;\tau)]$$

$$+(m-1)\sum_{(i=1)}^{v} log[F(g;\tau)](n-1)\sum_{(i=1)}^{v} log[1-F(g;\tau)] \tag{4.2}$$

Taking partial derivative of (30) with respect to $(m, n, \alpha, \beta, \gamma)$, we obtain

$$\frac{\delta L_{BMWR}(\tau)}{\delta m} = -vlog(m,n) + (m-1)\sum_{(g=1)}^{v} log[F(g;\tau] \tag{4.3}$$

$$\frac{\delta L_{BMWR}(\tau)}{\delta n} = -vlog(m,n) + (n-1)\sum_{(g=1)}^{v} log[1-F(g;\omega] \tag{4.4}$$

$$\frac{\delta L_{BMWR}(\tau)}{\delta \alpha} = \sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\alpha}[f(g;\tau)]}{[f(g;\tau)]}] + (m-1)\sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\alpha}[F(g;\tau)]}{[F(g;\tau)]}]$$

$$+(n-1)\sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\alpha}[1-F(g;\tau)]}{[1-F(g;\tau)]}] \tag{4.5}$$

$$\frac{\delta L_{BMWR}(\tau)}{\delta \beta} = \sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\beta}[f(g;\tau)]}{[f(g;\tau)]}] + (m-1)\sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\beta}[F(g;\tau)]}{[F(g;\tau)]}]$$

$$+(n-1)\sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\beta}[1-F(g;\tau)]}{[1-F(g;\tau)]}] \tag{4.6}$$

$$\frac{\delta L_{BMWR}(\tau)}{\delta \gamma} = \sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\gamma}[f(g;\tau)]}{[f(g;\tau)]}] + (m-1)\sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\gamma}[F(g;\tau)]}{[F(g;\tau)]}]$$

$$+(n-1)\sum_{(g=1)}^{v} log[\frac{\frac{\delta}{\delta\gamma}[1-F(g;\tau)]}{[1-F(g;\tau)]}] \tag{4.7}$$

Equations (29) to (33) can be solved using iteration method (Newton Raphson i.e an iterative algorithm for solving equations or an approach for finding the roots of nonlinear equations and is one of the most common root-finding algorithms due to its simplicity and speed. We therefore wrote R codes for the likelihood function and assigned initial value to each parameter in the distribution) to obtain $\widehat{m}, \widehat{n}, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ the MLE of $(m, n, \alpha, \beta, \gamma)$ respectively. This is done due to complexity, rigorous and tedious mathematics involved. The result of equation (29 - 33) are in Table 4 below.

5 Application to Real-life Data

5.1 Guinea Pig Data (Virulent Tubercle Disease)

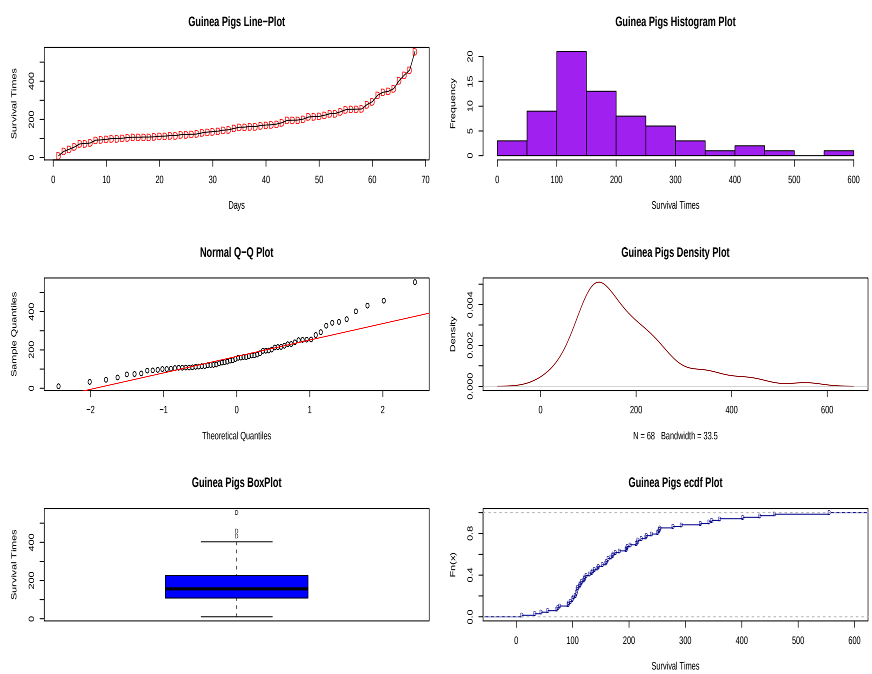
We fit BMWR and MWR distribution and compare the results. The data consists of 72 guinea pigs data and is given below:
10, 33, 44, 56, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 109, 112, 113, 115,116, 120, 121, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 251, 253, 254, 255, 278, 293, 327,342, 347, 361, 402, 432, 458, 555.

The Exploratory Data Analysis (EDA) is presented as follows:

Table 3: Summary: Descriptive Statistics

Min	1st Qut	Median	Mean	3rd Qut	Max	Skewness	Kurtosis
10.0	108.0	156.0	179.4	224.0	555.0	1.325493	4.896094.

Now, according to [16] and [14] in a symmetric distribution, as in the case of normal distribution, the coefficient of skewness of a normal distribution is 0, while the kurtosis value quantifies the weight of tails in comparison to the normal distribution for which the kurtosis equals 3 (i.e. in a normal distribution, the coefficient of kurtosis is 3). From table 3 above, the skewness is non-zero and kurtosis is greater than 3. This implies that the data is non-normal data (skewed data); and it requires flexible distribution.



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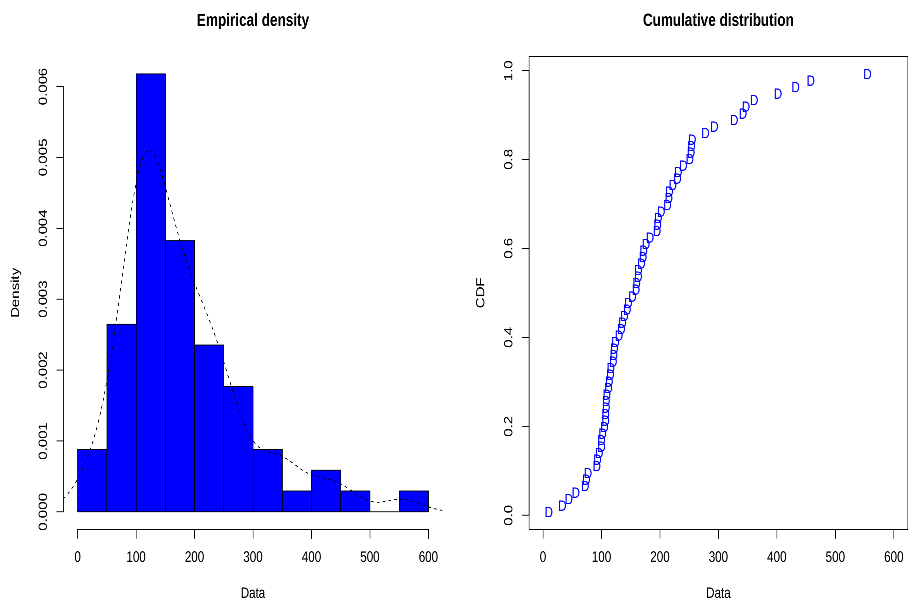

Figure 3: Contains guinea pig data line plot, normal QQ plot, box-plot, Empirical density and cumulative distribution.

5.2 Discussion and Results

Table 4: MLEs of the parameters from the baseline and proposed distributions (see equations 1, 2 and 5) fitted to the virulent tubercle disease data set, the corresponding SEs (given in parentheses)and p-value also [given in brace].

Distr/Para	m	n	α	β	γ	AIC	BIC	CAIC
BMWR	6.550	3.317	0.020	0.566	1.666			
	(0.188)	(0.036)	(0.001)	(0.010)	(0.032)	26084.96	26096.24	26097.24
MWR	–	–	3.012	-0.032	2.912			
	–	–	(0.036)	(0.004)	(0.201)	1658057	16558064	1658065

Table 3 contains the descriptive statistics of the data including the skewness and kurtosis. Unfortunately, the skewness and excess kurtosis are not normal as stated above. This is one of the reasons the data required skewed and flexible distribution to accommodate both skewness and excess kurtosis. We estimated the moments of the BMWR given some values to the parameters (m and n) in Table 2; and can be seen that the variance, skewness and kurtosis are reduced. Therefore, it implies that the proposed distribution has capability of accommodating the data.

Furthermore, Table 4 above shows the maximum likelihood estimates, standard error (in parenthesis), AIC, BIC and CAIC of both the BMWR and MWR. The log-likelihood of BMWR $\widehat{l}_{BMWR} = -13037.43$ and log-likelihood of MWR $\widehat{l}_{MWR} = -829025.70$. The likelihood ratio statistic for testing the hypothesis $H_0 : m = n = 1$ versus $H_1 : H_0$ is not true i.e., to compare the MWR and BMWR model is $w = 2(-\widehat{l}_{BMWR} - (-\widehat{l}_{MWR})) = 815988.30. Then, (p - value(< 2e - 16^{\star\star\star})$ all through while, the p-value of the baseline distribution are $(< 2e - 16^{\star\star\star})$ and $(< 6.89e - 14^{\star\star\star})$;

therefore, this gives favorable indications towards the BMWR model and MWR distribution should be rejected. Moreover, the BMWR distribution performed more than MWR distribution in fitting the data.

6 Conclusion

The Beta-modified Weighted Rayleigh distribution, having Lehmann Type II modified weighted Rayleigh and exponentiated modified weighted Rayleigh as special cases, is very useful for modeling survival data. Hence, we are able to introduce Beta-modified weighted Rayleigh (BMWR) distribution due to the wide usage of Rayleigh distribution and as a special case of Weibull distribution. In the study, we properly obtained various properties of the BMWR distributions, including reliability function, hazard rate function, reversed hazard function, moment generating function and the rth moment of the proposed distribution. We also estimated the parameters of the distribution using the method of maximum likelihood Estimation (MLEs). The variance, skewness and kurtosis were also estimated by giving different initial values to additional shape parameters (m and n) and fixed the existing shape parameters at some values (Table 2). Fortunately, the proposed distribution produced better fit than the baseline distribution and it can also provide a rather flexible mechanism for fitting a wide spectrum of real data sets. Finally, application of proposed model and baseline distribution is carried out using guinea pig data set. The superiority of the BMWR is reveals in Table 4 through information criteria results: the AIC, BIC and CAIC with lowest values. Therefore, the BMWR has better representation of the data than the MWR distribution.

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Competing financial interests

The author(s) declare no competing financial interests.

References

[1] Ramadan, M. M., A Class of Weighted Weibull distributions and its Properties.. *Studies in Mathematical Sciences,*6(1), 35-45, (2013).

[2] Jones, M. C. Families of distributions arising from distributions of order statistics test,.13,1 - 43,(2004).

[3] Nadarajah, S. & Kotz, S. The beta Gumbel distribution.. *Mathematics Probability Engineering.* **10**, 323 - 332, (2004).

[4] Nadarajah, S. & Kotz, S. On Some Recent Modifications of Weibull Distribution.. *IEEE Transactions Reliability* **54**, 561 - 562, (2005).

[5] Akinsete, A., Famoye, F. & Lee, C. The beta pareto distribution.. *Statistics.* **42**(6), 547-563, (2008).

[6] Zografos, K. & Balakrishnan, N. On families of beta- and generalized gamma-generated distributions and associated inference.. *Statistical Methodology.* **6**, 344 - 362, (2009).

[7] Barreto-Souza W.,Santos AHS. & Cordeiro G. M. The beta generalized exponential distribution.. *Journal of Statistical Computation and Simulation.* **80**(2), 159-172, (2010).

[8] Cordeiro G. M., Silva G. O. & Ortega E. M. M. The beta extended Weibull distribution.. *J Probab Stat Sci* **10**, 15-40, (2012).

[9] Singha, N., Jain, K. & Kumar, S. S. The Beta Generalized Weibull Distribution: Properties and Applications.. *Reliability Engineering and System Safety.* **102**, 5 - 15, (2012).

[10] Badmus, N. I. & Ikegwu, E. M. The Beta Weighted Weibull Distribution: Some Properties and Application to Bladder Cancer Data.. *Journal of Applied and Computational Mathematics.* **2**(5), 1- 6, (2013).

[11] Cordeiro G. M.,Silva G. O., Pescim R. R. & Ortega E. M .M. General properties for the beta extended half-normal distribution.. *J. Stat. Comput Simul.* **84**, 881 - 901, (2014).

[12] Cordeiro G., Gomes A., da Silva C. & Ortega M. The Beta Exponentiated Weibull Distribution.. *Journal of Statistical Computation and Simulations.* **38**(1), 114 - 13, (2013).

[13] Gupta, K. & Nadarajah, S. On the moments of the beta normal distribution.. *Comm. Statist. Theory Methods.* **33**(1), 1–13, (2004).

[14] Lemonte, A. .J. The beta log-logistic distribution.. *Braz. J. Probab. Statist.* **28**, 313-332, (2014).

[15] Badmus, N. I. & Bamiduro, T. A. Life Length of Components Estimates with Beta Weighted Weibull Distribution.. *Journal of Statistics: Advances in Theory and Applications.* **11**(2), 91-107, (2014).

[16] Alizadeh, M., Tahir, M.H., Cordeiro G. M., Zubair, M. & Hamedani, G. G. The Kumaraswamy Marshal-Olkin family of distributions.. *Journal of the Egyptian Mathematical Society, in press* (2015).

[17] Badmus, N. I., Bamiduro, T. A., Rauf-Animasaun, M. A. & Akingbade, A. A. The Beta Weighted Exponential Distribution: Theory and Applications.. *International Journal on Mathematical Analysis and Optimization: Theory and Application.* **2015**, 55 – 66, (2015).

[18] Mead M. E., Afify A. Z., Hamedani G. G. & Ghosh I. The Beta Exponential Frechet Distribution with Applications.. *Austrian Journal of Statistics.* (2016).

[19] Domma F. & Condino F. Use of the Beta-Dagum and Beta-Singh-Maddala distributions for modeling hydrologic data.. *Stoch Environ Res Risk Assess, published on line.* (2016).

[20] Aleem, M., Sufyan M. & Khan, N. S. A class of Modified Weighted Weibull Distribution and its properties.. *America Review of Mathematics and Statistics* **1**(1), 29 – 37, (2013).

[21] Gupta, R. D.& Kundu, D. Exponentiated exponential family: an alternative to gamma and weibull distributions.. *Biometrical Journal,* **43**, 117 - 130, (2001).

[22] Lai, C., Xie, M. & Murthy, D. A Modified Weibull Distribution.. *IEEE Transactions on Reliability* **52**, 33 – 37, (2003).

[23] Nadarajah, S. & Kotz, S. The Beta Exponential distribution.. *Reliability Engineering and System Safety.* **91**, 689 - 697, (2006).

[24] Carrasco M., Ortega E. M. & Cordeiro G. A Generalized Weibull Distribution for Lifetime Modeling.. *Computational Statistics and Data Analysis* **53**(2), 450 - 462, (2008).

[25] Silva, G., Ortega, E. M. & Cordeiro, G. The Beta Modified Weibull Distribution.. *Lifetime Data Analysis.* **16**, 409 - 430, (2010).

[26] Cordeiro, G. M.. & Alexandra de Castro, M. Generalized Beta Generated distributions. ICMA Centre.. *Discussion Papers in Finance DP 2011-05,* (2011).

[27] Baharith L.A., Mousa S. A., Atallah M. A. & Elgyar S. H. The beta generalized inverse Weibull distribution.. *British J. Math. Comput Sci.* **4**, 252-270, (2014).

[28] Hosking, J. R. M. L-moments Analysis and Estimation of distributions using Linear Combinations of Order Statistics.. *Journal Royal Statistical Society B.* **52**, 105-124, (1990).

[29] Ezequiel Uriel. Relaxing the Assumptions in the Linear Classical Model.. *University of Valencia.* **9**, 1 - 45, (2013).

[30] Delignette-Muller, M. L & Dutang, C. An R Package for Fitting Distributions.. *http://cran.r-project.org/package.* 1 - 23, (2014).