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Coefficient Bounds for Bazilevic Functions Involving Logistic Sigmoid Function Associated with Conic Domains

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Abstract

Let $k \geq 0$, $-1 \leq B < A \leq 1, \alpha > 0, n \in N_0 = N \cup \{0\}, 0 \leq \lambda \leq 1$ and $z \in U$. Also, let $M_{\lambda,k}^{\alpha,n}(\phi, A, B)$ denotes the class of Bazilevic functions involving logistic sigmoid function associated with conic domain and satisfying the condition that

$$\Re\left\{\frac{(B-1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A-1)}{(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)}\right\}$$

$$> k\left|\frac{(B-1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A-1)}{(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)}-1\right|.$$

We obtain sharp bounds of the first four initial coefficients $|a_2(\alpha)|$, $|a_3(\alpha)|$, $|a_4(\alpha)|$ and $|a_5(\alpha)|$ for functions belonging to the class $M_{\lambda,k}^{\alpha,n}(\phi, A, B)$ in the unit disk. Relevant connection of these coefficients to the classical Fekete-Szego functional is also considered.

Keywords: Analytic, Univalent, Bazilevic, Sigmoid functions, Conic domains.
MSC2010: Primary 30C45, 33E99

1 Introduction and Definitions

The concept of analytic functions has long been explored and found worthy of used in solving many physical problems such as in heat conduction, electrostatic potential and so on. So also is the theory of special functions such as sigmoid function. It is conventionally believed that activation function has to do with the way nervous system such as brain process information. In actual fact, activation function is composed of large number of highly interconnected processing element (neurons) working to perform a definite task. This function works in similar way the brain does. The most widely sigmoid function is the logistic function which has a lower bound of zero (0) and upper bound of one (1).

Many sigmoid functions have power series expansion which alternate in sign while some have inverse with hypergeometric series expansion. They can be evaluated differently especially by truncated series expansion. The logistic function has the following Mathematical formulation:

$$g(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

with the following properties.
(i.) It outputs real number between 0 and 1.
(ii.) It maps a very large input domain to a small range of outputs.
(iii.) It never loses information because it is a one- to- one function.
(iv.) It increases monotonically.
In view of the above properties, sigmoid function is highly referred in geometric function theory (see [1, 2, 3, 4, 5]).

Now, let A denote the class of all functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

normalized with $f(0) = f'(0) - 1 = 0$ that are analytic in the open unit disk $U = \{z : |z| < 1\}$ and by Ψ the class of univalent functions $f \in A$. Noor and Malik [6] had earlier studied the classes $k - ST[A, B]$ and $k - UCV[A, B]$ defined as follow:
A function $f(z) \in A$ is said to be in the class $k - ST[A, B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)} - 1\right|.$$

A function $f(z) \in A$ is said to be in the class $k - UCV[A, B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} - 1\right|.$$

It is not difficult to verify that

$$f(z) \in k - UCV[A, B] \Leftrightarrow zf'(z) \in k - ST[A, B].$$

Special cases of the above definitions can be found in [7, 8, 9, 10].

On the other hand, Bazilevic in 1955 [11] introduced and studied the Bazilevic function $I(z)$ given by

$$I(z) = \left\{\frac{\alpha}{1+\epsilon^2}\int_0^z \frac{p(v) - i\epsilon}{v^{\left(1+\frac{i\alpha\epsilon}{(1+\epsilon^2)}\right)}} g(v)^{\frac{\alpha}{1+\epsilon^2}}\, dv\right\}^{\frac{1+i\epsilon}{\alpha}}, \tag{1.3}$$

where $\alpha > 0$, ϵ is real, $p \in P$ (Class of Caratheodory functions) and $g \in \Psi^*$ (class of starlike functions).
The class of functions defined above is usually denoted by $B(\alpha, \epsilon)$. Very little is known about this class (I(z)), expect that Bazilevic showed that each function $f \in B(\alpha, \epsilon)$ is univalent. However, by simplifying (3) it is possible to understand and investigate the family better.

Abdulhalim in 1992 [12] studied certain generalised class of Bazilevic functions T_n^α say, satisfying the inequality

$$\Re\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\} > 0, \qquad z \in U \tag{1.4}$$

while Opoola in 1994 [13] gave a more generalised form of (4) as

$$\Re\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\} > \beta, \qquad z \in U, \tag{1.5}$$

and denoted it by $T_n^\alpha(\beta)$ where $\alpha > 0$ (α is real) and D^n is the well-known Salagean derivative Operator (see [2, 13, 14, 15, 16, 17]), $0 \le \beta < 1$ and $n \in N \cup \{0\}$. In the recent time, a little modification was made to (5) by Babalola [17] such that

$$T_n^\alpha(\beta) = \Re\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta, \qquad z \in U \tag{1.6}$$

It is noted here that

$$T_n^\alpha(0) = T_n^\alpha.$$

The study of Bazilevic functions has triggered the interest of both young and old researchers in the recent time. They have succeeded not only in simplifying the Bazilevic $I(z)$ as a whole in succinct mathematical terms but also in describing certain prescribed properties of Bazilevic functions such as coefficient inequalities, Bi-univalent conditions, Bazilevic functions in the space of harmonic functions, subordination conditions for Bazilevic function and so on (see [2, 14, 17] among others).

However, the studies of Bazilevic functions involving logistic sigmoid functions and their behaviour in conic regions were not famous in literatures. Consequently, for the purpose of the present investigation the authors define a modified Bazilevic function $F_{\alpha,n}(z) \in T_n^\alpha(\beta)$ by letting

$$F_{\alpha,n}(z) = z\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right)$$

such that

$$F_{\alpha,n}(z) = z + \sum_{m=2}^{\infty} \alpha_{n,m} a_m(\alpha) z^m, \tag{1.7}$$

where

$$\alpha_{n,m} = \left(\frac{\alpha + m - 1}{\alpha}\right)^n, \quad z \in U.$$

Definition: A function $F_{\alpha,n}(z) \in A$ is said to be in the class $M_{\lambda,k}^{\alpha,n}(\phi, A, B)$ if and only if

$$\Re\left\{\frac{(B-1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A-1)}{(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)}\right\}$$

$$> k\left|\frac{(B-1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A-1)}{(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)} - 1\right| \tag{1.8}$$

$$\big(k \ge 0,\ 0 \le \lambda \le 1,\ \alpha > 0,\ -1 \le B < A \le 1,\ n \in N_0 = N \cup \{0\} \ \ \textit{and} \ \ z \in U\big).$$

With various choices of the parameters involved several new classes of analytic functions as well as the existing ones studied by various authors are obtained. Here are few examples.

Example 1: $M_{0,\,k}^{\alpha,\,n}(\phi, A, B) \equiv$ the class of starlike functions of Bazilevic type associated with conic domains.

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Example 2: $M_{1,\,k}^{\alpha,\,n}(\phi, A, B) \equiv$ the convex class of Bazilevic type associated with conic domains.
Example 3: $M_{1/2,\,k}^{\alpha,\,n}(\phi, A, B) \equiv$ the class of bounded turning functions of Bazilevic type associated with conic domains.
Example 4: $M_{0,\,k}^{1,\,0}(\phi, A, B) \equiv$ the class of starlike functions associated with conic domains.
Example 5: $M_{1/2,\,k}^{1,\,0}(\phi, A, B) \equiv$ the class of bounded turning functions associated with conic domains.
Example 6: $M_{1,\,k}^{1,\,0}(\phi, A, B) \equiv$ the class of convex functions associated with conic domains.
Example 7: $M_{0,\,0}^{1,\,0}(\phi, 1, -1) \equiv$ the class of starlike functions.
Example 8: $M_{1,\,0}^{1,\,0}(\phi, 1, -1) \equiv$ the class of convex functions.
Example 9: $M_{1/2,\,0}^{1,\,0}(\phi, 1, -1) \equiv$ the class of bounded turning functions.
Example 10: $M_{0,\,0}^{\alpha,\,n}(\phi, 1, -1) \equiv$ the starlike class of Bazilevic type.
Example 11: $M_{1,\,0}^{\alpha,\,n}(\phi, 1, -1) \equiv$ the convex class of Bazilevic type.
Example 12: $M_{1/2,\,0}^{\alpha,\,n}(\phi, 1, -1) \equiv$ the class of bounded turning functions of Bazilevic type.

2 Coefficient Bounds

The following Lemmas shall be necessary for the purpose of our present investigation.
Lemma 2.1: Let h be a sigmoid function and

$$\phi(z)_{i,j} = 2h(z) = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{2^i} \left(\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \right)^i. \tag{2.1}$$

Then $\phi(z)_{i,j} \in P$, $|z| < 1$ where $\phi(z)_{i,j}$ is a sigmoid function (see [1, 2, 3]).
Lemma 2.2: Let

$$\phi(z)_{i,j} = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{2^i} \left(\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \right)^i. \tag{2.2}$$

Then

$$\left|\phi_{i,j}(z)\right| < 2(see[1, 2, 3]).$$

Lemma 2.3: If $\phi(z) \in P$ and it is starlike, then f is a normalised univalent function of the form (2) (see [1, 2, 3, 5]).
Suppose that $j = 1$ the following remark is due to Joseph *et al.* [1].
Remark A: Let

$$\phi(z)_{i,j} = 1 + \sum_{i=1}^{\infty} c_i z^i, \tag{2.3}$$

where

$$c_i = \frac{(-1)^{i+1}}{2i!}.$$

Then

$$|c_i| \leq 2, \quad i = 1,\ 2,\ 3,\ \ldots\ . \tag{2.4}$$

This result is sharp for each i (see [1, 2, 3]).
Theorem 2.4: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{\lambda,k}^{\alpha,n}(\phi, A, B)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\}$ *and* $z \in U$, then

$$\left|a_2(\alpha)\right| \leq \frac{\left|B - A\right|}{4(k+1)(-2\lambda^2 + 3\lambda + 1)\left(\frac{\alpha+1}{\alpha}\right)^n}, \tag{2.5}$$

$$\left|a_3(\alpha)\right| \leq \frac{\left|(B-A)D_1\right|}{32(k+1)^2(-2\lambda^2+3\lambda+1)(2\lambda^2+1)\left(\frac{\alpha+2}{\alpha}\right)^n}, \tag{2.6}$$

$$\left|a_4(\alpha)\right| \leq \frac{|B-A|\left\{3\left|D_1D_2\right|+8(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right\}}{384(k+1)^3(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)\left(\frac{\alpha+3}{\alpha}\right)^n}, \tag{2.7}$$

$$\left|a_5(\alpha)\right| \leq \frac{|B-A|\left\{\left|D_3\right|\left[3\left|D_1D_2\right|+8(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right]-8E\right\}}{1536(k+1)^4(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)(28\lambda^2-12\lambda+4)\left(\frac{\alpha+4}{\alpha}\right)^n}, \tag{2.8}$$

where

$$\begin{aligned}
D_1 &= 2(B+1)(2\lambda^2-\lambda+1)-(A+1)(6\lambda^2-5\lambda+1),\\
D_2 &= 3(B+1)(4\lambda^2-2\lambda+1)-(A+1)(8\lambda^2-6\lambda+1),\\
D_3 &= 4(B+1)(6\lambda^2-3\lambda+1)-(A+1)(10\lambda^2-7\lambda+1)
\end{aligned}$$

and

$$E = \left|D_1\right|(k+1)^3(2\lambda^2-3\lambda-1)(2\lambda^2+1)(14\lambda^2-5\lambda+3).$$

Proof: If $F_{\alpha,n}(z) \in A$, then by definition there exists $\phi \in P$ such that

$$(1+k)\left\{\frac{(B-1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A-1)}{(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)}\right\}-k=\phi(z), \tag{2.9}$$

where $\phi(z)$ is the modified sigmoid function given by

$$\phi(z) = 1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\frac{779}{20160}z^7-\dots\,. \tag{2.10}$$

Equation (17) can be re-expressed as

$$\begin{aligned}
&(1+k)\left\{(B-1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A-1)\right\}\\
&-k\left\{(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)\right\}\\
&=\phi(z)\left[(B+1)\left(\frac{zF'_{\alpha,n}(z)+(2\lambda^2-\lambda)z^2F''_{\alpha,n}(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zF'_{\alpha,n}(z)+(2\lambda^2-3\lambda+1)F_{\alpha,n}(z)}\right)-(A+1)\right].
\end{aligned} \tag{2.11}$$

It implies that

$$\begin{aligned}
&(B-A)z+\sum_{m=2}^{\infty}\Big[(1+k)S+kT\Big]\alpha_{n,m}a_m(\alpha)z^m\\
&=\left(1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\frac{779}{20160}z^7-\dots\right)\left[(B-A)z+\sum_{m=2}^{\infty}T\alpha_{n,m}a_m(\alpha)z^m\right],
\end{aligned} \tag{2.12}$$

where
$$S = m(B-1)\big[1+(m-1)(2\lambda^2-\lambda)\big] - (A-1)\big[m(2\lambda^2-\lambda)+2\lambda^2-3\lambda+1\big]$$
and
$$T = m(B+1)\big[1+(m-1)(2\lambda^2-\lambda)\big] - (A+1)\big[m(2\lambda^2-\lambda)+2\lambda^2-3\lambda+1\big].$$
Equating the coefficients of the like powers of z, z^2, z^3, z^4 and z^5. We have

$$a_2(\alpha) = \frac{(B-A)}{4(k+1)(2\lambda^2-3\lambda-1)\left(\frac{\alpha+1}{\alpha}\right)^n}, \tag{2.13}$$

$$a_3(\alpha) = \frac{(B-A)D_1}{-32(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\left(\frac{\alpha+2}{\alpha}\right)^n}, \tag{2.14}$$

$$a_4(\alpha) = \frac{(B-A)\left\{3[D_1D_2]+8(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right\}}{384(k+1)^3(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)\left(\frac{\alpha+3}{\alpha}\right)^n}, \tag{2.15}$$

$$a_5(\alpha) = \frac{(B-A)\left\{D_3\left[3D_1D_2+8(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right]-8E\right\}}{-1536(k+1)^4(2\lambda^2-3\lambda-1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)(28\lambda^2-12\lambda+4)\left(\frac{\alpha+4}{\alpha}\right)^n}, \tag{2.16}$$

where
$$D_1 = 2(B+1)(2\lambda^2-\lambda+1)-(A+1)(6\lambda^2-5\lambda+1),$$
$$D_2 = 3(B+1)(4\lambda^2-2\lambda+1)-(A+1)(8\lambda^2-6\lambda+1),$$
$$D_3 = 4(B+1)(6\lambda^2-3\lambda+1)-(A+1)(10\lambda^2-7\lambda+1)$$
and
$$E = D_1(k+1)^3(2\lambda^2-3\lambda-1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)$$
. This ends the proof of theorem 2.4.

Corollary 2.5: Let $F_{\alpha,n}(z) \in A$ be in the class $M^{\alpha,0}_{\lambda,0}(\phi,1,-1)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\}$ *and* $z \in U$, then

$$\left|a_2(\alpha)\right| \leq \frac{1}{2(-2\lambda^2+3\lambda+1)},$$

$$\left|a_3(\alpha)\right| \leq \frac{6\lambda^2-5\lambda+1}{8(-2\lambda^2+3\lambda+1)(2\lambda^2+1)},$$

$$\left|a_4(\alpha)\right| \leq \frac{3(6\lambda^2-5\lambda+1)(8\lambda^2-6\lambda+1)+2(2\lambda^2-3\lambda-1)(2\lambda^2+1)}{48(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)},$$

$$\left|a_5(\alpha)\right| \leq \frac{(10\lambda^2-7\lambda+1)\left[3(6\lambda^2-5\lambda+1)(8\lambda^2-6\lambda+1)+2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right]-F}{96(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)(28\lambda^2-12\lambda+4)},$$

where
$$F = 2(2\lambda^2+1)(2\lambda^2-3\lambda-1)(14\lambda^2-5\lambda+1)(14\lambda^2-5\lambda+3).$$

Corollary 2.6: Let $F_{\alpha,n}(z) \in A$ be in the class $M^{1,1}_{\lambda,k}(\phi,A,B)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\}$ *and* $z \in U$, then

$$\left|a_2(\alpha)\right| \leq \frac{|B-A|}{8(k+1)(-2\lambda^2+3\lambda+1)},$$

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$$\left|a_3(\alpha)\right| \leq \frac{\left|(B-A)D_1\right|}{96(k+1)^2(-2\lambda^2+3\lambda+1)(2\lambda^2+1)},$$

$$\left|a_4(\alpha)\right| \leq \frac{|B-A|\left\{3\left|D_1D_2\right|+8(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right\}}{1536(k+1)^3(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)},$$

$$\left|a_5(\alpha)\right| \leq \frac{|B-A|\left\{\left|D_3\right|\left[3\left|D_1D_2\right|+8(k+1)^2(2\lambda^2-3\lambda-1)(2\lambda^2+1)\right]-8E\right\}}{7680(k+1)^4(-2\lambda^2+3\lambda+1)(2\lambda^2+1)(14\lambda^2-5\lambda+3)(28\lambda^2-12\lambda+4)},$$

where D_1, D_2, D_3 and E are as earlier defined.

Corollary 2.7: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{1,0}^{\alpha,0}(\phi, 1, -1)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\}\ \ and\ \ z \in U$, then

$$\left|a_2(\alpha)\right| \leq \frac{1}{4},\quad \left|a_3(\alpha)\right| \leq \frac{1}{24},\quad \left|a_4(\alpha)\right| \leq \frac{1}{384},\quad and\quad \left|a_5(\alpha)\right| \leq \frac{39}{17280}.$$

Corollary 2.8: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{0,0}^{\alpha,0}(\phi, 1, -1)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\}\ \ and\ \ z \in U$ then

$$\left|a_2(\alpha)\right| \leq \frac{1}{2},\quad \left|a_3(\alpha)\right| \leq \frac{1}{8},\quad \left|a_4(\alpha)\right| \leq \frac{1}{144},\quad and\quad \left|a_5(\alpha)\right| \leq \frac{7}{1552}.$$

Remark B: The first three initial coefficients obtained in Corollary 2.8 are due to Hamzat and Olayiwola [2]; Murugusundaramoorthy and Janani [3].

Theorem 2.9: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{\lambda,k}^{\alpha,n}(\phi, A, B)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\},\ \ \mu \in \Re\ \ and\ \ z \in U$, then

$$\left|a_3(\alpha) - \mu a_2^2(\alpha)\right| \leq \left|B-A\right| \left|\frac{D_1(2\lambda^2-3\lambda-1)\left(\frac{\alpha+1}{\alpha}\right)^{2n} + 2\mu(B-A)(2\lambda^2+1)\left(\frac{\alpha+2}{\alpha}\right)^{n}}{32(k+1)^2(2\lambda^2-3\lambda-1)^2(2\lambda^2+1)\left(\frac{\alpha+1}{\alpha}\right)^{2n}\left(\frac{\alpha+2}{\alpha}\right)^{n}}\right|, \quad (2.17)$$

where

$$D_1 = 2(B+1)(2\lambda^2-\lambda+1) - (A+1)(6\lambda^2-5\lambda+1).$$

Proof: Using (21) and (22) the result is immediate.

Corollary 2.10: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{1,0}^{\alpha,0}(\phi, A, B)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\},\ \ \mu \in \Re\ \ and\ \ z \in U$, then

$$\left|a_3(\alpha) - \mu a_2^2(\alpha)\right| \leq \frac{1}{64}\left|B-A\right|\left|4 + 3\mu(B-A)\right|.$$

Corollary 2.11: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{0,0}^{\alpha,0}(\phi, A, B)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\},\ \ \mu \in \Re\ \ and\ \ z \in U$, then

$$\left|a_3(\alpha) - \mu a_2^2(\alpha)\right| \leq \frac{1}{16}\left|B-A\right|\left|1 + \mu(B-A)\right|.$$

Corollary 2.12: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{1,0}^{\alpha,0}(\phi, 1, -1)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\},\ \ \mu \in \Re\ \ and\ \ z \in U$, then

$$\left|a_3(\alpha) - \mu a_2^2(\alpha)\right| \leq \frac{1}{48}\left|(2 - 3\mu)\right|.$$

Corollary 2.13: Let $F_{\alpha,n}(z) \in A$ be in the class $M_{0,0}^{\alpha,0}(\phi, 1, -1)$. If $k \geq 0,\ 0 \leq \lambda \leq 1,\ \alpha > 0,\ -1 \leq B < A \leq 1,\ n \in N_0 = N \cup \{0\},\ \ \mu \in \Re\ \ and\ \ z \in U$, then

$$\left|a_3(\alpha) - \mu a_2^2(\alpha)\right| \leq \frac{1}{8}\left|(1 - 2\mu)\right|.$$

Remark C: If $\mu = 1$ in corollaries 2.12 and 2.13, then we have

$$\left|a_3(\alpha) - a_2^2(\alpha)\right| \leq \frac{1}{48}$$

and

$$\left|a_3(\alpha) - a_2^2(\alpha)\right| \leq \frac{1}{8}.$$

respectively. Finally, with further choices of the parameters involved several other results(known and new) are obtained.

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