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Three-Step Block Method for Solving Second Order Differential Equations

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Abstract
In this paper, we developed a three-step block Method for numerical solution of second order differential equations using Legendre polynomials as the basic function. Interpolation and collocation procedures are used by choosing interpolation points at $s = 2$ steps points using power series, while collocation points at $r = k$ step points, using a combination of power series and perturbation term gotten from the Legendre polynomials, giving rise to a polynomial of degree $r + s - 2$ and $r + s$ equations. All the analysis on the scheme derived shows that it is stable, convergent and has region of Absolute Stability. Numerical examples were provided to test the performance of the method. Results obtained when compared with existing methods in the literature, shows that the method is accurate and efficient.

Keywords: Three-step Block method, Legendre Polynomials and absolutely stable.

MSC2010: 65L05, 65L06

1. Introduction

Numerous problems in many field of application, notably in physics, chemistry, biology, engineering and social sciences are modeled mathematically by ordinary differential equation (ODEs) e.g. series circuits, mechanical systems with several springs attached in series lead to a system of differential equation. Abualnaja [1] and also in diverse fields like economics, medicine, psychology, operation research and even in anthropology are modeled mathematically. Anake [2] .
Interestingly, some differential equations arising from the modeling of physical phenomena, often do not have analytic solutions, hence the development of numerical method to obtain approximate solutions become necessary. Ehigie et al. [3] . To that extent several numerical methods such as one step method, linear multi-step methods, hybrid methods and block method have been developed based on the nature and type of the differential equation to be solved. Some researchers have attempted the solution of

$$y^{(n)} = f\left(x, y, y', y''..., y^{(n-1)}\right), y\left(x_0\right) = y_0, y'\left(x\right) = y_1, ..., y^{(n-1)} = y_{n-1} \tag{1.1}$$

using linear multistep methods (LMMs), without reduction to system of first order ODEs. Adeniyi and Adeyefa [4] . Ehigie et al. [3] proposed a generalized 2 – step continuous linear multistep method of hybrid type for the integration of second order ordinary differential equations.

Kayode and Adebeye [5] used Chebyshev polynomials without perturbation terms as the basic function for the development of the methods. The collocation and interpolation equations are generated at both grid and off – grid points for the development of continuous hybrid linear multistep method (CHLMM) for the solution of linear and non linear ODEs

Ademiluyi [6] , Anake [2] and Bolarinwa [7] have proposed single step hybrid methods for the direct numerical solution of initial value problem of second order and third orders ordinary differential equations. In these cases, their methods of implementation was in block mode with the proposed methods being efficient, adequate and suitable towards catering for the class of problem of higher order ordinary differential equations for which they were designed Osilagun et al. [8] , used four steps implicit method for the solution of general second order ODEs. Abdulganiy et al. [9] used a maximal order block trigonometrically fitted scheme for the numerical treatment of second order ODEs with ossillating solution. Peter and Ibrahim [10] , used defferential transform method in solving a typhoid fever model. However, Authors like. Zarina et al. . [11] , used block method for generalized multistep Adamas method and backward differentiation formula in solving first – order ODEs. Yahaya and Mohammed [12] used full implicit three points backward differentiation formulae for solving of first order initial value problems. Odekunle et al. [13] used a new block integrator for the solving of initial value problems of first order ODESs. Sunday et al. [14] used a computational approach to verhulst – pearl model of first order ODEs etc

Many of these methods have their own advantages and disadvantages over the other. Eg; One step method have low order of accuracy, time consuming for large scale problems. Awoyemi [15] . Linear Multistep Methods give high order system of accuracy and are suitable for the direct solution of (1.1) without necessarily reducing it to an equivalent of first order IVPs of ODEs. Adeniyi and Adeyefa [4] . Block method preserves the traditional advantages of one step methods of being self starting and permitting easy change of step length. Lambert [16] . Also the method generates simultaneous solution at all grids points.

In the light of this, Abualnaja [1] worked on “A block procedure with linear multi-step methods using Legendre polynomials for solving ODEs”. Here they derived a block for some k-step linear multi-step methods (for k = 1,2 and 3) using power series as the interpolation equation and power series with Legendre polynomial as the perturbation term as the collocation equation. Also Abhulimen and Aigbiremhon [17] did a similar work by taking K as 4 and 5. In their work, they considered the first order initial value problem.

These different methods have their very desirable qualities. However, in order to create a new line of research and to also improve on some of the existing methods, this paper device a mean for the direct solution of (1.1) without reduction to first order ODEs. In the next section, the methodology of the work is presented and the derived methods are specified.

The plan of the paper is as follows; section I, introduction, section 2, the derivation of the proposed methods is presented. In section 3, the stability and convergence analysis of the block schemes is given. In section 4, numerical examples are considered. The paper ends with conclusion in section 5.

2. Derivation of the methods

In this section, we derive discrete methods to solve (1.1) at a sequence of nodal points $x_n = x_0 + nh$ where h > 0 is the step – length or grid size defined by $h = x_{n+1} - x_n$ and $y(x)$ denotes the true solution to (1.1) while the approximate solution is denoted by the power series

$$y_{(x)} = c_0 x_n^0 + c_1 x_n^1 + c_2 x_n^2 + ... + c_k x_n^k \tag{2.1}$$

The proposed method depends on the perturbed collocation method with respect to the power series with the Legendre polynomials as the perturbation term. Interpolation and collocation procedures are used by choosing interpolation point at $s = 2$ grid points and collocation points at $r = k$ step points. We have a polynomial of degree $r + s - 2$ and $(r + s)$ equations.

In the first place, we consider the approximation solution of (1.1) in the power series.

$$p_i(x) = x^i, i = 0,1,...,k$$

Hence (2.1) becomes

$$y_k(x) = c_i p_i(x) = \sum_{i=0}^{k} c_i x^i \tag{2.2}$$

With the second derivatives as

$$y_k''(x) = c_i p_i''(x) = \sum_{i=0}^{k} i(i-1) c_i x^{i-2} \tag{2.3}$$

Combining equation (1.1) and (2.3), with the perturbation term, we have

$$\sum_{i=1}^{k} c_i p''(x) = f(x, y, y') + \lambda L_k(x_{n+i}), i = 1, (1)k \tag{2.4}$$

Where $L_k(x)$ is the Legendre polynomial of degree k, valid in $x_n \le x \le x_{n+k}$ and λ is a perturbed parameter. In particular, we shall be dealing with case k = 3 in (2.2) and (2.4), where (2.2) is the interpolation equations and (2.4) is the collocation equations. The well – known Legendre polynomials can be generated using the Rodrigues' formula $p_n(x) = \frac{1}{2^n n!} \frac{1}{dx^n}\left[(x^2-1)^n\right]$ where $L_0(x) = 1, L_1(x) = x,$. The rest are computed using the recurrence formula.

$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), i = 1,2,\ldots \text{giving}$$

$$L_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$L_3(x) = \frac{1}{2}(5x^3 - 3x), L_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$L_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \text{etc.} \tag{2.5}$$

In order to use these polynomials in the interval $[x_n, x_{n+k}]$ we define the shifted Legendre polynomials by introducing the change of variable.

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)}, \text{ Abualnaja [1]} \tag{2.6}$$

Interpolating (2.2) at s grid points and collocating (2.4) at k grid points respectively leads to the following systems of equations; (2.7) and (2.8)

$$\sum_{i=0}^{k} c_i p_i(x) = y_{n+s}, \ s = 0,1 \tag{2.7}$$

and

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$$\sum_{i=0}^{k} c_i p_i''(x) = f_{n+j} + \lambda L_k(x_{n+j}),\ j = 1(1)k \tag{2.8}$$

Three step method', (k = 3).

In this case, we take the polynomial $L_3(x) = \frac{1}{2}(5x^3 - 3x)$ from (2.5) and use (2.6) i.e. $x = \frac{2\bar{x} - (x_{n+k} + x_n)}{x_{n+k} - x_n}$, to obtain value for $L_3(x_{n+1}), L_3(x_{n+2})$ and $L_3(x_{n+3})$ to be

$$x = -\frac{1}{3}\ and\ L_3(x_{n+1}) = \frac{11}{27}$$
$$x = \frac{1}{3}\ and\ L_3(x_{n+2}) = -\frac{11}{27}$$
$$x = 1\ and\ L_3(x_{n+3}) = 1$$

In addition, from (2.3) $c_0 p_0''(x) = 0, c_1 p_i''(x) = 0,\ c_2 p_2''(x) = 2c_2$ and $c_3 p_3''(x) = 6c_3 x_n$ .Then (2.8) will reduced to the form.

$$0 + 0 + 2c_2 + 6c_3 x_n = f(x, y, y') + \lambda L_3(x_{n+i})\ i = 1,2,3 \tag{2.9}$$

We now collocate Equation (2.9) at $x_{n+i}, i = 1,2,3$ and interpolate (2.1) at $x_{n+i},\ i = 0,1$. We get a system of 5 equations with $c_i\ i = 0,\ 1,\ 2,\ 3$ and λ, which in matrix form is:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & .0 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & 0 \\ 0 & 0 & 2 & 6x_{n+1} & -\frac{11}{27} \\ 0 & 0 & 2 & 6x_{n+2} & \frac{11}{27} \\ 0 & 0 & 2 & 6x_{n+3} & -1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \lambda \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{2.10}$$

Equation (2.10) is solved by Gaussian elimination method to obtain the value of the unknown parameters.

$c_i, (i = 0,1,2\,3)$ and λ,

Which is substituted into (2.1) yields a continuous implicit three step method in the form of a continuous linear multistep method describe by the formula

$$y_{(x)} = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + h^2 \sum_{i=1}^{k} \beta_i(x) f_{n+i} \tag{2.11}$$

Where

$$\alpha_0(t) = -1 - t, \qquad \alpha_1(t) = 2 + t$$

$$\beta_1(t) = -\frac{h^2}{360}\left[294 + 365t + 33t^2 - 38t^3\right]$$

$$\beta_2(t) = \frac{h^2}{180}\left[66 + 125t + 57t^2 + 8t^3\right]$$

$$\beta_3(t) = \frac{h^2}{360}\left[-66 - 55t + 33t^2 + 22r^3\right] \tag{2.12}$$

Where

$t = \dfrac{x_n - x_{n+2}}{h}$. Evaluating (2.12) at t = 0 and t = 1 and substituting into (2.11) gives

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{60}\left[-11f_{n+3} + 22f_{n+2} + 49f_{n+1}\right] \tag{2.13}$$

$$y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{60}\left[-11f_{n+3} + 82f_{n+2} + 109f_{n+1}\right]$$

Evaluating the first derivative of (2.12) at $t = -2, -1, 0$ and 1 and substituting into (2.11) gives

$$\begin{aligned}
&360hy_n' - 360y_{n+1} + 360y_n = -h^2\left[-77f_{n+3} + 34f_{n+2} + 233f_{n+1}\right]\\
&72hy_{n+1}' - 72y_{n+1} + 72y_n = h^2\left[-11f_{n+3} + 10f_{n+2} + 37f_{n+1}\right]\\
&72hy_{n+2}' - 72y_{n+1} + 72y_n = h^2\left[-11f_{n+3} + 46f_{n+2} + 73f_{n+1}\right]\\
&360hy_{n+3}^1 - 360y_{n+1} - 360y_n = h^2\left[77f_{n+3} + 506f_{n+2} + 317f_{n+1}\right]
\end{aligned} \tag{2.14}$$

Now we obtained the modified block formulae from (2.13) and (2.14) as

$$\begin{bmatrix} -120 & 60 & 0 & 0 & 0 & 0 \\ -180 & 0 & 60 & 0 & 0 & 0 \\ -360 & 0 & 0 & 0 & 0 & 0 \\ -72 & 0 & 0 & 72h & 0 & 0 \\ -72 & 0 & 0 & 0 & 72h & 0 \\ -360 & 0 & 0 & 0 & 0 & 360h \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+1}' \\ y_{n+2}' \\ y_{n+3}' \end{bmatrix} = \begin{bmatrix} -60 & 0 \\ -120 & 0 \\ -360 & -360h \\ -72 & 0 \\ -72 & 0 \\ 360 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_n' \end{bmatrix}$$
$$+ \begin{bmatrix} 49h^2 & 22h^2 & -11h^2 \\ 109h^2 & 82h^2 & -11h^2 \\ -233h^2 & -34h^2 & 77h^2 \\ 37h^2 & 10h^2 & -11h^2 \\ 73h^2 & 46h^2 & -11h^2 \\ 317h^2 & 506h^2 & 77h^2 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{2.15}$$

And taking the normalized version of (2.15), we obtain the block Solution

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y'_{n+1}\\y'_{n+2}\\y'_{n+3}\end{bmatrix}=\begin{bmatrix}1&h\\1&2h\\1&3h\\0&1\\0&1\\0&1\end{bmatrix}\begin{bmatrix}y_n\\y'_n\end{bmatrix}+\begin{bmatrix}\frac{223}{360}h^2&\frac{17}{180}h^2&-\frac{77}{360}h^2\\\frac{37}{18}h^2&\frac{5}{9}h^2&-\frac{11}{18}h^2\\\frac{147}{40}h^2&\frac{33}{20}h^2&-\frac{33}{40}h^2\\\frac{17}{15}h&\frac{7}{30}h&-\frac{11}{30}h\\\frac{49}{30}h&\frac{11}{15}h&-\frac{11}{30}h\\\frac{3}{2}h&\frac{3}{2}h&0\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \tag{2.16}$$

to simultaneously obtain values for $y_{n+1,}\ y_{n+2},\ y_{n+3},\ y'_{n+1},\ y'_{n+2}$ and y'_{n+3}.
Equation (2.16) can be written explicitly as

$$\begin{aligned}
y_{n+1} &= y_n + hy'_n + \frac{h^2}{360}\left[-77f_{n+3} + 34f_{n+2} + 223f_{n+1}\right]\\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{18}\left[-11f_{n+3} + 10f_{n+2} + 37f_{n+1}\right]\\
y_{n+3} &= y_n + 3hy'_n + \frac{h^2}{40}\left[-33f_{n+3} + 66f_{n+2} + 147f_{n+1}\right]\\
y'_{n+1} &= y'_n + \frac{h}{30}\left[-11f_{n+3} + 7f_{n+3} + 34f_{n+1}\right]\\
y'_{n+2} &= y'_n + \frac{h}{30}\left[-11f_{n+3} + 22f_{n+2} + 49f_{n+1}\right]\\
y'_{n+3} &= y'_n + \frac{3}{2}h\left[f_{n+2} + f_{n+1}\right]
\end{aligned} \tag{2.17}$$

3. Analysis of the method

Basic properties of the block method and their associated main method are analyzed to establish their validity. These properties help to show the nature of convergence of the methods. These properties includes; order and error constant, consistency and zero stability. All these put together reveal the nature of convergence of the method. Also the regions of absolute stability of the methods have also been established in this section. However a brief introduction of these properties are made for a better understanding of the section.

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Order and Error Constant

Order of the method

Let the linear difference operator L associated with the continuous multi-step method (2.11) be defined as

$$L\left[y(x)_j h\right]=\sum_{j=0}^{k}\left\{\alpha_j y\left(x_n+jh\right)-h^2\beta_j y''\left(x_n+jh\right)\right\}; j=0,1,2,..,\mathrm{k}$$

Lambert [16] (3.1)

Where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n+jh)$ and $y''(x_n+jh), j=0,1,2,3,...,\mathrm{k}$ in Taylor series about x_n and collecting like terms in h and y gives

$$L.[y(x);h]=C_0 y(x)+C_1 h y^{(1)}(x)+C_2 h^2 y^{(2)}(x)+\ldots+Cph^{(p)}(x)+\ldots \tag{3.2}$$

Definition 1

The difference operator L and the associated implicit multi step method (2.11) are said to be of order p if in (3.2) $C_0=C_1=C_2=\ldots=C_p=C_{P+1}=0, C_{p+2}\neq 0$

Definition 2

The term C_{p+2} is called the error constant and it implies that the local truncation error is given by

$$t_{n+k}=C_{p+2}h^{P+2}y^{(P+2)}\left(x_n\right)+0\left(h^{p+3}\right)$$

Order of the Block

The order of the block will be defined following the method of Chollon et al. [18] , however, with some modification to accommodate general higher order ordinary differential equations and step points,

Definition 3

The term $\overline{C}_{P+2}$ is called the error constant and implies that the local truncation error for the implicit block formular is given by

$$t_{n+k}=\overline{C}_{P+2}h^{p+2}y^{(p+2)}(x_n)+0(h^{p+3}) \tag{3.3}$$

Order and Error constant of the new method (k=3)

From eqn (2.13)

$$y_{n+3}=3y_{n+1}-2y_n+\frac{109}{60}h^2 f_{n+1}+\frac{41}{30}h^2 f_{n+2}-\frac{11}{60}h^2 f_{n+3}$$

Can be rewritten in the form

$$y_{n+3}-3y_{n+1},+2y_n-h^2\left[\frac{109}{60}f_{n+1}+\frac{41}{30}f_{n+2}-\frac{11}{60}f_{n+3}\right]=0 \tag{3.4}$$

Expanding (3.4) in Taylor series form, we have

$$\sum_{j=0}^{\infty}\left(\frac{3^j h^j}{j!}\right)y_n^{(j)}-3\sum_{j=0}^{\infty}\frac{(1)^j h^j}{j!}y_n^{(j)}+2y_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{109}{60}(1)^j+\frac{41}{30}(2)^j-\frac{11}{60}(3)^j\right]=0$$

And collecting terms in powers of h and y, leads to

$$c_0=1-3+2=0$$
$$c_1=3-3=0$$
$$c_2=\frac{9}{2}-\frac{3}{2}-\left[\frac{109}{60}+\frac{41}{30}-\frac{11}{60}\right]=0$$
$$c_3=\frac{27}{6}-\frac{3}{6}-\left[\frac{109}{60}(1)+\frac{41}{30}(2)-\frac{11}{60}(3)\right]=0$$
$$c_4=\frac{81}{24}-\frac{3}{24}-\frac{1}{2!}\left[\frac{109}{60}(1)^2+\frac{41}{30}(2)^2-\frac{11}{60}(3)^2\right]=\frac{13}{30}$$

Hence, the method (2.13) is of order P = 2, with error constant $c_{p+2}=\frac{13}{30}$ (3.5)

Order and error constant of the new block method (k=3)

Rewrite the block form of Eqn (2.16), in this form:

$$\begin{bmatrix}0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y'_{n+1}\\y'_{n+2}\\y'_{n+3}\end{bmatrix}-\begin{bmatrix}1&h\\1&2h\\1&3h\\0&1\\0&1\\0&1\end{bmatrix}\begin{bmatrix}y_n\\y'_n\end{bmatrix}-\begin{bmatrix}\frac{223}{360}h^2&\frac{17}{180}h^2&-\frac{77}{360}h^2\\\frac{37}{18}h^2&\frac{5}{9}h^2&-\frac{11}{18}h^2\\\frac{147}{40}h^2&\frac{33}{20}h^2&-\frac{33}{40}h^2\\\frac{17}{15}h&\frac{7}{30}h&-\frac{11}{30}h\\\frac{49}{30}h&\frac{11}{15}h&-\frac{11}{30}h\\\frac{3}{2}h&\frac{3}{2}h&0\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}=0$$

(3.6)

And writing (3.6) in explicit form, we have

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$$
\begin{aligned}
&y_{n+1} - y_n - hy_n' - h^2\left[\frac{223}{360}f_{n+1} + \frac{17}{180}f_{n+2} - \frac{77}{360}f_{n+3}\right] = 0 \\
&y_{n+2} - y_n - 2hy_n' - h^2\left[\frac{37}{18}f_{n+1} + \frac{5}{9}f_{n+2} - \frac{11}{18}f_{n+3}\right] = 0 \\
&y_{n+3} - y_n - 3hy_n' - h^2\left[\frac{147}{40}f_{n+1} + \frac{33}{20}f_{n+2} - \frac{33}{40}f_{n+3}\right] = 0 \\
&y_{n+1}' - y_n' - h\left[\frac{17}{15}f_{n+1} + \frac{7}{30}f_{n+2} - \frac{11}{30}f_{n+3}\right] = 0 \\
&y_{n+2}' - y_n' - h\left[\frac{49}{30}f_{n+1} + \frac{11}{15}f_{n+2} - \frac{11}{30}f_{n+3}\right] = 0 \\
&y_{n+3}' - y_n' - h\left[\frac{3}{2}f_{n+1} + \frac{3}{2}f_{n+2} + 0\right] = 0
\end{aligned}
\tag{3.7}
$$

Using Taylor's series expansion on (3.7)
We have;

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\frac{(1)^j h^j}{j!}y_n^{(j)} - y_n - hy_n^{(1)} - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{223}{360}(1)^j + \frac{17}{180}(2)^j - \frac{77}{360}(3)^j\right] \\
&\sum_{j=0}^{\infty}\frac{(2)^j h^j}{j!}y_n^{(j)} - y_n - 2hy_n^{(1)} - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{37}{18}(1)^j + \frac{5}{9}(2)^j - \frac{11}{18}(3)^j\right] \\
&\sum_{j=0}^{\infty}\frac{(3)^j h^j}{j!}y_n^{(j)} - y_n - 3hy_n^{(1)} - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{147}{40}(1)^j + \frac{33}{20}(2)^j - \frac{33}{40}(3)^j\right] \\
&\sum_{j=0}^{\infty}\frac{(1)^j h^j}{j!}y_n^{(j+1)} - y_n^{(1)} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+2)}\left[\frac{17}{15}(1)^j + \frac{7}{30}(2)^j - \frac{11}{30}(3)^j\right] \\
&\sum_{j=0}^{\infty}\frac{(2)^j h^j}{j!}y_n^{(j+1)} - y_n^{(1)} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+2)}\left[\frac{49}{30}(1)^j + \frac{11}{15}(2)^j - \frac{11}{30}(3)^j\right] \\
&\sum_{j=0}^{\infty}\frac{(3)^j h^j}{j!}y_n^{(j+1)} - y_n^{(1)} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+2)}\left[\frac{3}{2}(1)^j + \frac{3}{2}(2)^j\right]
\end{aligned}
$$

And collecting terms in h and y leads to the following;

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$$\bar{C}_4 = \begin{bmatrix} c_4 = \frac{(1)^4}{4!} - \frac{1}{2!}\left(\frac{223}{360}(1)^2 + \frac{17}{180}(2)^2 - \frac{77}{360}(3)^2\right) \\ c_4 = \frac{(2)^4}{4!} - \frac{1}{2!}\left(\frac{37}{18}(1)^2 + \frac{5}{9}(2)^2 - \frac{11}{18}(3)^2\right) \\ c_4 = \frac{(3)^4}{4!} - \frac{1}{2!}\left(\frac{147}{40}(1)^2 + \frac{33}{20}(2)^2 - \frac{33}{40}(3)^2\right) \\ c_4 = \frac{(1)^3}{3!} - \frac{1}{2!}\left(\frac{17}{15}(1)^2 + \frac{7}{30}(2)^2 - \frac{11}{30}(3)^2\right) \\ c_4 = \frac{(2)^3}{3!} - \frac{1}{2!}\left(\frac{49}{30}(1)^2 + \frac{11}{15}(2)^2 - \frac{11}{30}(3)^2\right) \\ c_4 = \frac{(3)^3}{3!} - \frac{1}{2!}\left(\frac{3}{2}(1)^2 + \frac{3}{2}(2)^2\right) \end{bmatrix} \Rightarrow \begin{bmatrix} c_4 = \frac{91}{180} \\ c_4 = \frac{22}{18} \\ c_4 = \frac{39}{20} \\ c_4 = \frac{49}{60} \\ c_4 = \frac{7}{10} \\ c_4 = \frac{3}{4} \end{bmatrix}$$

Hence the block method (2.16) is of order $p = (2, 2, 2, 2, 2, 2)^T$ (3.8)

With error constant : Cp+2 = (91/180,22/18,39/20,49/60,7/10,3/4)

Consistency

Definition 4

Given a continuous implicit multi step method (2.11) the first and second characteristics polynomials are defined as;

$$\rho(z) = \sum_{j=0}^{k} \alpha_j Z^J \quad (3.9)$$

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j \quad (3.10)$$

Where z is the principle root, $\alpha_k \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$

Definition 5

The continuous implicit multi step method (2.11) is said to be consistent if it satisfies the following conditions;

i. The order $p \geq 1$
ii. $\sum_{j=0}^{k} \alpha_j = 0$
iii. $\rho(1) = \rho'(1) = 0$ and

iv. $\rho''(1) = 2!\sigma(1)$

Remark
Condition (i) is sufficient for the associated block method to be consistent i.e $p \geq 1$ Jator [19] .

Recall the main method; (2.13)

$$y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{60}\left(-11f_{n+3} + 82f_{n+2} + 109f_{n+1}\right)$$

The first and second characteristics polynomials of the method are given by

$$\rho(z) = z^3 - 3z + 2$$

And

$$\sigma(z) = \frac{-11z^3 + 82z^2 + 109z}{60}$$

And by definition 5, the method (2.13) is consistent since it satisfies the following;

i. The order of the method is $P = 2 \geq 1$

ii. $\alpha_0 = 2,\ \alpha_1 = -3,\ \alpha_3 = 1$

Thus, $\sum_{j=0}^{3} \alpha_j,\ j = 0,1,3,\ \sum_{j=0}^{3} \alpha_j = 2 - 3 + 1 = 0$

iii.
$$\rho(z) = z^3 - 3z + 2$$
$$\rho(1) = (1)^3 - 3(1) + 2 = 0$$
$$\rho^{'}(z) = 3z^2 - 3$$
$$\rho^{'}(1) = 3(1)^2 - 3 = 0$$
$$\therefore p(1) = p'(1) = 0$$

iv.
$$\rho^{"}(z) = 6z$$
$$\rho^{"}(1) = 6(1) = 6$$
$$\sigma(z) = \frac{-11z^3 + 82z^2 + 109z}{60}$$
$$\sigma(1) = \frac{-11(1)^3 + 82(1)^2 + 109(1)}{60} = \frac{180}{60} = 3$$
$$2!\sigma(1) = 2\,\mathrm{X}\,3 = 6$$
$$p^{"}(z) = 2!\sigma(z) = 6$$

The conditions (i-iv) are satisfied, hence the method is consistence

Similarly, the block method (2.16) is consistent since the order of each method in the block method is greater than 1 as shown in Equation (3.8)

Zero Stability

Definition 6

The continuous implicit multi step method (2.11) is said to be zero – stable if no root of the first characteristics polynomial $\rho(z)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than two Lambert [20] .

Definition 7

The implicit block method (2.16) is said to be zero stable if the roots $Z_s, s = 1,\ldots,n$ of the first characteristics polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z) = \det\left[Z\bar{A} - \bar{E}\right] \tag{3.11}$$

Satisfies $|Z_s| \leq 1$ and every root with $|Z_s| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$

Zero stability of the block method (2.16)

From (2.16) using the definition in (7) as $h \to 0$

$$p(z) = \det\left[z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right] = \det\begin{bmatrix} z & 0 & -1 & 0 & 0 & 0 \\ 0 & z & -1 & 0 & 0 & 0 \\ 0 & 0 & z-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z \end{bmatrix} = (z-1)z^5$$

$$(z-1)z^5 \tag{3.12}$$

Solving for Z in (3.12) i.e. $(z-1)z^5 = 0$ gives $z_2 = z_3 = z_4 = z_5 = 0, z_1 = 1$

Hence the block method is stable

Zero stability of new main method (2.13)

The first characteristics polynomial of (2.13) i.e.

$y_{n+3} - 3y_{n+1} + 2y_n = h^2\left[\frac{109}{60}f_{n+1} + \frac{41}{30}f_{n+3} - \frac{11}{60}f_{n+3}\right]$ is given by

$$p(z)=z^3-3z+2 \tag{3.13}$$

Equating (3.13) to zero and solving for z, gives $z=-2, z=1, z=1$

The root of z of (3.13) for which $|z|=1$ is simple, hence the method is zero stable as $h \to 0$ as defined by 6,and by the stability of the block method (2.16).

Convergence

The convergence of the continuous implicit multi step method (2.11) is considered in the light of the basic properties, in conjunction with the fundamental theorem of Dahlquist , Henrici [21] for linear multistep methods] . In what follows, we state Dahlquist's theorem without proof.

Theorem 3.1: Dahlquist theorem. Lambert **[16]**
The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

Remark

The numerical methods derived here are considered to be convergent by theorem 3.1 as $h \to 0$
Following theorem 3.1, the method (2.13) is convergence since it satisfies the necessary and sufficient conditions of consistency and zero stability

Region of Absolute Stability of the block method

Definition 8

If the first and second characteristics polynomials of Linear Multistep Method (LMM) are ρ and σ respectively, then the polynomial equation can be written as

$$\pi(r,\mathrm{\bar{h}}) \Rightarrow \rho(r)-\bar{h}\sigma(r)=0 \tag{3.14}$$

Where $\bar{h}=(\lambda h)^2$

Then $\pi(r,\bar{h})$ is called the stability polynomial of the method defined by ρ and σ , and $\bar{h}=(\lambda h)^2$ is the test equation.

To get the region of absolute stability, we use the Routh –Hurwitz criterion by substituting into 4.10

$$r=\frac{1+z}{1-z} \tag{3.15}$$

On evaluating the coefficient of the resulted polynomials, gives the region of absolute stability.

To get the graph of the stability region, we make $\bar{h}$ the subject of the formular from (3.14) to get

$$\bar{h}(r)=\frac{\rho(r)}{\sigma(r)} \tag{3.16}$$

Which is then plotted in MATLAB environment to produce the required absolute stability region of the method that will be plotted in a graph.

Using definition (8), and expressing the first and second characteristics polynomial of Equation (2.13) as $\rho(r)=r^3-3r+2$ and $\sigma(r)=\frac{1}{60}\left(-11r^3+82r^2+109r\right)$ and substituting into Equation (3.14) gives

$$r^3-3r+2-\frac{(\lambda h)^2}{60}\left(-11r^3+82r^2+109r\right)=0$$

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$$\left(1+11\frac{(\lambda h)^2}{60}\right)r^3-\frac{82}{60}(\lambda h)^2r^2\left(3+\frac{109}{60}(\lambda h)\right)r+2=0 \tag{3.17}$$

Therefore,(3.17) is the stability polynomial.

To get the region of absolute stability, we use the Routh-Hurwitz criterion by substituting $r=\frac{1+z}{1-z}$ into (3.17) to get

$$\left(1+\frac{11}{60}(\lambda h)^2\right)\left(\frac{1+z}{1-z}\right)^3-\frac{82}{60}(\lambda h)^2\left(\frac{1+z}{1-z}\right)^2-\left(3+\frac{109}{60}\lambda h\right)\left(\frac{1+z}{1-z}\right)+2=0$$

Simplifying and collecting like terms, we have

$$\left(-4-\frac{4}{15}(\lambda h)^2\right)z^3+\left(12+\frac{56}{15}(\lambda h)^2\right)+(\lambda h)^2z-3(\lambda h)^2=0$$

Using the coefficients of z^3, z^2, z^1 and z^0 we have

$$-4-\frac{4}{15}(\lambda h)^2>0.$$

$$12+\frac{56}{12}(\lambda h)^2>0$$

$$(\lambda h)^2>0 \tag{3.18}$$

and $$-3(\lambda h)^2>0$$

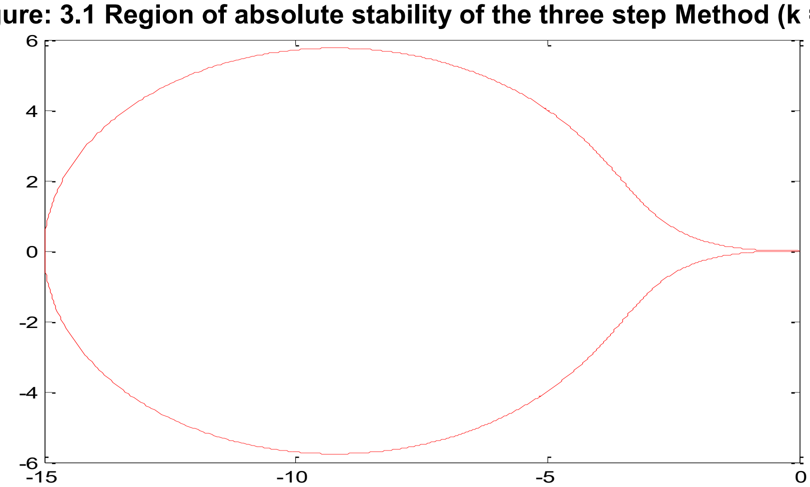
and simplifying (3.18) gives an interval of $(-15,0)$

To get the graph of the absolute stability region, using Equation (3.16) to get

$$\bar{h}(r)=\frac{\rho(r)}{\sigma(r)}=\frac{60(r^3-3r+2)}{-11r^3+82r^2+109r} \tag{3.19}$$

Which is then plotted in MATLAB environment to produce the required absolute stability region of the method that will be plotted in a graph as shown below

Figure: 3.1 Region of absolute stability of the three step Method (k = 3)



The region is A-stable

Table: 3.1 Summary of the Analysis of the Methods

Method	Order	Error constant	Zero Stability	Consistency	Interval of absolute stability
3SM	2	4.33x10-1	Zero stable	Consistent	-15, 0

4. Numerical Examples

In order to study the efficiency of the developed method, we present some numerical examples with the following three problems. The continuous implicit multi step method 3SM was applied to solve the following test problems.

1. $y'' = y',\ y^{(0)} = 0,\ y'(0) = -1,\ h = 0.1$

   exact solution: $y(x) = 1 - \exp(x)$;

   Source: Ehige et al. [3]

Table: 4.1
SHOWING THE EXACT SOLUTION AND THE COMPUTED RESULTS FROM THE PROPOSED METHODS FOR PROBLEM ONE AND IT COMPARISM WITH Ehigie et al. **[3]**

X values	y_{cx}	3SM	[8]	Error in 3SM	Error in [8]
0.1	-0.1051709180756	-0.1051708452556	-0.1048333333	7.281999e-08	3.38e-04
0.2	-0.2214027581601	-0.2214024359613	-0.2206078733	3.221988e-07	7.95e-04
0.3	-0.3498588075760	-0.3498580230810	-0.3484633860	7.844950e-07	1.40e-03
0.4	-0.4918246976412	-0.491823195479	-0.4896604103	1.502293e-06	2.16e-03
0.5	-0.6487212707001	-0.6487187471265	-0.6455911064	2.523574e-06	3.13e-03
0.6	-0.8221188003905	-0.8221148961235	-0.8177929079	3.904267e-06	4.33e-03
0.7	-1.0137527074704	-1.0137469980411	-1.0079636772	5.709429e-06	5.79e-03
0.8	-1.2255409284924	-1.2255329162964	-1.2179784459	8.012196e-06	7.56e-03
0.9	-1.4596031111569	-1.4595922138418	-1.4499079018	1.089732e-05	9.70e-03
1.0	-1.7182818284590	-1.7182673656339	-1.7060388057	1.446283e-05	1.22e-02

Note: The new method perform better than Ehigie et al. **[3]**

2. $y'' - x(y')^2 = 0,\quad y(0) = 1,\ y'(0) = \frac{1}{2}, h = \frac{0.1}{40}$

   Exact solution: $y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$.

   Source: Osilagun et al. [8]

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Table 4.2

SHOWING THE EXACT SOLUTION AND THE COMPUTED RESULTS FROM THE PROPOSED METHODS FOR PROBLEM TWO AND ITS COMPARISM WITH Osilegun et al. **[8]**

X values	y_{cx}	3SM	[14]	Error in 3SM	Error in [14]
0.0025	1.00125000065104	1.00125000065098	1.001250000186	5.440093e-14	4.650e-10
0.0050	1.00250000520835	1.00250000520821	1.002500003997	1.416645e-13	1.211e-09
0.0075	1.00375001757828	1.00375001757805	1.003750013174	2.142730e-13	4.030e-09
0.0100	1.00500004166729	1.00500004166666	1.005000047982	6.268319e-13	6.314e-09
0.0125	1.00625008138211	1.00625008138098	1.006250080358	1.131317e-12	8.462e-09
0.0150	1.00750014062974	1.00750014062814	1.007500025790	1.598943e-12	1.148e-09
0.0175	1.00875022331755	1.00875022331474	1.008750239662	2.809530e-12	1.993e-09
0.0200	1.01000033335334	1.01000033334916	1.010000078382	4.170220e-12	2.550e-09
0.0225	1.01125047464541	1.01125047463994	1.011250489037	5.472733e-12	4.256e-09
0.0250	1.01250065110271	1.01250065109478	1.012500610101	7.921441e-12	4.100e-08

Note: The new method perform better than Osilagun et al. **[8]**

3. $y'' = y + xe^{3x}$, $y(0) = \dfrac{-3}{32}$, $y'(0) = \dfrac{-5}{32}$, $h = 0.1$

Exact solution: $y(x) = \dfrac{4x-3}{32e^{-3x}}$.

Source: Osilagun et al. [8]

Table: 4.3

SHOWING THE EXACT SOLUTION AND THE COMPUTED RESULTS FROM THE PROPOSED METHODS FOR PROBLEM THREE AND ITS COMPARISM WITH Osilegun et al. **[8]**

X values	y_{cx}	3SM	[14]	Error in 3SM	Error in [14]
0.0025	-0.094140915761848	-0.094140915880459	-0.0941409131568	1.186113e-10	2.61e-09
0.0050	-0.094532404142338	-0.094532404442303	-0.0945324074753	2.999645e-10	3.33e-09
0.0075	-0.094924451608388	-0.094924452066225	-0.0949244224215	4.578383e-10	2.92e-08
0.0100	-0.095317044390700	-0.095317045147706	-0.0953170247449	7.570066e-10	2.02e-08
0.0125	-0.095710168480980	-0.095710169602148	-0.0957101793552	1.121167e-09	1.08e-08
0.0150	-0.096103809629113	-0.096103811090129	-0.0961039982252	1.461016e-09	1.88e-07
0.0175	-0.09649533403163	-0.096497955287523	-0.0964952920355	1.947207e-09	4.23e-08
0.0200	-0.096892584872264	-0.096892587372978	-0.0968923659413	2.500714e-09	2.18e-07
0.0225	-0.097289689232184	-0.097287692261238	0.0972874625827	3.029054e-09	2.22e-06
0.0250	-0.097683251173919	-0.09683254882864	-0.0976830958236	3.708945e-09	1.55e-07

Note: The new method perform better than Osilagun et al. **[8]**

Discussion of the Results

The computer programs written for the implementation of the continuous implicit multi step method 3SM, was tested on numerical examples which are respectively, nonlinear, linear and stiff initial value problems of general second order ordinary differential equations in the least section.
Generally, the performance of our method as notice in table 4.1 are superior to those from methods implemented by Ehigie et al. [3] , that used a 2 – step continuous linear multistep method of hybrid type on moderately stiff problem one. It is observed that our method 3SM, in table 4.2, performed far better than Osilegun et al. [8] , method of four steps implicit method on non – linear problem two. Also, our method 3SM, perform better than Osilegun et al. [8] , method of four steps implicit method on linear problem three in table 4.3
Finally, our scheme have been demonstrated to be more efficient in stiff problems as shown in table 4.1 of problem one.

5. Conclusion

This paper illustrates the derivation, analysis and implementation of block method for solving second order initial value problem of ordinary differential equations directly.
Numerical experiments have been carried out using appropriate step size as required by each problem. Such problem which are stiff, non-linear and linear. In general, the results from numerical experiment so presented in this paper show that the new method performed effectively well when compared with other methods in the literature.

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