International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 104 - 122
https://doi.org/10.5281/zenodo.8218022

EXPONENTIATED WEIBULL INVERSE RAYLEIGH DISTRIBUTION

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Article Info
Received: 22 April 2022 Revised: 22 July 2023
Accepted: 23 July 2023 Available online: 31 July 2023

Abstract

This work focuses on the study of a new four-parameter Exponentiated Weibull Inverse Rayleigh Distribution (EWIR) using Exponentiated Weibull-G family of distribution as the generator. Statistical properties of the distribution (like, Moment, Quantile, Skewness & Kurtosis, Moment, Mgf) were derived along with its asymptotic behaviour. The parameters of the new distribution were estimated using Maximum Likelihood Estimation (MLE) methods. The performance of the EWIR distribution was compared with other related distribution from the literature using the Akaike Information Criterion (AIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) methods comparison. A simulation study was conducted to evaluate the MLE estimates, bias, and standard error for various parameter combinations at different sample sizes. Application of the distribution was made using a real dataset, the data set contains carbon fiber strength (20mm). The MLEs, Standard Errors (SEs), and –log-likelihood for the new distribution and five other related distributions were fitted to the data set. Goodness-of-fit measures based on AIC, BIC, Kolmogorov-Smirnov test (K-S) values and their corresponding ranks (in parentheses) for the dataset was also presented. Hence, the new EWIR model provided the best fit among the other models for the data set, since it has the lowest values of AIC, BIC, and K-S Values.

Keywords: Exponentiated; Maximum Likelihood Estimation; Parameter Estimation; Simulation Study; Weibull Inverse Rayleigh.
MSC2010: 26A18.

1 INTRODUCTION

In recent years, the attention of researchers has been shifted to different methods of generalization of probability distribution theory, which have the ability to fit any kind of data with some degree of flexibility. Statistical distributions are widely applied to model and analyze data in different

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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https://doi.org/10.5281/zenodo.8218022

disciplines such as engineering, biology, economics, finance and medical sciences. [1] introduced a method of an extra parameter to cumulative distribution function

$$F(x) = [G(x)]^a$$

which produced a more flexible distribution compared to the baseline distribution.

$$FF(x) = 1 - [R(x)]^a$$

where R(x) in above equation is the reliability function and several distributions were developed such as; Weibull-G [2], Exponentiated Weibull-G [3], Weibull-Normal [4], Odd Lomax-Exponential Distribution [5], exponentiated-exponential-Dagum{lomax} distribution developed by [6], arcsine exponentiated-X [7], Type II Exponentiated Half Logistic generated family of distributions with applications [8], Weibull Inverse Rayleigh Distribution [9], and Gamma-Power{log-logistic} by [10, 11].

This research aims to introduce an Exponentiated version of the Weibull Inverse Rayleigh Extended-G (EX-G) distributions which is called as Exponentiated Weibull Inverse Rayleigh distribution. The remaining part of the article is organized as thus: In section 2, the Exponentiated Weibull Inverse Rayleigh distribution is derived, the mixture representation of the EWIR distribution pdf in terms of base line CDF and PDF. Some mathematical properties including r^{th} moment, moment generating function, the density of i^{th} order statistics is given and enthropy of the new distribution are derived. In section 3, model parameters are estimated by ML method. In section 4, application is carried out on two real data sets. In section 5, conclusion is made.

2 RESULTS

2.1 2.1 Derivation of EWIR Distribution

In this section, the four-parameter EWIR distribution is obtained based on the EW-G family of distribution and some of its mathematical properties are explained.

2.1.1 The Exponentiated Weibull Inverse Rayleigh Distribution Given Exponentiated Weibull-G family of distribution [3]. The cumulative distribution function of the family is defined by

$$F(x) = \left[1 - exp\left(-b\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right)\right]^{a}; a, b, \beta > 0, \tag{2.1}$$

Where a, β are shape parameters, $b > 0$ is a scale parameter and $G(x)$ is the CDF of the baseline distribution. The probability density function (pdf) of the EW-G family of distribution is given below

$$f(x) = \frac{ab\beta g(x)(G(x))^{\beta-1}}{(1-G(x))^{\beta+1}} e^{-b\left[\frac{G(x)}{1-G(x)}\right]^{\beta}} \left[1 - exp\left(-b\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right)\right]^{a-1}, \tag{2.2}$$

where $a, b, \beta > 0$ or Weibull Inverse Rayleigh Distribution (WIR) [9]:

$$F(x) = 1 - \exp\left\{-b\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}$$

The PDF and CDF of a random variable X having the Inverse Rayleigh distribution with scale parameter λ are given by

$$g(x, \lambda)) = \frac{2\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2}; x, \lambda > 0 \tag{2.3}$$

$$G(x, \lambda) = e^{-\left(\frac{\lambda}{x}\right)^2} \tag{2.4}$$

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Substituting (2.4) into (2.1) (the Exponentiated Weibull-G family), we get the CDF of EWIR distribution as follows

$$F(x) = \left[1 - exp\left(-\alpha\left[\frac{e^{-\left(\frac{\lambda}{x}\right)^2}}{1 - e^{-\left(\frac{\lambda}{x}\right)^2}}\right]^{\beta}\right)\right]^{a}$$

If further simplified, it will give

$$F(x, \Psi) = \left(1 - \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}\right)^{a} \tag{2.5}$$

where $F(x, \Psi) \equiv (a, \alpha, \beta, \lambda)$ are the set parameters vector.

Differentiate the CDF of EWIR distribution in equation (2.5), the pdf of EWIR distribution is obtained as follows

$$f(x, \Psi) = 2a\alpha\beta\lambda^2 x^{-3} \exp\left[-\beta\left(\frac{\lambda}{x}\right)^2\right]\left\{1 - \exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1} \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}$$
$$\times\left(1 - \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}\right)^{a-1} \tag{2.6}$$

where $x \geq 0, a, \alpha, \beta, \lambda > 0$

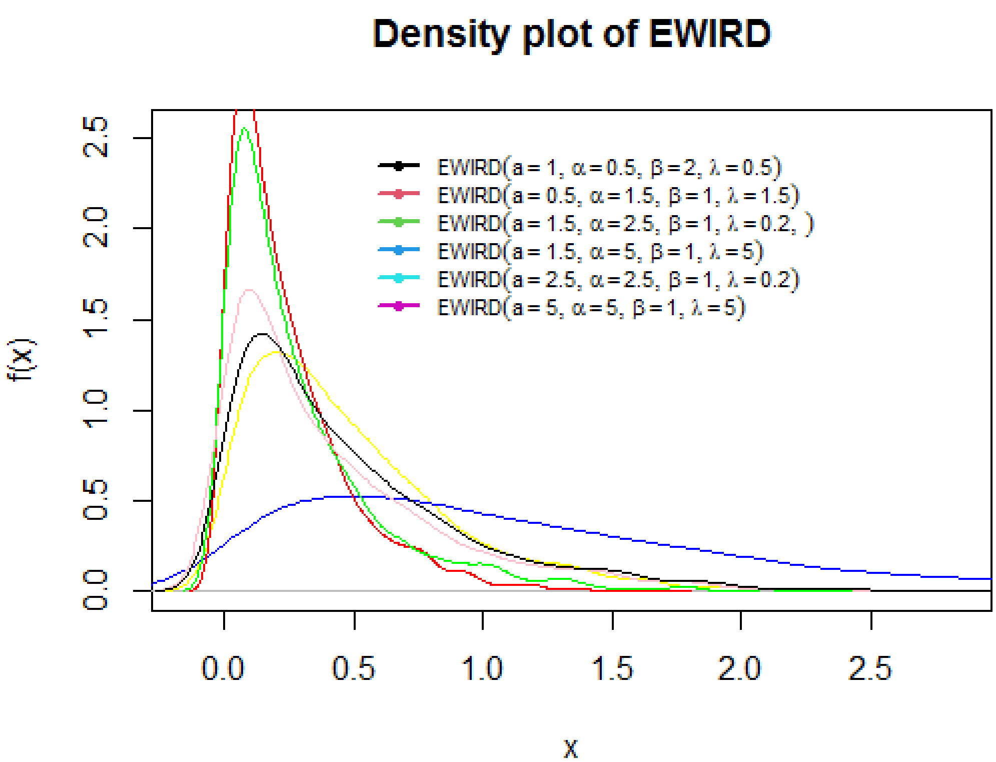

Fig.1. Illustration of the PDF of EWIR Distribution

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Figure 1 shows that the PDF of EWIR distribution is left skewed with heavy tail. The skewness and the kurtosis of the new distribution are demonstrated from the figure above. The PDF of WIR distribution is demonstrated with different shape and scale parameters are shown in Figure 1.

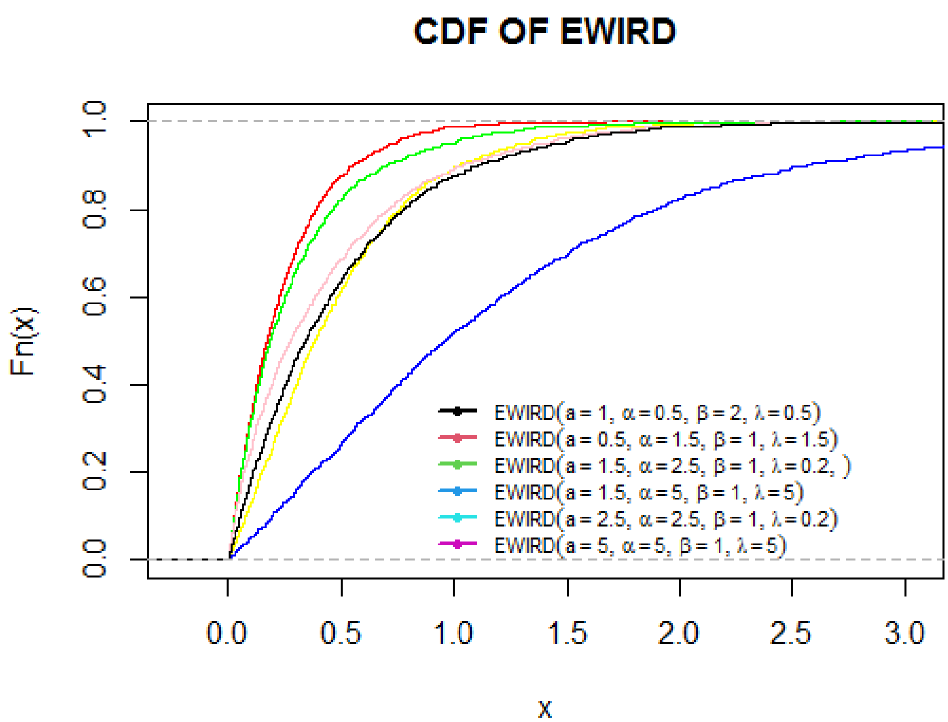

Fig.2. CDF of EWIR distribution different values of $a, \alpha, \beta, \lambda$

Figure 2 shows that the sum of probability values of EWIR distribution with its domain is equal to one.

2.2 2.1.2 Mathematical Properties of EWIR

Some mathematical properties such as survival function, hazard function, cumulative hazard function and moment of the exponentiated Weibull inverse Rayleigh distribution are derived

Survival function

The survival function is given as
$S(x, \Psi) = 1 - F(x, \Psi)$, hence the Survival function of EWIR distribution is as thus:

$$S(x, \Psi) = 1 - \left(1 - \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}\right)^a \tag{2.7}$$

Hazard function

The Hazard function is given as;

$$h(x,\Psi) \;=\; \frac{f(x,\Psi)}{S(x,\Psi)}$$

$$h(x,\Psi)=\frac{2a\alpha\beta\lambda^2x^{-3}\exp\left[-\beta\left(\frac{\lambda}{x}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1}\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right\}}{1-\left(1-\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right\}\right)^{a}}$$

$$\times\left(1-\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]\right\}\right)^{a-1} \tag{2.8}$$

Cumulative hazard function

The cumulative hazard function of EWIR distribution are respectively defined as

$$H_X(x)=-\log[S_X(x)]$$

$$H(x,\Psi)=-\log\left[1-\left(1-\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right\}\right)^{a}\right] \tag{2.9}$$

Reversed hazard function of EWIR distribution:

By definition, the reverse hazard function is given by

$$\tau(x,\;\Psi)=\;\frac{f(x,\Psi)}{F(x,\;\Psi)}$$

Thus, the reverse hazard function of EWIR distribution is given by

$$\tau(x,\Psi)=\frac{f(x,\Psi)=2a\alpha\beta\lambda^2x^{-3}\exp\left[-\beta\left(\frac{\lambda}{x}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1}\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right\}}{1-exp\left(-\alpha\left[\frac{e^{-\frac{\lambda^2}{x^2}}}{1-\;e^{-\frac{\lambda^2}{x^2}}}\right]^{\beta}\right)}$$

$$\times\left(1-\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right\}\right)^{a-1} \tag{2.10}$$

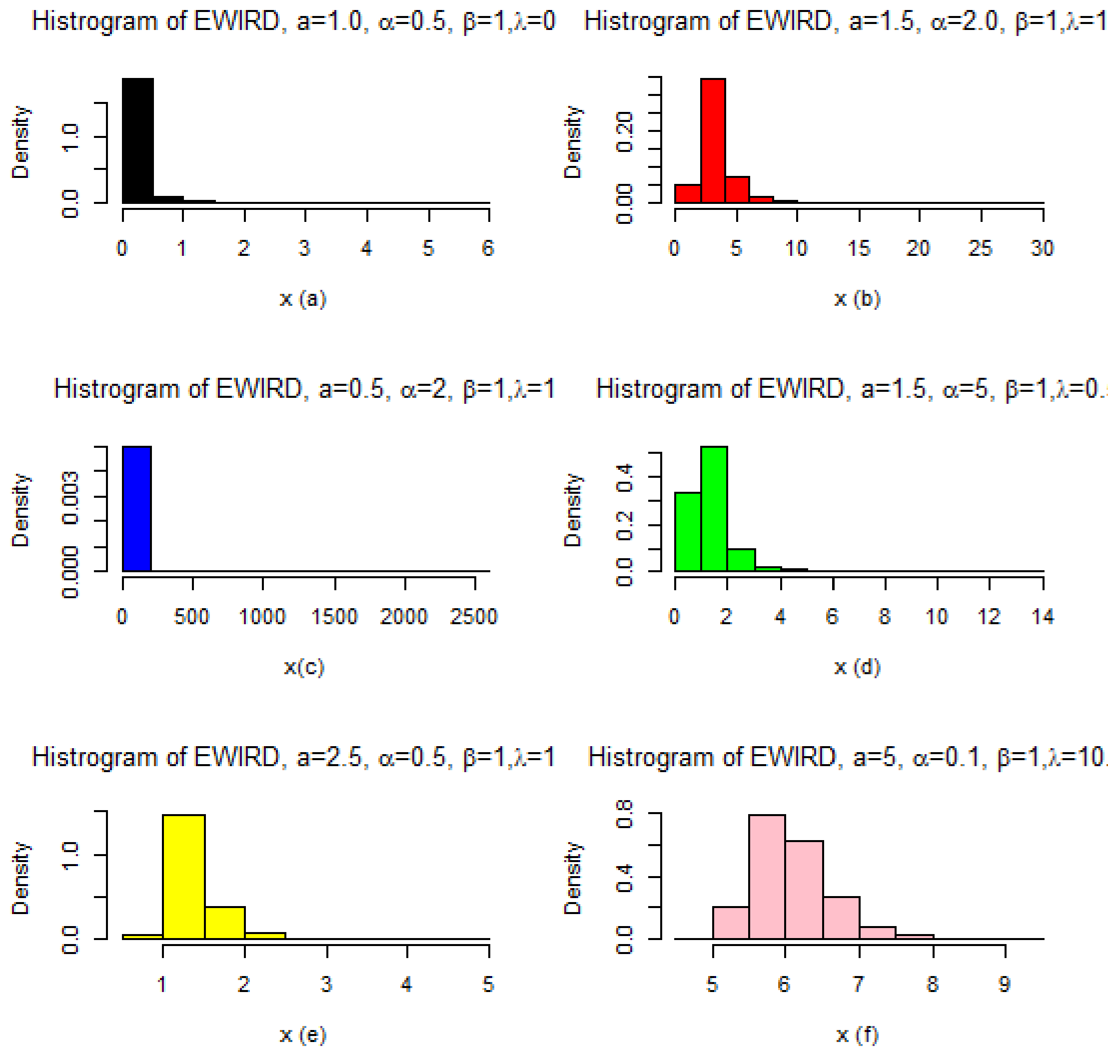

Fig. 3. Histogram of EWIRD with different values of $a, \alpha, \beta, \lambda$

2.3 2.1.3 Quantile Function

Given a random variable x with continues and strictly monotonic probability density function $f(x)$, a quantile function (x) assigns to each probability p attained by the value x for which $\Pr(X < x) = p$. [12].

The quantile function, say$Q(u) = F^{-1}(u)$, of X is obtained by inverting (5) the CDF of EWIR distribution.

So we have,

$$Q(u) = \sqrt{\frac{\lambda^2}{\ln\left[1 - \left(\frac{-\alpha}{ln(1-u)}\right)^{\frac{1}{a}}\right]}} \tag{2.11}$$

where $\left(\frac{-\alpha}{ln(1-u)}\right)$ <1 and $\beta, \lambda^2 > 0$.
If $Q(u)$ is uniform (0, 1), then $Q(u)$is EWIR random variable. Therefore, one can simulate numbers from the EWIR distribution by using (2.11). Given u =0.25, 0.75, and 0.5, in (2.11), the 1stquantile,

the 3rd quantile, and the median can be obtained respectively for the EWIR distribution. The median will be

$$Med \ = Q(0.5) = \sqrt{\frac{\lambda^2}{\ln\left[1-\left(\frac{-\alpha}{ln(1-0.5)}\right)^{\frac{1}{a}}\right]}} \tag{2.12}$$

2.1.4 Linear Mixture Representation: Useful expansions

The linear mixture of PDF and CDF of EWIR distribution is obtained below:
Recall the CDF of EWIR distribution in (5) given as

$$F(x,\ \Psi) = \left(1-\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right\}\right)^a$$

Using the generalized binomial expansions;

$$(1-h)^d \ = \sum_{i=0}^{\infty}(-1)^i\binom{d}{i}h^i \tag{2.13}$$

Now by using (2.13), we obtain the expansion for CDF raised to the power of m, where m is an integer.

$$F(x,\Psi) \ = \sum_{q=0}^{\infty}(-1)^q\binom{a}{q}e^{-\alpha q\left(e^{-\left(\frac{\lambda}{x}\right)^2}-1\right)^{\beta}} \tag{2.14}$$

Using power series expansion for the inverse Rayleigh function gives

$$\begin{aligned} F(x,\Psi) \ &= \sum_{q,w=0}^{\infty}\binom{a}{q}\frac{(-1)^{q+w}(\alpha q)^w}{w!}\left[e^{\left(\frac{\lambda}{x}\right)^2}-1\right]^{\beta w} \\ &= \sum_{q,w=0}^{\infty}\binom{a}{q}\frac{[\ (-1)]^{q+w}[(\alpha q)]^w}{w!}\left[1-e^{\left(\frac{\lambda}{x}\right)^2}\right]^{\beta w} e^{\beta w\left(\frac{\lambda}{x}\right)^2} \end{aligned} \tag{2.15}$$

Using equation (2.13) again and the identity $\binom{-r}{k}=\binom{r+k-1}{k}(-1)^k$, we have

$$e^{\beta w\left(\frac{\lambda}{x}\right)^2} \ = \left(1-\left[1-e^{\beta w\left(\frac{\lambda}{x}\right)^2}\ \right]\right)^{\beta w} = \sum_{j=0}^{\infty}\binom{\beta w+j-1}{j}\left(1-e^{\left(\frac{\lambda}{x}\right)^2}\right)^j$$

Thus, $F(x,\ \Psi)$ is written as

$$F(x,\ \Psi) = \sum_{k=0}^{\infty}\eta_{k,q,w}\ e^{-k\left(\frac{\lambda}{x}\right)^2}$$

where

$$\eta_{k,q,w} \ = \sum_{i,j=0}^{\infty}\frac{(-1)^{q+w+k}(\alpha q)^w}{w!}\binom{a}{q}\binom{\beta w+j-1}{j}\binom{\beta w+j}{k} \tag{2.16}$$

Also, we find an expansion for $f(x,\ \Psi)$

$$f(x,\ \Psi) = \ 2a\alpha\beta\lambda^2x^{-3}e^{-\left(\frac{\lambda}{x}\right)^2}\ \left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{(\beta-1)}\ exp-\alpha\left[\frac{e^{-\left(\frac{\lambda}{x}\right)^2}}{1-e^{-\left(\frac{\lambda}{x}\right)^2}}\right]^{\beta}$$

$$\times\left[1-exp\left(-\alpha\left[\frac{e^{-\left(\frac{\lambda}{x}\right)^2}}{1-e^{-\left(\frac{\lambda}{x}\right)^2}}\right]^{\beta}\right)\right]^{(a-1)}$$

Using equation (6) we have

$$f(x,\Psi)=2a\alpha\beta\lambda^2x^{-3}e^{-\left(\frac{\lambda}{x}\right)^2}\sum_{q=0}^{\infty}(-1)^q\binom{(a-1)}{q}e^{-\alpha q\left(e^{\left(\frac{\lambda}{x}\right)^2}-1\right)^{\beta}} \tag{2.17}$$

$$=2a\alpha\beta\lambda^2x^{-3}e^{\left(\frac{\lambda}{x}\right)^2}\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\frac{[(-1)]^{j+q}[\alpha q]^j}{j!}\binom{(a-1)}{q}\left(e^{\left(\frac{\lambda}{x}\right)^2}-1\right)^{\beta j} \tag{2.18}$$

Using identity

$$e^{\left(\frac{\lambda}{x}\right)^2}-1=\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)\ e^{\left(\frac{\lambda}{x}\right)^2} \tag{2.19}$$

we have

$$f(x,\Psi)=2a\alpha\beta\lambda^2x^{-3}e^{-\left(\frac{\lambda}{x}\right)^2}$$

$$\times\sum_{q,j=0}^{\infty}\frac{(-1)^{j+q}[\alpha q]^j}{j!}\binom{(a-1)}{q}\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\beta j}e^{-\left(\frac{\lambda}{x}\right)^2\beta(j+\varepsilon)} \tag{2.20}$$

Considering equation (13) and using equation (18), So, the PDF of EWIR can be rewritten as

$$f(x,\Psi)\ =\left(2a\beta\lambda^2x^{-3}\right)^{\varepsilon}$$

$$\times e^{-\left(\frac{\lambda}{x}\right)^2}\sum_{q,j=0}^{\infty}\frac{(-1)^{j+q}\left[\alpha(q+\varepsilon)\right]^j}{j!}\binom{(a-1)\varepsilon}{q}\binom{\beta(j+\varepsilon)+\varepsilon+w-1}{w}\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\beta(j+\varepsilon)-\varepsilon+w} \tag{2.21}$$

$$2a\alpha\beta\lambda^2x^{-3}\sum_{q,j,w,k=0}^{\infty}\frac{(-1)^{j+q+k}[\alpha q]^j}{j!}\binom{(a-1)}{q}\binom{\beta j+w-1}{w}\binom{\beta j+w-1}{k}e^{-(k+1)\left(\frac{\lambda}{x}\right)^2} \tag{2.22}$$

From the above relations, we arrive at

$$f(x,\Psi)=2a\alpha\beta\lambda^2\sum_{q,j,w,k}^{\infty}\eta_{q,j,w,k}{}^{*}x^{-3}e^{-k\left(\frac{\lambda}{x}\right)^2} \tag{2.23}$$

where

$$\eta_{q,j,w,k}{}^{*}=2a\alpha\beta\lambda^2\sum_{q,j,w,k=0}^{\infty}\frac{(-1)^{j+q+k}[bq\ ]^j}{j!}\binom{(a-1)}{q}\binom{\beta j+w-1}{w}\binom{\beta j+w}{k-1} \tag{2.24}$$

Equation (23) can be used as an alternative for the PDF of EWIR distribution after setting $\epsilon=1$

2.1.5 Moment

The r^{th} moment of the EWIR model is given by

$$u_r^{'}=\ E(X^r)\ =\ \int_0^{\infty}x^rf(x,\Psi)dx \tag{2.25}$$

Using equation (25) the r^{th} moment of EWIR distribution is defined as

Substituting (23) into (25)

$$u_r^{'}=E(X^r=\ \eta_{q,j,w,k}{}^{*}\int_0^{\infty}x^{r-3}e^{-k\left(\frac{\lambda}{x}\right)^2} \tag{2.26}$$

then $u_r^{'}$, using gamma function (26) becomes

$$u_r^{'} = \frac{\eta_{q,j,w,k}{}^{*}}{2} \frac{\Gamma\left(1-\binom{r}{2}\right)}{(k\lambda^2)^{\left(1-\binom{r}{2}\right)}} \; ; r = 1,2 \tag{2.27}$$

Where

$$\eta_{q,j,w,k}{}^{*} = 2a\alpha\beta\lambda^2 \sum_{q,j,w,k=0}^{\infty} \frac{(-1)^{j+q+k}[\alpha q\,]^j}{j!} \binom{(a-1)}{q} \binom{\beta j + w - 1}{w} \binom{\beta j + w}{k-1} \tag{2.28}$$

And $\Gamma(.)$ is a gamma function.
In particular, the mean and variance of EWIR distribution are obtained, respectively, as follows

$$\mu = E(X) = \frac{\eta_{q,j,w,k}{}^{*}}{2} \frac{\Gamma\left(1-\binom{r}{2}\right)}{(k\lambda^2)^{\left(1-\binom{r}{2}\right)}} = \frac{\eta_{q,j,w,k}{}^{*}}{2} \sqrt{\frac{\pi}{k\lambda^2}}$$

$$E(X^2) = \frac{\eta_{q,j,w,k}{}^{*}}{2}$$

$$Var(X) = E\left(X^2\right) - [E(X)]^2$$

$$Var(X) = \frac{\eta_{q,j,w,k}{}^{*}}{2} - \frac{\eta_{q,j,w,k}{}^{*}}{2} \left[\sqrt{\frac{\pi}{k\lambda^2}}\right]^2$$

$$Var(X) = \frac{\eta_{q,j,w,k}{}^{*}}{2} \left[1 - \frac{\pi}{2k\lambda^2}\right] \tag{2.29}$$

2.1.6 Moment Generating Function

The moment generating function is the expectation of a function of the random variable [13]. Mathematically, the moment generating function (MGF) of a random variable X is a function $M_{x(s)}$ defined as $M_{x\,(t)} = E[e^{tX}]$.

The moment generating function of EWIR model is given by

$$M\ x\ (t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x, \Psi) dx \tag{2.30}$$

$$M\ x\ (t) = \int_0^{\infty} e^{tx} 2a\alpha\beta\lambda^2 x^{-3} \exp\left[-\beta\left(\frac{\lambda}{x}\right)^2\right] \left\{1 - \exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1} \times \left(1 - \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}\right)^{a-1\}} \tag{2.31}$$

$$M\ x\ (t) = (2\beta\ \lambda^2) \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}\alpha^i}{i!(ai\ +\ j)} \binom{-aj}{j} (ai\ +\ j) \int_0^{\infty} x^{r-3} e^{\frac{-k\lambda^2(ai+j)}{x^2}} dx$$

Then the moment generating function of EWIR distribution is given by

$$M\ x\ (t) = \frac{\omega_{i,j,w,q,k}{}^{*}}{2} \frac{t^r\ \Gamma\left(1-\binom{r}{2}\right)}{r!\left(\ k\left(\frac{\lambda}{x}\right)^2\right)^{1-\binom{r}{2}}} \tag{2.32}$$

Where

$$\omega_{i,j,w,q,k}{}^{*} = 2a\alpha\beta\lambda^2 \sum_{q,j,w,k=0}^{\infty} \frac{(-1)^{j+q+k}[\alpha q\,]^j}{j!(ai\ +\ j)} \binom{-aj}{j} \binom{(a-1)}{q} \binom{\beta j + w - 1}{w} \binom{\beta j + w}{k-1}$$

IJMSO

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
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7. Skewness and Kurtosis

Skewness is a measure of the degree of asymmetric of a probability distribution. Kurtosis is a statistical technique that measures the degree of peakness of a distribution. Different methods are used to find skewness and kurtosis in certain distributions. The most common method is the one that uses moments of the distribution. However, in EWIR distribution we have only the first moment. Due to this reason, the appropriate method of finding kurtosis and skewness is by using quantiles. The skewness and kurtosis base on quantile function for EWIR distribution are obtained numerically. [14,15] proposed skewness base on quantiles called the Bowley skewness which is defined as follows:

$$Sk = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{4}\right) \ + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}$$

Further, the kurtosis proposed by [16] base on quantile called Moors kurtosis is defined as follows:

$$Ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

Table 1: The Galtons skewness and Moors kurtosis for some values of $(a, \alpha, \lambda,)$ of the new EWIR distribution when β=1

a	α	λ	Skewness	Kurtosis	a	α	λ	Skewness	Kurtosis
0.1	0.1	10	0.9366	6.2942	1	0.5	0.1	0.3182	1.2194
0.1	0.1	20	0.9366	6.2942	1	1	1	0.1241	1.1021
0.1	0.5	0.1	0.2781	1.1488	1	1	10	0.1241	1.1021
0.1	0.5	10	0.2781	1.1488	1	10	1	-0.0712	1.1933
0.1	1	0.1	0.1061	1.0892	1	10	10	-0.0712	1.1933
0.1	1	0.5	0.1061	1.0892	1	20	0.5	-0.0821	1.2050
0.1	10	0.1	-0.0713	1.1934	1	20	20	-0.0821	1.2050
0.1	10	1	-0.0713	1.1934	10	0.1	0.1	0.1177	1.1704
0.1	20	10	-0.0821	1.2050	10	0.1	0.5	0.1177	1.1704
0.1	20	20	-0.0821	1.2050	10	0.5	0.5	0.1123	1.1589
0.5	0.1	10	0.9407	6.3073	10	0.5	20	0.1123	1.1589
0.5	0.1	20	0.9407	6.3073	10	1	0.1	0.0742	1.1307
0.5	0.5	0.1	0.3176	1.1688	10	1	0.5	0.0742	1.1307
0.5	0.5	0.5	0.3176	1.1688	10	10	1	-0.0711	1.1933
0.5	0.5	20	0.3176	1.1688	10	10	20	-0.0711	1.1933
0.5	1	1	0.1227	1.0921	10	20	0.1	-0.0821	1.2050
0.5	1	10	0.1227	1.0921	10	20	0.5	-0.0821	1.2050
0.5	1	20	0.1227	1.0921	20	0.5	1	0.0728	1.1463
0.5	10	20	-0.0712	1.1933	20	0.5	20	0.0728	1.1463
0.5	10	0.1	-0.0712	1.1933	20	1	0.1	0.0540	1.1347
0.5	20	0.1	-0.0821	1.2050	20	1	20	0.0540	1.1347
0.5	20	20	-0.0821	1.2050	20	10	0.1	-0.0711	1.1933
1	0.1	0.1	0.8983	6.6209	20	10	20	-0.0711	1.1933
1	0.1	0.5	0.8983	6.6209	20	20	0.1	-0.0821	1.2050
1	0.5	20	0.3182	1.2194	20	20	20	-0.0821	1.2050

Table 1 show the Galtons skewness and Moors kurtosis for some values of $(a, \alpha, \lambda,)$ of the new EWIR distribution when β=1, however, the result shows that the value of λ does not affect the shape of the EWIR distribution. The skewness is only affected by the parameters α and λ.

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2.1.8 Renyi Entropy of EWIR Distribution

The entropy of a random variable quantifies its associated uncertainty [17]. The Renyi entropy has numerous applications in information-theoretic statistics such as classification, distribution identification problems, and statistical inference.

The Renyi entropy is defined by:

$$I_\omega(X) = \frac{1}{1-\omega}\log\int_{-\infty}^{\infty} f^\omega(x)dx,\ \omega > 0\ and\ \omega \neq 1 \tag{2.33}$$

Therefore, using (33) the Renyi entropy of a random variable X follows the EWIR is given by

$$I_\omega(X) = \frac{1}{1-\omega}\log\int_{-\infty}^{\infty} f^\omega(x)dx \tag{2.34}$$

Substituting equation (23) into

$$I_\omega(X) = \frac{1}{1-\omega}\log\left[2\left(\frac{\lambda}{x}\right)^\varepsilon \sum_{k=1}^{\infty} \eta_{q,j,w,k}{}^* e^{-k\left(\frac{\lambda}{x}\right)^2}\right]$$

Where $\eta_{q,j,w,k}{}^*$ is defined in (24)

2.1.9 Order Statistics of EWIR Distribution

Order statistics make their appearance in many areas of statistical theory and practice. Let X_1, $X_2, \ldots,\ \ldots,\ X_n$ be a random sample from EWIR distribution and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics. The pdf of the k$^{\text{th}}$ order statistics is given by:

$$G_{(k:n)}(x) = \frac{1}{B(k, n-k+1)} g_x(x)\left(G_x(x)\right)^{k-1} 1 - G_x(x)^{n-k}\ 1 < k < n \tag{2.35}$$

Using the binomial series expansion of $[1 - G_x(x)^{n-k}]$, we obtain

$$[1 - G_x(x)^{n-k}] = \sum_{j=0}^{n-1} (-1)^j \binom{n-j}{j} \left(G_x(x)\right)^j \tag{2.36}$$

Therefore equation (35) can be rewritten as follows

$$G_{(k:n)}(x) = \frac{1}{B(k, n-k+1)} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} g_x(x)\left(G_x(x)\right)^{k+j-1} \tag{2.37}$$

Substituting equation (3) and (17) into equation (37), we get the pdf of the k$^{\text{th}}$ order statistics given as

$$G_{(k:n)}(x) = \frac{2ab\beta\lambda^2 x^{-3}}{B(k, n-k+1)} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \left\{1 - exp\left\{-a\left[\frac{e^{\frac{-\lambda^2}{x^2}}}{1 -\ e^{\frac{-\lambda^2}{x^2}}}\right]^a\right\}\right\}^{k+j-1} \tag{2.38}$$

Since,

$$1\ -\ exp\left\{\left\{-a\left[\frac{e^{\frac{-\beta\lambda^2}{x^2}}}{1 -\ e^{\frac{-\beta\lambda^2}{x^2}}}\right]^b\right\}\right\}^{k+j-1} = \sum_{j=0}^{n-k} (-1)^q \binom{k+j-1}{q} \times exp\left\{\left\{-aq\left[\frac{e^{\frac{-\beta\lambda^2}{x^2}}}{1 -\ e^{\frac{-\beta\lambda^2}{x^2}}}\right]^b\right\}\right\}$$

Then,

$$G_{(j,k,:n)}(x) = \frac{2ab\beta\lambda^2 x^{-3}}{B(k, n-k+1)} \sum_{j=0}^{n-k}\sum_{q=0}^{\infty} (-1)^{j+q} \binom{n-k}{j}\binom{k+j-1}{q}$$

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$$\times exp\left\{-a(q+1)\left[\frac{e^{\frac{-\beta\lambda^2}{x^2}}}{1-\ e^{\frac{-\beta\lambda^2}{x^2}}}\right]^b\right\}\left[1-e^{\frac{-\beta\lambda^2}{x^2}}\right]^{-b-1} e^{\frac{-b\beta\lambda^2}{x^2}} \tag{2.39}$$

Since,

$$Exp\left\{-a(q+1)\left[\frac{e^{\frac{-\beta\lambda^2}{x^2}}}{1-\ e^{\frac{-\beta\lambda^2}{x^2}}}\right]^b\right\} = \sum_{s=0}^{\infty}\frac{(-1)^s(P)^s(q+1)^s}{s!}\left[\frac{e^{\frac{-\beta\lambda^2}{x^2}}}{1-\ e^{\frac{-\beta\lambda^2}{x^2}}}\right]^{bs} \tag{2.40}$$

Then

$$G_{(k:n)}(x) = \frac{2b\beta\lambda^2x^{-3}}{B(k,n-k+1)}\sum_{j=0}^{n-k}\sum_{q=0}^{\infty}(-1)^{j+q+s}\binom{n-k}{j}\binom{k+j-1}{q}$$
$$\times\sum_{s=0}^{\infty}\frac{(P)^{s+1}(q+1)^s}{s!}\left[1-e^{\frac{-\beta\lambda^2}{x^2}}\right]^{b(s-1)-1} e^{\frac{-b\beta\lambda^2(s+1)}{x^2}} \tag{2.41}$$

Again, using the binomial theorem

$$\left[1-e^{\frac{-\beta\lambda^2}{x^2}}\right]^{b(s-1)-1} = \sum_{q=0}^{\infty}(-1)^w\binom{b(s-1)-1}{w}e^{\frac{-\lambda^2 w}{x^2}} \tag{2.42}$$

Then, the probability density function of the k$^{\text{th}}$ order statistics $X_{(1:n)}$ from EWIR distribution takes the following form

$$G_{(k:n)}(x) = \sum_{j=0}^{n-k}\sum_{q,w,s=0}^{\infty}E_{j,q,w,s}x^{-3}e^{\frac{-\lambda^2(b(s+1)+w)}{x^2}} \tag{2.43}$$

where,

$$E_{j,q,w,s} = \frac{1}{B(k,n-k+1)}\sum_{j=0}^{n-k}\sum_{q,w,s=0}^{\infty}(-1)^{j+q+s+w}\binom{n-k}{j}\binom{k+j-1}{q}\binom{b(s-1)-1}{w}$$
$$\times\frac{2b\beta\lambda^2(P)^{s+1}(q+1)^s}{s!}$$

In particular, the pdf of the smallest order statistics $X_{(1:n)}$ is obtained from (43) as:

$$G_{(1:n)}(x) = \sum_{j=0}^{n-k}\sum_{q,w,s=0}^{\infty}E_{j,q,w,s}x^{-3}e^{\frac{-\lambda^2(b(s+1)+w)}{x^2}} \tag{2.44}$$

where

$$E_{1,q,w,s} = \frac{1}{B(1,n)}\sum_{j=0}^{n-1}\sum_{q,w,s=0}^{\infty}(-1)^{j+q+s+w}\binom{n-1}{j}\binom{j}{q}\binom{b(s-1)-1}{w}$$
$$\times\frac{2\alpha\beta\lambda^2(P)^{s+1}(q+1)^s}{s!}$$

Also, the pdf of the largest order statistics $X_{(n:n)}$ is obtained as

$$G_{(k:n)}(x) = \sum_{q,w,s=0}^{\infty}E_{j,q,w,s}x^{-3}e^{\frac{-\lambda^2(b(s+1)+w)}{x^2}} \tag{2.45}$$

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where

$$E_{n,q,w,s}=\frac{1}{B(n,1)}\sum_{q,w,s=0}^{\infty}(-1)^{j+q+s+w}\binom{n+j-1}{j}\binom{j}{q}\binom{b(s-1)-1}{w}\frac{2\lambda^{2}(P)^{s+1}(q+1)^{s}}{s!}$$

2.1.10 Asymptotic Behavior of EWIR Distribution

2.3.1 The asymptotic behavior of the proposed distribution model EWIRD, when $x \to 0$ and when $x \to \infty$

The value x of the function EWIR distribution when $f_X(x)$ approaches∞.

$\lim_{x\to\infty} f_X(x)=\infty$

The functions $f_X(x)$ and $h_X(x)$ of EWIR distribution will be undefined if

$$\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)^{2}\right]\right\}^{-\beta-1}=0 \tag{2.46}$$

and $x=\infty$. Thus, the vertical asymptote of $f_X(x)$ is given by

$$\lim_{x\to\infty} f_X(x)=\infty \tag{2.47}$$

and the vertical asymptote of $h_X(x)$ is given by

$$\lim_{x\to\infty} h_X(x)=\infty \tag{2.48}$$

So, equations (47) and (48) are the vertical asymptotes of the PDF and hazard function of EWIR distribution respectively for $x \geq 0, a, \alpha, \beta, \lambda, > 0$.

2.2 Estimation of Parameters of EWIR Distribution

There exist many parameter estimation methods such as maximum likelihood estimation (MLE) Bayes Estimators, methods of moment estimators, etc, however, in this study the methods of MLE are considered as it is the classical frequentist approach to parameter estimation. Most of the other parameter estimation methods are derived from these classical methods. [9], but [18] used quantile estimation method

2.2.1 Maximum Likelihood Estimation Method

The maximum likelihood estimate (MLE) is the value $\widehat{\theta}$ which maximizes the function $L(\theta)$ given by

$$L(\theta)=f(x,\ \Psi)$$

Where $f(x,\ \Psi)$ is the pdf of EWIR distribution

Let $X_1, X_2, \ldots,\ X_n$ be a random sample of size n from EWIR distribution with observed values $x_1,\ x_2, \ldots,\ x_n$. the likelihood function of EWIR can be given by

$$L(\theta)=\prod_{i=1}^{n} f(x,\ \Psi)$$

$$f(x,\ \Psi)=2a\alpha\beta\lambda^{2}x^{-3}\exp\left[-\beta\left(\frac{\lambda}{x}\right)^{2}\right]\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)^{2}\right]\right\}^{-\beta-1}\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^{2}-1\right]^{-\beta}\right\}$$
$$\times\left(1-\exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^{2}-1\right]^{-\beta}\right\}\right)^{a-1};$$

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$x \geq 0, a, \alpha, \beta, \lambda, > 0$

$$L(\theta) = \left(2a\alpha\beta\lambda^2\right)^n x^{-3n} \left[e^{-\sum_{i=1}^{n} \left(\beta\left(\frac{\lambda}{x}\right)^2\right)} \right] \prod_{i=1}^{n} \begin{array}{c} \left\{1 - \exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1} \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\} \\ \times \left(1 - \exp\left\{-\alpha\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right\}\right)^{a-1} \end{array} \tag{2.49}$$

By taking the logarithm of (49) we find the log-likelihood function

$$\begin{aligned} L = lnf(x,\Psi) &= n\log(2) + n\log(a\alpha\beta) + nlog\left(\lambda^2\right) - (2n)sum\left(\log(x)\right) - \beta\sum_1^n\left(\left(\frac{\lambda}{x}\right)^2\right) + (-\beta-1) \\ &\times \sum_1^n\left(\log\left(1 - \exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)\right) - \beta\sum_1^n\left(-\alpha \times \left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right) \\ &+ (a-1)\sum_1^n\left(\log\left(1 - \exp\left(-\alpha\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right)\right)\right) \end{aligned} \tag{2.50}$$

Taking a partial differentiation of equation (50) for a, α and β, λ respectively and equate them to zero

$$\frac{\partial \updownarrow}{\partial a} = \frac{n}{a} + \sum_1^n\left(\log\left(1 - \exp\left(-\alpha\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right)\right)\right) \tag{2.51}$$

$$\frac{\partial \updownarrow}{\partial a} = \frac{n}{a} - \beta\sum_1^n\left(\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right) - (a-1)\sum_1^n\left(\log\left(1 - \exp\left(\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right)\right)\right)^{-\beta} \tag{2.52}$$

$$\begin{aligned} \frac{\partial \updownarrow}{\partial \beta} &= \frac{n}{\beta} - \sum_1^n\left(\left(\frac{\lambda}{x}\right)^2\right) - \sum_1^n\left(\log\left(1 - \exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)\right) + \sum_1^n\left(-\alpha\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right) \\ &\times \left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right) + (a-1)\sum_1^n\left(\log\left(1 - \exp\left(-\alpha\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right)\right)\right)\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right) \end{aligned} \tag{2.53}$$

$$\frac{\partial \updownarrow}{\partial \lambda^2} = \frac{2n}{\lambda} + A_a \tag{2.54}$$

where

$$\begin{aligned} A_a = -2\sum_1^n\left(\left(\frac{\lambda}{x}\right)^2\right) - 2(-\beta-1)\sum_1^n\left(\log\left(1 - \exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)\right) \\ + 2\beta\sum_1^n\left(-\alpha\left(\exp\left(\frac{\lambda}{x}\right)^2 - 1\right)^{-\beta}\right) \end{aligned}$$

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$$-2(a-1)\sum_{1}^{n}\left(\log\left(1-\exp\left(-\alpha\left(\exp\left(\frac{\lambda}{x}\right)^{2}-1\right)^{-\beta}\right)\right)\right)$$

The solution of the non-linear system of equations obtained by differentiating (51), (52), (53) and (54) for a, α, β and λ gives the maximum likelihood estimates of the model parameters. The solution can also be obtained directly by using R software when data sets are available.

2.2.2 Simulation Study

A simulation study was conducted to evaluate the MLE estimates, bias, and standard error for various parameter combinations and different sample sizes. We consider the values a = (0.5, 1.5, 2.0, 2.5) for the parameter a = (0.2. 1, 1.5, 2. 5), for β= (0. and 0.5, for the parameter β when λ= (0.5, 0.1, 1, 1.5). The process is repeated 1000 times. Four different sample sizes n = 10, 100, 500, and 1000 are considered. The estimates, bias, and the standard error are presented in Table 2 below.

Table 2. The estimates, bias, and the standard error of EWIR from the Simulation study

	n=10			100			500			1000		
	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error
a=0.5	0.781	-0.281	0.068	0.621	-0.121	0.334	0.523	-0.023	0.089	0.471	0.029	0.0326
α=0.2	0.317	-0.117	0.215	0.167	0.033	0.163	0.167	0.033	0.119	0.184	0.016	0.072
β=0.5	0.592	-0.092	0.497	0.606	-0.106	0.133	0.512	-0.012	0.047	0.495	0.005	0.345
λ=0.1	0.145	-0.045	0.102	0.201	-0.101	0.092	0.089	0.011	0.278	0.11	-0.01	0.039
a=1	1.247	-0.247	0.324	1.713	-0.713	0.016	1.219	-0.219	0.418	1.017	-0.017	0.044
α=1	1.312	-0.312	2.225	1.19	-0.19	0.013	0.821	0.179	0.982	1.042	-0.042	0.074
β=0.5	0.598	-0.098	1.442	0.708	-0.208	0.213	0.877	-0.377	0.771	0.489	0.011	0.298
λ=0.5	0.569	-0.069	0.068	0.321	0.179	0.301	0.432	0.068	0.041	0.544	-0.044	0.243
a=1.5	1.678	-0.178	0.591	1.71	-0.21	0.056	2.012	-0.512	0.327	1.56	-0.06	0.043
α=1.5	1.821	-0.321	0.749	2.012	-0.512	1.289	2.133	-0.633	0.812	1.223	0.277	0.331
β=0.5	0.553	-0.053	0.422	0.879	-0.379	1.241	0.664	-0.164	0.019	0.516	-0.016	0.082
λ=1	1.257	-0.257	0.387	0.822	0.178	0.025	0.849	0.151	0.059	0.927	0.073	0.042
a=2	2.243	-0.243	0.457	2.291	-0.291	0.761	2.721	-0.721	0.331	2.033	-0.033	0.087
α=2.5	2.088	0.412	3.426	2.832	-0.332	1.664	2.048	0.452	0.889	2.539	-0.039	0.211
β=2	3.2	-1.2	0.477	2.71	-0.71	0.319	2.417	-0.417	0.068	2.112	-0.112	0.098
λ=1.5	1.766	-0.266	0.322	1.554	-0.054	0.029	1.509	-0.009	0.079	1.502	-0.002	0.0012

2.3 Application

In this section, real data sets are utilized to show that the EWIR model outperforms some other models. The data set contains sample of size 69 carbon fiber strength (20mm)

2.3.1 Data Description

The strength data was originally reported by [19] where the strength is measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows at gauge lengths of 20 mm. These data set were fitted to the Half- Logistics Inverse Rayleigh (HLIR) distribution by [20] and the Type II Topp-Leone Inverse Rayleigh (T2TLIR) distribution was fitted to the data. Other distributions that have been fitted to these same data are the Transmuted Inverse Rayleigh distribution (TIR), the Odd Frechet Inverse Rayleigh (OFIR) distribution, one parameter Inverse Rayleigh (IR) distribution and Weibull Inverse Rayleigh (WIR) distribution was discussed by [9].

Data: carbon fibers Strength (20mm) Data set

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629,

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2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

The MLE of the parameters (with their standard errors) is presented in Table 3 and their corresponding log-likelihood values. The Akaike Information Criterion (AIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) are used to compare the performances of all the models on the data sets employed. The results are shown in Table 3.

Table 3. The MLEs, (SEs in parentheses) and -log likelihood of the new EWIR distribution and those of the other five existing related distributions on the Strength (20mm) data set

Models	MLE(SE)				$-\log$ likelihood
EWIR($\alpha, \alpha, \beta, \lambda$)	0.4239	0.2213	0.1786	3.223	47.129
	(0.1422)	(0.291)	(0.3284)	(0.1019)	
WIR(α, β, δ)	0.3232	2.1396	1.7896		48.8915
	(0.9469)	(0.5469)	(0.8746)		
HLIR(α, λ)	3.6538	10.2773	-		50.5018
	(0.2197)	(2.5587)			
T2TLIR(α, θ)	2.7966	10.2992	-		52.0685
	(0.1574)	(2.8538)			
TIR(θ, λ)	7.5093	0.8891	-		71.9390
	(19.4108)	(0.0182)			
OFIR(θ, α)	2.9540	1.3910	-		71.7113
	(0.1862)	(0.1231)			
IR(α)	2.2827	-	-		88.4130
	(0.1374)				

Table 3 shows the MLE parameter estimates with their standard errors for the competing models EWIR, WIR, HLIR, TIR, OFIR, and IR distributions. Their –log likelihood values are also displayed.

Table 4. Goodness-of-fit measures based on AIC, BIC, HQIC, K-S values for the Strength (20mm) data set

Models	AIC	BIC	HQIC	K-S Value	P-value
EWIR($\alpha, \alpha, \beta, \lambda$)	102.258	111.001	105.1144	0.021	0.9211
WIR(α, β, δ)	103.7831	110.4854	106.4421	0.029	0.9437
HLIR(α, λ)	105.0030	109.4720	106.7764	0.0596	0.9668
T2TLIR(α, θ)	108.1371	112.6053	109.9098	0.0776	0.7993
TIR(θ, λ)	145.8787	148.1128	146.7651	0.254	0.0002
OFIR(θ, α)	147.4228	151.8910	149.1955	0.1801	0.0227
IR(α)	178.8262	181.0603	179.7125	0.3549	0.0000

Table 4 shows the goodness of fit criteria for the competing models. The Akaike Information Criterion (AIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion

(HQIC) are used to compare the performances of all the models on the data sets employed. Base on The MLEs, SEs (parentheses), and -log likelihood of the new EWIR distribution and those of the other five existing related distributions on the Strength (20mm) data set. These five distributions are fitted to the data using maximum likelihood estimation. Based on the criteria displayed, the new EWIR model provides the best fit among the other models for data as shown in Table 4, since it has the lowest values of AIC, BIC, and K-S Values.

3 DISCUSSION

In this study, a new continuous distribution is developed named Exponentiated Weibull Inverse Rayleigh (EWIR) distribution. Different properties of EWIR were derived. The maximum likelihood estimation method was used to estimate the parameters of the distribution. A simulation study was carried out to show the consistency of the MLEs. Table 2 shows that when $n \geq 100$ the biases are a bit high but as the value of n increases to infinity the biases converge to zero. Table 3 shows the MLE parameter estimates with their standard errors for the competing models EWIR, WIR, HLIR, TIR, OFIR, and IR distributions. Their -log likelihood values are also displayed. Table 4 shows the goodness of fit criteria for the competing models. The Akaike Information Criterion (AIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) are used to compare the performances of all the models on the data sets employed. Base on The MLEs, SEs (parentheses), and $-l$og likelihood of the new EWIR distribution and those of the other five existing related distributions on the Strength (20mm) data set. These five distributions are fitted to the data using maximum likelihood estimation. Based on the criteria displayed, the new EWIR model provides the best fit among the other models for data as shown in Table 4, since it has the lowest values of AIC, BIC, and K-S Values.

4 CONCLUSION

In this study, a four-parameter Exponentiated Weibull Inverse Rayleigh is introduced. Some certain properties of the proposed distribution are discussed. This model includes some new special distributions, nevertheless, the relevance of the new model is clarified through the application of real data, where the EWIR yields the best fit among the other related models. We conclude that EWIR distribution can be regarded as a more flexible model for modeling real-life data. Believing that the new EWIR distribution may serve as the most preferred model for life distribution and has application in lots of scientific fields. The MLE method was used for the parameter estimation base on the simulation and application of real data, but it will be imperative to employ other estimation methods to compare their performance.

ACKNOWLEDGMENTS

The authors would like to extend their heartfelt gratitude to the reviewers and the editor of this article for their valuable input, which played a significant role in the improvement of this research work.

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