International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2019, No. 1 , pp. 455 - 467

Characterization of a class of Symmetric Group

E. E. Edeghagba[1]⋆, K. I. Rabiu[2] and G. I. Tajudeen[3]

1 *, 2, Department of Mathematical Science, Bauchi State University, Gadau, Nigeria.
3, Department of Mathematics and Statistics, School of Science and Tecgnology, Abubakar Tatari Ali Polytechnic, Bauchi, Nigeria.
⋆Corresponding Author email: eghosaelijah@yahoo.co.uk

Article Info
Received: 19 January 2018 Revised: 21 January 2019
Accepted: 25 April 2019 Available online: 13 May 2019

Abstract

This paper studies a class of permutation group on a nonempty set $X = \{1, 2, 3, \ldots, n\}$ that maps even integer to even integer and odd integer to odd integer. We concluded that this collection of permutations, $\mathcal{B}_n$ is a group and that if n is even $\mathcal{B}_n \cong (S_{n/2})^2$ but if n is odd $\mathcal{B}_n \cong S_{(n+1)/2} \times S_{(n-1)/2}$.

Keywords: Symmetric group; Permutation; Dihedral group; Generators; Relations; Cayley's Theorem.
MSC2010: 20B35

1 Introduction

The notion of deriving subgroups of a symmetric group have appeared in several research works carried out by different researchers [1, 2, 3, 4, 5].
A subgroup of a symmetric group is called a permutation group. The notion of subgroups of symmetric group is used with several slightly different meanings, all related to the act of permuting (rearranging in an ordered fashion) objects or values. Informally, a permutation of a set of values is an arrangement of those values into a particular order.
It is often convenient to represent group elements as "words" in a few symbols, having certain relations.
The problem of describing all groups of order n for a positive integer n by giving one presentation by generators and relations for each isomorphism type of group of order n was initiated by Cayley [6] in 1878. He called it the general problem for finite groups.
Then W. Von Dyck (1856-1934), an American mathematician in the 1880s was the one who systematically invented the notion of 'presentation' of a group, where he derived the presentation of S_4 given by $\langle x, y : x^4 = y^2 = (xy)^3 = e\rangle$, as an example. This work has led to a branch of group theory called Combinatorial Group Theory.
Good introductions to group theory are provided in the following [7, 8, 9, 10, 11]. Important concepts we use in describing the computations in this paper are the following. A finite presentation

http://ijmao.unilag.edu.ng/article/view/450

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$\langle A|R\rangle$ consists of a finite set A of generators and a finite set R of relators, which are words in the generators and their inverses.
If G_1 and G_2 are groups presented as follows $G_1 = \langle A_1|R_1\rangle$ and $G_2 = \langle A_2|R_2\rangle$, it is well known that their direct product has a presentation $G_1 \times G_2 = \langle A_1, A_2|R_1, R_2, xy = yx(x \in A_1, y \in A_2)\rangle$. An immediate consequence of this is that $G_1 \times G_2$ is finitely generated if and only if both G_1 and G_2 are finitely generated, and is finitely presented if and only if both G_1 and G_2 are finitely presented.
Thus the study of the structure of groups via their generators and relators have appeared in several literatures. In [12], the authors in their study of endomorphisms of groups of order 36, presented the 14 pairwise non-isomorphic groups of order 36 using their presentations.
Our research deals with identification of certain kind of subset of the symmetry group such that each element of this subset, which of course are permutations, preserves parity of the integers (i.e. even integer are mapped to even integer and odd integer are mapped to odd integer) and then characterize this subset by group presentation. Thus, in this work we were able to prove that this subset as derived turns out to be a permutation group of the given symmetric group. This subset is denoted by $\mathcal{B}_n$.
In the section of main result we give some characterizations of this subset, $\mathcal{B}_n$ and that $\mathcal{B}_n$ is isomorphic to an existing group.
Section two gives some preliminary notions, while in section three we presented the main result of the paper.
Finally, in section four we give some description and construction of a general way of deriving this permutation group for some finite positive integer n.
The aim of this research is to study certain collection of permutations of the symmetric group which turns out to be a subgroup of the symmetric group and isomorphic to existing group.

2 Preliminary

We give some basic definitions and results for the benefit of all.

Definition 1. *[9] A permutation of a set A is simply a bijection from A to itself.*

Definition 2. *[13] The family of all the permutations of a set X, denoted by S_X , is called the symmetric group on X. When $X = \{1, 2, \dots, n\}$, S_X is usually denoted by S_n, and it is called the symmetric group on n letters.*

Definition 3. *A transposition is a permutation which exchanges any two element and keeps all others fixed.*

Definition 4. *[14] Let $\pi \in S_n$, so that π is a permutation of the set $\{1, 2, ..., n\}$. The support of π is defined to be the set of all i such that $\pi(i) \neq i$, in symbols supp(π). Now let r be an integer satisfying $1 \leq r \leq n$. Then π is called an r-cycle if $supp(\pi) = \{i_1, i_2, ..., i_r\}$, with distinct i_j , where $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_{r-1}) = i_r$ and $\pi(i_r) = i_1$. So π moves the integers $i_1, i_2, \dots, i_r$ anticlockwise around a circle, but fixes all other integers: often π is written in the form $\pi = (i_1 i_2 \cdots i_r)(i_{r+1}) \cdots (i_n)$ where the presence of a 1-cycle (j) means that $\pi(j) = j$.*

Definition 5. *[14] Permutations $\pi, \tau \in S_n$ are called disjoint if their supports are disjoint, i.e., they do not both move the same element.*

Definition 6. *The group of symmetries of a regular polygon of n sides is called dihedral group of degree n, denoted D_{2n}. It is generated by two elements r (rotation) and s (reflection) satisfying the relations*

$$r^n = 1,\ s^2 = 1,\ \text{and}\ \ srs = r^{-1}. \tag{2.1}$$

Definition 7. *[15] Let $(G, \circ)$ be a group and $a \in G$. If there exists a positive integer n such that $a^n = e$, then the smallest such positive integer is called the order of a. If no such positive integer n exists, then we say that a is of infinite order.*

Definition 8. *[9] A subset S of elements of a group G with the property that every element of G can be written as a (finite) product of elements of S and their inverses is called a set of generators of G. We shall indicate this notationally by writing $G = \langle S \rangle$ and say G is generated by S or S generates G.*

Definition 9. *[15] Let $(G, *)$ and $(G_1, *_1)$ be groups and ϕ a mapping from G into G_1. Then ϕ is called a homomorphism of G into G_1 if for all $a, b \in G$,*

$$\phi(a * b) = \phi(a) *_1 \phi(b).$$

Thus ϕ is said to be an isomorphism if ϕ is bijective. In this case, we write $G \simeq G_1$ and say that G and G_1 are isomorphic.

Theorem 1. *$(Cayley'sTheorem)$[11]. Every group is isomorphic to a group of permutations.*

Theorem 2. *[15]. Let D_{2n} be the dihedral group of degree n. Then the following assertions hold.*
(i) Every element of D_{2n} is of the form $r^i s^j$, $0 \le i < n, 0 \le j < 2$.
(ii) D_{2n} has exactly $2n$ elements, i.e., $o(D_{2n}) = 2n$.
(iii) D_{2n} is a noncommutative group.

Theorem 3. *Let G be generated by elements a and b where $a^n = 1$ for some $n \ge 3$, $b^2 = 1$, and $bab^{-1} = a^{-1}$. Then there is a subjective homomorphism $D_{2n} \to G$ and if G has order $2n$, then this homomorphism is an isomorphism.*

Theorem 4. *[9] Let $\pi, \lambda \in S_n$ such that π and λ are disjoint. Then $\pi \circ \lambda = \lambda \circ \pi$, i.e., π and λ commute.*

Theorem 5. *[16] The order of any permutation σ is the least common multiple of the length of its disjoint cycles.*

Proposition 1. *[13] Let G be a finite group. Let H and K be subgroups of G, then*

$$o(HK) = \frac{o(H)o(K)}{o(H \cap K)},$$

where $HK = \{hk : h \in H,\ k \in K\}$.

Proposition 2. *[13] If G is a group containing normal subgroups H and K with $H \cap K = \{1\}$ and $HK = G$, then $G \cong H \times K$.*

3 Main Results

Remark 1. *The fact that a permutation α on a nonempty set X is usually written as $\alpha : X \to X$, even though in general the nature of the set X is not of much importance. But since we consider X to be finite and countable then X can be put in a 1-1 correspondence with a subset of the natural numbers (counting numbers). Therefore, our notion of preserving parity of counting numbers is not out of place.*

Definition 10. *The set $\mathcal{B}_n \subset S_n$ is the set whose elements are permutations that maps even integer to even integer and odd integer to odd integer.*

Proposition 3. *Let S_n be the symmetric group on $X = \{1, 2, 3, \ldots, n\}$. Then $\mathcal{B}_n$ is a subgroup of S_n.*

Proof. Let $\alpha, \beta \in \mathcal{B}_n$. We show that $\mathcal{B}_n$ is closed under the composition operation. We look at three cases:
Case I: Suppose α and β fix every even integer. Then since α and β move at least one odd integer, it follows that for x an odd number in X, there exists an odd integer say y in X such that $\beta(x) = y$. Thus

$$\alpha\beta(x) = \alpha(\beta(x)) = \alpha(y),$$

which is also an odd integer. Thus $\alpha\beta \in \mathcal{B}_n$.
Case II: Suppose α and β fix every odd integer and move at least one even integer. The argument follows from Case I.
Case III. We can assume WLOG that α moves at least one even integer and β moves at least one odd integer. Then for an odd integer i, we have that

$$\alpha\beta(i) = \alpha(\beta(i)) = \alpha(j) = j$$

for some odd integer j, where $\beta(i) = j$. Now suppose i is an even integer, then

$$\alpha\beta(i) = \alpha(\beta(i)) = \alpha(i) = k$$

for some even integer k, where $\alpha(i) = k$. Hence $\alpha\beta \in \mathcal{B}_n$. Next, let $\alpha \in \mathcal{B}_n$ and α^{-1} be any given permutation on X. Then for $x, y \in X$ such that

$$\alpha(x) = y \text{ and } \alpha^{-1}(y) = x\ .$$

Clearly, $\alpha^{-1} \in \mathcal{B}_n$, since if x is an odd integer then y must also be an odd integer where α is one to one. Similar argument holds for when x is an even integer. Hence since α is one to one and onto it follows that

$$\mathcal{I}(y) = y = \alpha(x) = \alpha(\alpha^{-1}(y)) = \alpha\alpha^{-1}(y) = (\alpha\alpha^{-1})(y)$$

and

$$\mathcal{I}(x) = x = \alpha^{-1}(y) = \alpha^{-1}(\alpha(x)) = \alpha^{-1}\alpha(x) = (\alpha^{-1}\alpha)(x)$$

Where $\mathcal{I}$ is the identity permutation. Thus

$$\alpha^{-1}\alpha = \mathcal{I} = \alpha\alpha^{-1}.$$

Hence α^{-1} is the inverse of α. Associativity follows immediately. Since if every element is fixed, then even integer is mapped to even integer while odd integer is mapped to odd integer. Hence $\mathcal{I} \in \mathcal{B}_n$. So we have shown that $\mathcal{B}_n$ is a group and therefore a subgroup of S_n. □

Proposition 4. *If $n = 4$ or 5, then the maximum order of the elements in $\mathcal{B}_n$ is equal to $o(\mathcal{B}_n)/2$.*

Corollary 1. *If $n \leqslant 5$, the $\mathcal{B}_n \cong D_{2\times o(\mathcal{B}_n)/2}$.*

We present a characterization for $n = 5$.
$\mathcal{B}_5 = \{\alpha_0 = (1),\ \alpha_1 = (13),\ \alpha_2 = (15),\ \alpha_3 = (35),\ \alpha_4 = (24),\ \beta_1 = (135),\ \beta_2 = (153),\ \tau_1 = (13)(24),\ \tau_2 = (15)(24),\ \tau_3 = (35)(24),\ \sigma_1 = (135)(24),\ \sigma_2 = (153)(24)\}$.

Proposition 5. *Given the group $\mathcal{B}_5$. Then $\mathcal{B}_5$ can be generated by two elements subject to the following relations*

$$\beta^2 = 1,\ \alpha^6 = 1 \text{ and } \alpha\beta = \beta\alpha^{-1}$$

for $\alpha, \beta \in \mathcal{B}_5$.

Proof. First we show that $\mathcal{B}_5$ contains an element of order 2. Let $\beta \in \mathcal{B}_5$ such that β fixes all odd integer, clearly β is of order 2. Now let $\beta = \beta_1 \circ \beta_2$ for $\beta_1, \beta_2 \in \mathcal{B}_5$, for which β_1 fixes all even numbers and one odd number, and β_2 fixes all odd numbers. Clearly, $\beta = \beta_1 \circ \beta_2$ is the composition of disjoint permutations, β_1 and β_2 and by Theorem 5, β is of order 2. Since $\mathcal{B}_5$ is a group then $\beta \in \mathcal{B}_5$. Next, if $\beta \in \mathcal{B}_5$ fixes all even numbers and fixes one odd number in $\mathcal{B}_5$, certainly β is of order 2. Hence there exists an element of order 2 in $\mathcal{B}_5$. Next we show that $\mathcal{B}_5$ contains an element of order 6. Let $\alpha_1, \alpha_2 \in \mathcal{B}_5$ where α_1 fixes all even numbers and α_2 fixes all odd numbers in $\mathcal{B}_5$. Clearly, $\alpha = \alpha_1 \circ \alpha_2$ is the composition of disjoint permutations α_1 and α_2. Then again by Theorem 5, α is of order 6 and since $\mathcal{B}_5$ is a group then $\alpha \in \mathcal{B}_5$. Hence there exists an element of order 6 in $\mathcal{B}_5$. Therefore, there exist permutations α and β in $\mathcal{B}_5$ such that $\beta^2 = 1$, and $\alpha^6 = 1$. Hence we have the following distinct permutations; $\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6 = 1$. Let

$$\beta \neq \alpha^i, \ \forall i \in \{1, 2, 3, 4, 5, 6\}. \tag{3.1}$$

From equation (3.1) it must be that β is a cycle of odd numbers. This follows from the fact that α^i for $i = 3$ is a 2-cycle of even numbers which must be the cycle (24). Thus if α^i is a transposition disjoint from β then by Theorem 5

$$o(\alpha^i \beta) = 2.$$

Implying that

$$\alpha^i \beta = \beta^{-1} \alpha^{-i} = \beta \alpha^{-i} \Rightarrow \alpha^i \beta = \beta \alpha^{-i}.$$

Next, let α^i be a composition of disjoint cycles. Then since α is a composition of a 3-cycle of odd integer and a 2-cycle of even numbers. It follows that any power of α that is a composition of disjoint cycles can only be a 3-cycles of odd numbers and 2-cycles of even numbers. Hence if

$$\alpha^i = \alpha_1 \alpha_2.$$

Then by associative property it follows that

$$\alpha^i \beta = (\alpha_1 \alpha_2)\beta = \alpha_1(\alpha_2 \beta). \tag{3.2}$$

So from (3.2) if α_2 is the 3-cycle of odd numbers then $\alpha_2 \beta$ is a 2-cycle of odd numbers. Thus by Theorem 5

$$o(\alpha_1(\alpha_2 \beta)) = 2,$$

and so

$$(\alpha_1 \alpha_2)\beta = ((\alpha_1 \alpha_2)\beta)^{-1}.$$

Hence

$$\alpha^i \beta = (\alpha^i \beta)^{-1} = \beta^{-1} \alpha^{-i} = \beta \alpha^{-i} \Rightarrow \alpha^i \beta = \beta \alpha^{-i}.$$

Now for the case where α^i is a 3-cycle, then it must be a cycle of odd numbers. Thus $\alpha^i \beta$ is a 2-cycle and so

$$o(\alpha^i \beta) = 2.$$

Hence

$$\alpha^i \beta = \beta \alpha^{-i}.$$

Therefore we have shown that the relation

$$\alpha^i\beta = \beta\alpha^{-i}.$$

holds in $\mathcal{B}_5$. In particular for $i = 1$

$$\alpha\beta = \beta\alpha^{-1}.$$

Now for $\alpha, \beta \in \mathcal{B}_5$ we show that α and β generates $\mathcal{B}_5$ such that $\alpha^6 = 1$ and $\beta^2 = 1$. Therefore, from our argument above we have that $\alpha^i, i \in \{1, 2, \ldots, 6\}$ are distinct elements. Thus

$$\beta\alpha^i \neq \beta\alpha^j, \text{ for } 0 \leq i \leq 5,$$

with $i \neq j$ so

$$\mathcal{B}_5 = \{\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \beta\alpha^0, \beta\alpha^1, \beta\alpha^2, \beta\alpha^3, \beta\alpha^4, \beta\alpha^5\}.$$

Therefore $\mathcal{B}_5 = \langle\alpha, \beta\rangle$.

□

Clearly, from above we have that $\mathcal{B}_5$ is not commutative. For example the elements $\alpha_1, \tau_3 \in \mathcal{B}_5$ do not commute with themselves;
$\alpha_1\tau_3 = (13)((35)(24)) = (135)(24) = \sigma_1$
and
$\tau_3\alpha_1 = ((35)(24))(13) = (153)(24) = \sigma_2$.
Hence, let $\alpha = \sigma_2$ and $\beta = \alpha_3$ therefore $\mathcal{B}_5 = \langle\sigma_2, \alpha_3\rangle$. Where

$$\langle\sigma_2\rangle = \{(1), (153)(24), (135), (24), (153), (135)(24)\}.$$

and

$$\langle\alpha_3\rangle = \{(1), (35)\}.$$

Therefore,

$$\langle\sigma_2\rangle\langle\alpha_3\rangle = \{(1), (153)(24), (135), (24), (153), (135)(24), (35), (13)(24), (15), (24)(35), (13), (15)(24)\} = \mathcal{B}_5.$$

Next we show that $\sigma_2\alpha_3 = \alpha_3\sigma_2^{-1}$

$\sigma_2\alpha_3 = ((153)(24))(13) = \tau_2$ and $\alpha_3(\sigma_2)^{-1} = (13)((153)(24))^{-1} = \tau_2$.

Hence the relation, $\sigma_2\alpha_3 = \alpha_3\sigma_2^{-1}$ exists between the generators.
From the above results and computations it follows that $\mathcal{B}_5$ is a group and of course a subgroup of S_5 which is generated by two elements σ_2 and α_3 satisfying the relation

$$\sigma_2^6 = \alpha_0, \quad \alpha_3^2 = \alpha_0, \quad \sigma_2\alpha_3 = \alpha_3\sigma_2^{-1}.$$

where α_0 is the identity permutation.
Hence, by Cayley's Theorem and Corollary 3 we assert the existence of an isomorphic copy of the dihedral group D_{12} in the symmetric group S_5. Therefore the following corollary.

Corollary 2. *The subgroup $\mathcal{B}_5$ of the symmetric group S_5 is isomorphic the dihedral group D_{12}.*

Therefore, the group $\mathcal{B}_5$ is give by the presentation

$$\mathcal{B}_5 = \langle\alpha, \beta \mid \beta^2 = \alpha^6 = 1, \; \beta\alpha = \alpha^{-1}\beta\rangle.$$

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where $\beta = (13)$ and $\alpha = (135)(24)$.
Below we give the Cayley's table for the group $\mathcal{B}_5$.

$\circ$	α_0	α_3	α_4	α_2	α_1	β_1	β_2	τ_3	τ_1	τ_2	σ_1	σ_2
α_0	α_0	α_3	α_4	α_2	α_1	β_1	β_2	τ_3	τ_1	τ_2	σ_1	σ_2
α_3	α_3	α_0	τ_3	β_1	β_2	α_2	α_1	α_4	σ_2	σ_1	τ_2	τ_1
α_4	α_4	τ_3	α_0	τ_2	τ_1	σ_1	σ_2	α_3	α_1	α_2	β_1	β_2
α_2	α_2	β_2	τ_2	α_0	β_1	α_1	α_3	σ_2	σ_1	α_4	τ_1	τ_3
α_1	α_1	β_1	τ_1	β_2	α_0	α_3	α_2	σ_1	α_4	σ_2	τ_3	τ_2
β_1	β_1	α_1	σ_1	α_3	α_2	β_2	α_0	τ_1	τ_2	τ_3	σ_2	α_4
β_2	β_2	α_2	σ_2	α_1	α_3	α_0	β_1	τ_2	τ_3	τ_4	α_1	σ_1
τ_3	τ_3	α_4	α_3	σ_1	σ_2	τ_2	τ_1	α_0	β_2	β_1	α_2	α_1
τ_1	τ_1	σ_1	α_1	σ_2	α_4	τ_3	τ_2	β_1	α_0	β_2	α_3	α_2
τ_2	τ_2	σ_2	α_2	α_4	σ_1	τ_1	τ_3	β_2	β_1	α_0	α_1	α_3
σ_1	σ_1	τ_1	β_1	τ_3	τ_2	σ_2	α_4	α_1	α_2	α_3	β_2	α_0
σ_2	σ_2	τ_2	β_2	τ_1	τ_3	α_4	σ_1	α_2	α_3	α_1	α_0	β_1

Figure 1: Cayley's Table for $\mathcal{B}_5$

Next we present a characterization for $n = 6$.
The set $\mathcal{B}_6$ contains the following elements:
(1), (13), (15), (35), (24), (26), (46), (13)(24), (13)(26), (13)(46), (15)(24), (15)(26), (15)(46), (35)(24), (35)(26), (35)(46), (135), (153), (246), (264), (135)(24), (135)(26), (135)(46), (153)(24), (153)(26), (153)(46), (246)(13), (246)(15), (246)(35), (264)(13), (264)(15), (264)(35), (135)(246), (135)(264), (153)(246), (153)(264).

Clearly, $\mathcal{B}_6$ is not isomorphic to any dihedral group. The obvious reason for this is because in $\mathcal{B}_6$, there does not exist any element (permutation) of order half the order of $\mathcal{B}_6$. Therefore the group $\mathcal{B}_6$, can not be generated by two elements, α, β subject to the relations,

$$\alpha^n = 1 \ , \ \beta^2 = 1 \ , \ \alpha\beta = \beta\alpha^{-1}$$

$\mathcal{B}_6$ is of order 36, but clearly by Theorem 5 the maximum order of elements in $\mathcal{B}_6$ is 6 which is less than 18.

We claim that the group $\mathcal{B}_6$ is given by the presentation $G = \langle a, b, c, d | \ a^2 = b^2 = c^3 = d^3 = 1, \ da = ad^{-1}, \ cb = bc^{-1}, \ ab = ba, \ ac = ca, \ bd = db, \ cd = dc \rangle$. Proving this we set $a \mapsto (13)$, $b \mapsto (24)$, $c \mapsto (246)$, $d \mapsto (135)$. Since every relations satisfied by $\{a, b, c, d\}$ are satisfied by $\{(13), (24), (246), (135)\}$, clearly we have a (unique) homomorphism $\varphi : G \to \mathcal{B}_6$. Hence G is a homomorphic image of $\mathcal{B}_6$. Therefore, we need show that φ is injective.

Proposition 6. *Let H be a subgroup of $\mathcal{B}_6$. If H is an isomorphic copy of D_6 in $\mathcal{B}_6$ such that each of the generators of H are not permutations of disjoint cycles in $\mathcal{B}_6$, then H is a normal subgroup in $\mathcal{B}_6$.*

Proof. Suppose $H \leq \mathcal{B}_6$ and $H \cong D_6$. Let $H = \langle a, b \rangle$, then $H = \{a^i b^k : 0 \leqslant i \leqslant 1, 0 \leqslant k \leqslant 2\}$. Suppose $\alpha \in H$, then $\alpha = a^i b^k$ for some i and k. For a nontrivial $\beta \in \mathcal{B}_6$, we must show that $\beta\alpha\beta^{-1} \in H$.
Case I : Assume $\beta \in H$ then $\beta = a^j b^l$ for some $0 \leqslant j \leqslant 1, 0 \leqslant l \leqslant 2$. It follows that

$$\begin{aligned}
\beta\alpha\beta^{-1} &= (a^j b^l)(a^i b^k)(a^j b^l)^{-1} = (a^j b^l)(a^i b^k)(b^{-l} a^j) \\
&= (a^j b^l)(a^i b^k b^{-l})a^j = (a^j b^l)(a^i b^{k-l})a^j = a^j (b^l b^{-(k-l)})a^i a^j \\
&= a^j b^{l-(k-l)} a^{i+j} = a^j a^{i+j} b^{-(l-(k-l))} = a^{(i+2j)} b^{(k-2l)} \\
&= a^m b^n,
\end{aligned}$$

where $m \equiv i + 2j \ (mod\ 2)$ and $n \equiv k - 2l \ (mod\ 3)$.

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The formulation above holds since every element of H has the unique representation, $a^ib^k, 0 \leqslant i \leqslant 1, 0 \leqslant k \leqslant 2$ and any product of two elements in this form can be reduced to this form, where all exponents are reduced accordingly. Thus $\beta\alpha\beta^{-1} \in H$.

Case II : Assume $\beta \notin H$, it suffices to prove that for $\alpha \in H$, $\beta\alpha\beta^{-1} = a^ib^k \in H$ for some $0 \leqslant i \leqslant 1, 0 \leqslant k \leqslant 2$, for $\alpha = a^ib^k$. Since the generators a and b are not products of disjoint cycles then we consider first, the case were β and α are disjoint permutations, then it follows that

$$\beta\alpha\beta^{-1} = \beta(a^ib^k)\beta^{-1} = \beta\beta^{-1}(a^ib^k) = (a^ib^k).$$

Thus $\beta\alpha\beta^{-1} \in H$.

Next we consider the case were α and β are not disjoint. Therefore, the permutation, β must be product of disjoint cycles of order 3 or 6. If β should be otherwise (i.e. a 2-cycle or 3-cycle permutation), then it must be that β is disjoint from α or an element of H, a contradiction.

Thus, let $\beta = \beta_1\beta_2$, where β_1 and β_2 are disjoint permutations. Since α and β are not disjoint permutations it follow that either α and β_1 or α and β_2 are not disjoint. WLOG we assume α and β_1 are not disjoint.

$$\begin{aligned}\beta\alpha\beta^{-1} &= (\beta_1\beta_2)^{-1}\alpha(\beta_1\beta_2) = \beta_1^{-1}\beta_2^{-1}\alpha\beta_1\beta_2 \\ &= \beta_2^{-1}\beta_2\beta_1^{-1}\alpha\beta_1 = \beta_1^{-1}\alpha\beta_1.\end{aligned}$$

Therefore, by definition of H, it follows that $\beta_1^{-1}\alpha\beta_1 \in H$ since $\beta_1^{-1}\alpha\beta_1$ must either be a 2-cycle or a 3-cycle permutation. Hence $\beta^{-1}\alpha\beta \in H$. This completes the prove. □

The condition in Proposition 6, that the generators of H are not permutations of disjoint cycles in $\mathcal{B}_6$, is very essential. For example the subgroup $K = \{(1), (13)(24), (15)(26), (35)(46),$
$(135)(246), (153)(264)\}$ generated by $\{(13)(24), (135)(246)\}$ (permutations of which each is a product of disjoint cycles) is isomorphic to D_6 but not normal in $\mathcal{B}_6$. For $(35)(24) \notin K$, then
$((35)(24))(13)(24)((35)(24))^{-1} = (15)(24) \notin K$.

Corollary 3. *The group $\mathcal{B}_6$ contains isomorphic copies of the dihedral group D_6 which are normal in $\mathcal{B}_6$.*

Therefore, from Proposition 6 above, the subgroups
$G_1 = \{(1), (13), (15), (35), (135), (153)\}$ and
$G_2 = \{(1), (24), (26), (46), (246), (264)\}$ of $\mathcal{B}_6$ generated by the sets $\{(13), (135)\}$ and $\{(24), (246)\}$ respectively are subject to the following relations $\{(13)^2 = (135)^3 = ((13)(135))^2 = 1\}$ and $\{(24)^2 = (246)^3 = ((24)(246))^2 = 1\}$ respectively. We have that $o(G_1) = 6$ and $o(G_2) = 6$, and clearly $G_1 \cap G_2 = (1)$.

Thus, by Proposition 1 it follows that $o(G_1G_2) = 36$ and by Proposition 6 we have that G_1 and G_2 are isomorphic to D_6 and clearly are normal in $\mathcal{B}_6$.

Hence, by Proposition 2, $G_1G_2 \cong G_1 \times G_2$, hence $o(G_1 \times G_2) = 36$. Therefore, since G_1 and G_2 are isomorphic to D_6 then,

$$D_6 = \langle a, d \mid a^2 = d^3 = 1,\ da = ad^{-1}\rangle \cong G_1, \text{ for } a \mapsto (13), d \mapsto (135)$$

and

$$D_6 = \langle b, c \mid b^2 = c^3 = 1,\ cb = bc^{-1}\rangle \cong G_2, \text{ for } b \mapsto (24), c \mapsto (246)$$

Thus,
$D_6 \times D_6 = \langle a, b, c, d \mid\ a^2 = b^2 = c^3 = d^3 = 1,\ da = ad^{-1},\ cb = bc^{-1},\ ab = ba,\ ac = ca,\ bd = db,\ cd =$

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$dc\rangle \cong G_1 \times G_2 = G$.

Hence we have that φ is injective. Therefore, $\mathcal{B}_6 \cong D_6 \times D_6$.

Next we present a characterization for $n = 7$.
The set $\mathcal{B}_7$ contains the following elements:
(1), (13), (15), (17), (35), (37), (57), (24), (26), (46), (13)(24),
(13)(26), (13)(46), (15)(24), (15)(26), (15)(46), (17)(24), (17)(26),
(17)(46), (35)(24), (35)(26), (35)(46), (37)(24), (37)(26), (37)(46),
(57)(24), (57)(26), (57)(46), (13)(57), (15)(37), (17)(35), (13)(57)(24),
(13)(57)(26), (13)(57)(46), (15)(37)(24), (15)(37)(26), (15)(37)(46),
(17)(35)(24), (17)(35)(26), (17)(35)(46), (135), (153), (137), (173),
(157), (175), (357), (375), (246), (264), (135)(24), (135)(26),
(135)(46), (153)(24), (153)(26), (153)(46), (137)(24), (137)(26),
(137)(46), (173)(24), (173)(26), (173)(46), (157)(24), (157)(26),
(157)(46), (175)(24), (175)(26), (175)(46), (357)(24), (357)(26),
(357)(46), (375)(24), (375)(26), (375)(46), (246)(13), (246)(15),
(246)(17), (246)(35), (246)(37), (246)(57), (264)(13), (264)(15),
(264)(17), (264)(35), (264)(37), (264)(57), (135)(246),
(135)(264), (153)(246), (153)(264). (137)(246), (137)(264),
(173)(246), (173)(264), (157)(246), (157)(264), (175)(246),
(175)(264), (357)(246), (357)(264) (375)(246), (375)(264),
(13)(57)(246), (13)(57)(264), (15)(37)(246), (15)(37)(264),
(17)(35)(246), (17)(35)(264), (1357), (1573), (1735), (1537),
(1375), (1753), (1357)(24), (1357)(26), (1357)(46), (1375)(24),
(1375)(26), (1375)(46), (1537)(24), (1537)(26), (1537)(46),
(1573)(24), (1573)(26), (1573)(46), (1735)(24), (1735)(26),
(1735)(46), (1753)(24), (1753)(26), (1753)(46), (1357)(246),
(1357)(264), (1375)(246), (1375)(264), (1537)(246), (1537)(264),
(1573)(246), (1573)(264), (1735)(246), (1735)(264),
(1753)(246), (1753)(264).
For the symmetric group S_7, $\mathcal{B}_7$ has 144 elements and the maximum order of elements in $\mathcal{B}_7$ is $12 < 72$ and so $\mathcal{B}_7$ is not isomorphic to any dihedral group.

We claim that the group $\mathcal{B}_7$ is given by the presentation $H = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^4 = (ad)^3 = 1,\ ab = ba,\ cb = bc^{-1},\ ac = ca,\ bd = db,\ cd = dc\rangle$. Proving this we set $a \mapsto (13)$, $b \mapsto (24)$, $c \mapsto (246)$, $d \mapsto (1357)$. Since every relations satisfied by $\{a, b, c, d\}$ are satisfied by $\{(13), (24), (246), (1357)\}$, clearly we have a (unique) homomorphism $\phi : H \to \mathcal{B}_7$. Hence H is a homomorphic image of $\mathcal{B}_7$. Next we show that ϕ is injective.

Corollary 4. *The group $\mathcal{B}_7$ contains isomorphic copies of the symmetric groups S_4 and S_3 which are normal in $\mathcal{B}_7$.*

Therefore, from Corollary 4 above we have that $\mathcal{B}_7$ contains subgroups;
$H_1 = \{$(1), (13), (15), (17), (35), (37), (57), (13)(57), (15)(37),
(17)(35), (135), (137), (153), (157), (173), (175), (357), (375), (1357),
(1375), (1537), (1573), (1735), (1753), $\}$
and
$H_2 = \{$(1), (24), (26), (46), (246), (264)$\}$,
which are generated by the sets $\{(13), (1357)\}$ and $\{(24), (246)\}$ respectively. Clearly, these generators are subject to the following relations $\{(13)^2 = (1357)^4 = ((13)(1357))^3 = 1\}$ and $\{(24)^2 = (246)^3 = 1,\ (24)(246) = (246)^{-1}(24)\}$ respectively. It follows then that $o(H_1) = 24$ and $o(H_2) = 6$. Thus,

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$H_1 \cap H_2 = (1)$.
Thus by Proposition 1 it follows that $o(H_1H_2) = 144$ and by Corollary 4 we have that H_1 and H_2 are isomorphic to S_4 and S_3 respectively. Clearly they are normal in $\mathcal{B}_7$.
Hence, by Proposition 2, $H_1H_2 \cong H_1 \times H_2$, hence $o(H_1 \times H_2) = 144$. Therefore, since H_1 is isomorphic to S_4 and H_2 is isomorphic to S_3 then,

$$S_4 = \langle a, d \mid a^2 = d^4 = (ad)^3 = 1 \rangle \cong H_1, \text{ for } a \mapsto (13), d \mapsto (1357)$$

and

$$S_3 = \langle b, c \mid b^2 = c^3 = 1, \ cb = bc^{-1} \rangle \cong H_2, \text{ for } b \mapsto (24), c \mapsto (246)$$

Thus,
$S_4 \times S_3 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^4 = (ad)^3 = 1, \ ab = ba, \ cb = bc^{-1}, \ ac = ca, \ bd = db, \ cd = dc \rangle \cong H_1 \times H_2 = H$.

Hence we have shown that ϕ is injective. Therefore, $\mathcal{B}_7 \cong S_4 \times S_3$.

Next we present a characterization for $n = 8$ and $n = 9$.
For the symmetric groups S_8 and S_9, $\mathcal{B}_8$ and $\mathcal{B}_9$ have 576 and 2880 elements respectively. Of course they are not isomorphic to any dihedral group.

Corollary 5. *The group $\mathcal{B}_8$ contains isomorphic copies of the symmetric group S_4 which are normal in $\mathcal{B}_8$.*

The subgroups: $H_1 = \{(1), (13), (15), (17), (35), (37), (57), (13)(57), (15)(37), (17)(35), (135), (137), (153), (157), (173), (175), (357), (375), (1357), (1375), (1537), (1573), (1735), (1753), \}$ and
$H_2 = \{(1), (24), (26), (28), (46), (48), (24)(68), (26)(48), (28)(46), (246), (248), (264), (268), (284), (286), (468), (486), (2468), (2486), (2648), (2684), (2846), (2864)\}$, of $\mathcal{B}_8$ generated by the sets $\{(13), (1357)\}$ and $\{(24), (2468)\}$, respectively and subject to the following relations $\{(13)^2 = (1357)^4 = ((13)(1357))^3 = 1\}$ and $\{(24)^2 = (2468)^4 = (24)(2468)^3 = 1\}$ respectively are normal in $\mathcal{B}_8$. Clearly, $o(H_1) = o(H_2) = 24$ and $H_1 \cap H_2 = (1)$.
Thus by Proposition 1 it follows that $o(H_1H_2) = 576$. By Corollary 5 we have that H_1 and H_2 are both isomorphic to S_4.

Therefore, we have that;

$$S_4 = \langle a, d \mid a^2 = d^4 = (ad)^3 = 1 \rangle \cong H_1, \text{ for } a \mapsto (13), d \mapsto (1357)$$

and

$$S_4 = \langle b, c \mid b^2 = c^4 = (bc)^3 = 1 \rangle \cong H_2, \text{ for } b \mapsto (24), c \mapsto (2468)$$

Thus,
$S_5 \times S_4 = \langle a, b, c, d \mid a^2 = b^2 = c^4 = d^4 = (ad)^3 = (bc)^3 = 1, \ ab = ba, \ ac = ca, \ bd = db, \ cd = dc \rangle \cong H_1 \times H_2$.
Clearly, $H_1H_2 = \mathcal{B}_8$. Therefore, by Proposition 2, we have that $H_1H_2 \cong H_1 \times H_2$.

Hence, $\mathcal{B}_8 \cong S_4 \times S_4$.

Corollary 6. *The group $\mathcal{B}_9$ contains isomorphic copies of the symmetric groups S_4 and S_5 which are normal in $\mathcal{B}_9$.*

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Let H_1 and H_2 subgroups be subgroups of $\mathcal{B}_9$ generated by the sets $\{(13),(13579)\}$ and $\{(24),(2468)\}$, respectively and subject to the following relations
$\{(13)^2 = (13579)^5 = ((13)(13579))^4 = ((13579)(13)(13579)^{-2}(13)$
$(13579))^2 = 1\}$ and $\{(24)^2 = (2468)^4 = (24)(2468)^3 = 1\}$ respectively.
Thus by Proposition 1 it follows that $o(H_1H_2) = 2880$. By Corollary 6 we have that H_1 is isomorphic to S_5 and H_2 is isomorphic to S_4.

Therefore, we have that;

$$S_5 = \langle a,d \mid a^2 = d^5 = (ad)^4 = (dad^{-2}ad)^2 = 1\rangle \cong H_1, \text{ for } a \mapsto (13),\ d \mapsto (13579)$$

and

$$S_4 = \langle b,c \mid b^2 = c^4 = (bc)^3 = 1\rangle \cong H_2, \text{ for } b \mapsto (24),\ c \mapsto (2468)$$

Thus,
$S_5 \times S_4 = \langle a,b,c,d \mid a^2 = b^2 = c^4 = d^5 = (ad)^4 = (bc)^3 = (dad^{-2}ad)^2 = 1,\ ab = ba,\ ac = ca,\ bd = db,\ cd = dc\rangle \cong H_1 \times H_2$.
Clearly, $H_1H_2 = \mathcal{B}_9$. Therefore, by Proposition 2, we have that $H_1H_2 \cong H_1 \times H_2$.

Hence, $\mathcal{B}_9 \cong S_5 \times S_4$.

From our constructions above, where $S_3 \cong D_6$ and $D_{12} \cong D_6 \times C_2$, for $C_2 \cong S_2$, we have the following result.

Proposition 7. *For the group $\mathcal{B}_n$, if n is even, then $\mathcal{B}_n \cong S_{n/2} \times S_{n/2}$ and if n is odd, then $\mathcal{B}_n \cong S_{(n+1)/2} \times S_{(n-1)/2}$.*

4 Discussion

In this section we present some generalizations, which will be useful and considered as a starting point for further research in this direction. We present some combinatoric and set theoretic notions enabling us to deal with these structures easily.
In the previous section we were able to give some characterization of $\mathcal{B}_n$, where we obtained $\mathcal{B}_n$ by collecting the permutations on the set $X = \{1,2,3,4,5,\ldots,n\}$ which maps even number to even number and odd number to odd number. But in some sense we could see that this is a partition of the set X into even numbers and odd numbers. Then we obtain the symmetric groups arising from each of the partitions and after which the product of these symmetric groups is taken i.e. $S_AS_B = \{ab : a \in A, b \in B\}$. Clearly this contains the identity element. This product turns out to be the permutation group $\mathcal{B}_n$ of the symmetric group, S_n on the given nonempty set. Therefore given a set $X = \{a_1,a_2,\ldots,a_n\}$, X can be partitioned into two disjoint subset such that

$$X = A \cup B$$

Where subset A is the set containing the elements indexed with the even numbers and subset B the set containing the elements indexed with odd numbers. Therefore, we have the following two cases in obtaining the elements of the subset A and subset B for a given n.
Case 1: For n an odd positive integer;

$$A = \{a_2,a_4,\ldots,a_{n-1}\} \text{ and } B = \{a_1,a_3,\ldots,a_n\}$$

Thus $o(A) = (n-1)/2$ and $o(B) = (n+1)/2$.
Case II: if n is an even integer;

$$A = \{a_2,a_4,\ldots,a_n\} \text{ and } B = \{a_1,a_3,\ldots,a_{n-1}\}$$

Thus $o(A) = o(B) = n/2$
Therefore, the collection of all the permutations on each of the subset is the corresponding symmetric group, donated by S_A and S_B for the subsets A and B respectively. Clearly,

$$S_A \cap S_B = (a_1) \quad \text{(The identity permutation)}.$$

Hence, since S_A and S_B are disjoint up to the identity permutation and by our product, we have that,

$$o(S_A S_B) = ((n+1)/2)! \times ((n-1)/2)!, \quad \text{for } n \text{ an odd number},$$

and

$$o(S_A S_B) = ((n/2)!)^2, \quad \text{for } n \text{ an even number}.$$

For example, if $n = 5$ clearly, $o(A) = (5-1)/2 = 2$ and $o(B) = (5+1)/2 = 3$ and the order of $S_A S_B$ is 12. That is, in cycle form
$S_A S_B = \{(a_1), (a_3a_5), (a_1a_5), (a_1a_3), (a_1a_5a_3), (a_1a_3a_5), (a_2a_4), (a_2a_4)(a_3a_5),$
$(a_2a_4)(a_1a_5), (a_2a_4)(a_1a_3), (a_2a_4)(a_1a_5a_3), (a_2a_4)(a_1a_3a_5)\}$
where $S_A = \{(a_1), (a_2a_4)\}$ and $S_B = \{(a_1), (a_3a_5), (a_1a_5), (a_1a_3), (a_1a_5a_3),$
$(a_1a_3a_5)\}$. Clearly, $S_A S_B$ is equivalent to $\mathcal{B}_5$ as presented above. Should $a_i, i \in \{1, 2, \ldots, 5\}$ be replace by the set $\{1, 2, \ldots, 5\}$ then $S_A S_B$ is equal to $\mathcal{B}_5$. Therefore in replacing the set $\{a_1, \ldots, a_n\}$ with the set $\{1, 2, \ldots, n\}$, $\mathcal{B}_n$ can be represented as product of the two symmetric groups obtained from the partition of the set $\{1, 2, \ldots, n\}$ into two subsets of even and odd numbers (i.e., disjoint subsets) respectively.

5 Conclusion

In this paper we studied a particular class of permutation on a given finite set. Our study considered permutations that map even integer to even integer and likewise odd integer to odd integer. It was established that this collection of permutations, $\mathcal{B}_n$ is a permutation group. In particular, it was observe that this group, $\mathcal{B}_n$ is isomorphic to $S_{(n+1)/2} \times S_{(n-1)/2}$ if n is odd and to $S_{n/2} \times S_{n/2}$ if n is even.
Our next task which is ongoing, is intended to investigate more on the algebraic and combinatoric properties of this group and then investigating its subclasses using some activation functions[17].

Acknowledgment

The authors are most grateful for the constructive and instructive comments of the reviewer which greatly improved this work.

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