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ρ-T-Stability of some fixed point iterations with respect to operators satisfying contractive conditions of integral type in modular function spaces

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Abstract

Recently, Okeke and Khan [1] extended the concepts of T-stability, almost T-stability and summably almost T-stability to modular function spaces and proved some fixed point theorems in the framework of modular function spaces. The purpose of the present paper is to continue in this research direction. We prove that some fixed point iterations is ρ-T-stable with respect to a generalized operator satisfying contractive conditions of integral type in modular function spaces.

Keywords: Mann iterations, multistep iterations, Noor iterations, contractive condition of integral type, modular function spaces.
MSC2010:47H09; 47H10; 49M05; 54H25

1 Introduction

Inview of the applications and consequences of approximating fixed points of an operator, several authors have developed and studied various iterative processes to approximate the fixed point of certain operators (see e.g. [1], [2], [3], [4], [5], [6]). Therefore, it is of great interest to know whether these methods are numerically stable or not (see, e.g. [3], [1]). Several mathematicians have studied the stability of various fixed point iterative processes (see, e.g. [7], [2], [8], [9], [10], [11], [12], [13]). It is our purpose in this paper to extend the concept of T-stability of a fixed point iterative process to modular function spaces. We prove that the Mann, Ishikawa, Noor, multistep, Kirk and Picard-Mann hybrid iterative processes are ρ-T-stable with respect to a generalized operator satisfying contractive conditions of integral type in modular function spaces.

The theory of modular spaces was initiated in 1950 by Nakano [14] in connection with the theory of ordered spaces which was further generalized by Musielak and Orlicz [15]. Modular function spaces are natural generalizations of both function and sequence variants of several important, from application perspective, spaces like Musielak-Orlicz, Orlicz, Lorentz, Orlicz-Lorentz, Kothe, Lebesgue, Calderon-Lozanovskii spaces and several others. Interest in quasi-nonexpansive mappings in modular function spaces stems mainly in the richness of structure of modular function spaces, that - besides being Banach spaces (or F-spaces in a more general settings)- are equipped with modular equivalents of norm or metric notions and also equipped with almost everywhere convergence and convergence in submeasure. It is known that modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts, particulary in applications to integral operators, approximation and fixed point results. Moreover, there are certain fixed point results that can be proved only using the apparatus of modular function spaces. Hence, fixed point theory results in modular function spaces, in this perspective, should be considered as complementary to the fixed point theory in normed and metric spaces (see, e.g. [16]).

Several authors have proved very interesting fixed points results in the framework of modular function spaces, (see, e.g. [17], [18], [12]). Recently, Khan and Abbas initiated the study of approximating fixed points of multivalued nonlinear mappings in the framework of modular function spaces [19]. A very recent work was done by Khan, Abbas and Ali [18]. They approximated the fixed points of $\rho-$quasi-nonexpansive multivalued mappings in modular function spaces using a three step iterative process, where ρ satisfies the so-called Δ_2-condition. Their results improves and generalizes the results of Khan and Abbas [19].

Recently, Okeke *et al.* [12] proved some convergence and stability results for $\rho-$quasi-nonexpansive mappings using the Picard-Krasnoselskii hybrid iterative process. Their results improve, extend and generalize several known results, including the recent results of Khan *et al.* [18], in the sense that the restriction that ρ satisfies the so-called Δ_2- condition in [18] is removed. Moreover, it is known (see, [5]) that the Picard-Krasnoselskii hybrid iterative process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes. More recently, Okeke and Khan [1] extended the recent results of Okeke *et al.* [12] to the class of multivalued ρ-quasi-contractive mappings in modular function spaces. They iteratively approximated fixed points of this class of nonlinear multivalued mappings in modular function spaces. Moreover, they extended the concepts of T-stability, almost T-stability and summably almost T-stability to modular function spaces and proved some fixed point theorems in the framework of modular function spaces. The purpose of this paper is to continue in this research direction. We prove that some fixed point iterations is ρ-T-stable with respect to a generalized operator satisfying contractive conditions of integral type in modular function spaces.

2 Preliminaries

In this study, we let Ω denote a nonempty set and Σ a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E\cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ-finite measure space). By 1_A, we denote the characteristic function of the set A in Ω. By ε we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we denote the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \varepsilon$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for each $\omega \in \Omega$.

Definition 2.1. *Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*
(1) $\rho(0) = 0$;
(2) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(3) ρ is orthogonally subadditive, i.e., $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;
(4) ρ has Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$,

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where $f \in \mathcal{M}_\infty$;
(5) ρ *is order continuous in* ε, *i.e.,* $g_n \in \varepsilon$ *and* $|g_n(\omega)| \downarrow 0$ *implies* $\rho(g_n) \downarrow 0$.

A set $A \in \Sigma$ is said to be $\rho-$null if $\rho(g1_A) = 0$ for every $g \in \varepsilon$. A property $p(\omega)$ is said to hold $\rho-$almost everywhere ($\rho-$a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is $\rho-$null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a $\rho-$null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho\text{-}a.e.\},$$

where $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists, we shall write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

The following definitions were given in [19].

Definition 2.2. *Let* ρ *be a regular function pseudomodular;*
(a) we say that ρ *is a regular convex function modular if* $\rho(f) = 0$ *implies* $f = 0$ $\rho-a.e.$
(b) we say that ρ *is a regular convex function semimodular if* $\rho(\alpha f) = 0$ *for every* $\alpha > 0$ *implies* $f = 0$ $\rho-a.e.$

It is known (see, e.g. [20]) that ρ *satisfies the following properties:*
(1) $\rho(0) = 0$ *iff* $f = 0$ $\rho-a.e.$
(2) $\rho(\alpha f) = \rho(f)$ *for every scalar* α *with* $|\alpha| = 1$ *and* $f \in \mathcal{M}$.
(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ *if* $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ *and* $f, g \in \mathcal{M}$.

ρ is called a convex modular if, in addition, the following property is satisfied:
$(3')$ $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.

The class of all nonzero regular convex function modulars on Ω is denoted by $\Re$.

Definition 2.3. *The convex function modular* ρ *defines the modular function space* L_ρ *as*

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq \alpha\right\}.$$

In the case ρ is convex modular,

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

defines a norm on the modular space L_ρ, and it is called the Luxemburg norm.

Lemma 2.4. *[20] Let* $\rho \in \Re$. *Defining* $L^0_\rho = \{f \in L_\rho; \rho(f, .)$ *is order continuous*$\}$ *and* $E_\rho = \{f \in L_\rho; \lambda f \in L^0_\rho$ *for every* $\lambda > 0\}$, *we have*
(i) $L_\rho \supset L^0_\rho \supset E_\rho$;
(ii) E_ρ *has the Lebesgue property, i.e.,* $\rho(\alpha f, D_k) \to 0$, *for* $\alpha > 0$, $f \in E_\rho$ *and* $D_k \downarrow \emptyset$;
(iii) E_ρ *is the closure of* ε *(in the sense of* $\|.\|_\rho$*).*

The following uniform convexity type properties of ρ can be found in [20].

Definition 2.5. *Let* ρ *be a nonzero regular convex function modular defined on* Ω.
(i) Let $r > 0$, $\epsilon > 0$. *Define*

$$D_1(r, \epsilon) = \{(f, g) : f, g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho(f - g) \geq \epsilon r\}.$$

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Let

$$\delta_1(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_1(r,\epsilon)\right\} \quad \text{if } D_1(r,\epsilon) \neq \emptyset,$$

and $\delta_1(r,\epsilon) = 1$ *if* $D_1(r,\epsilon) = \emptyset$. *We say that* ρ *satisfies* $(UC1)$ *if for every* $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. *Observe that for every* $r > 0$, $D_1(r,\epsilon) \neq \emptyset$, *for* $\epsilon > 0$ *small enough.*
(ii) We say that ρ *satisfies* $(UUC1)$ *if for every* $s \geq 0$, $\epsilon > 0$, *there exists* $\eta_1(s,\epsilon) > 0$ *depending only on* s *and* ϵ *such that* $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ *for any* $r > s$.
(iii) Let $r > 0$, $\epsilon > 0$. *Define*

$$D_2(r,\epsilon) = \left\{(f,g) : f,g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho\left(\frac{f-g}{2}\right) \geq \epsilon r\right\}.$$

Let

$$\delta_2(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_2(r,\epsilon)\right\}, \quad \text{if } \ D_2(r,\epsilon) \neq \emptyset,$$

and $\delta_2(r,\epsilon) = 1$ *if* $D_2(r,\epsilon) = \emptyset$. *We say that* ρ *satisfies* $(UC2)$ *if for every* $r > 0$, $\epsilon > 0$, $\delta_2(r,\epsilon) > 0$. *Observe that for every* $r > 0$, $D_2(r,\epsilon) \neq \emptyset$, *for* $\epsilon > 0$ *small enough.*
(iv) We say that ρ *satisfies* $(UUC2)$ *if for every* $s \geq 0$, $\epsilon > 0$, *there exists* $\eta_2(s,\epsilon) > 0$ *depending only on* s *and* ϵ *such that* $\delta_2(r,\epsilon) > \eta_2(s,\epsilon) > 0$ *for any* $r > s$.
(v) We say that ρ *is strictly convex* (SC), *if for every* $f,g \in L_\rho$ *such that* $\rho(f) = \rho(g)$ *and* $\rho\left(\frac{f+g}{2}\right) = \frac{\rho(f)+\rho(g)}{2}$, *there holds* $f = g$.

Proposition 2.6. *([20]) The following conditions characterize relationship between the above defined notions:*
(i) $(UUCi) \implies (UCi)$ *for* $i = 1,2$.
(ii) $\delta_1(r,\epsilon) \leq \delta_2(r,\epsilon)$.
(iii) $(UC1) \implies (UC2)$.
(iv) $(UUC1) \implies (UUC2)$.
(v) If ρ *is homogeneous (e.g. it is a norm), then all the conditions* $(UC1)$, $(UC2)$, $(UUC1)$, $(UUC2)$ *are equivalent and* $\delta_1(r,2\epsilon) = \delta_1(1,2\epsilon) = \delta_2(1,\epsilon) = \delta_2(r,\epsilon)$.

Definition 2.7. *([20]) A nonzero regular convex function modular* ρ *is said to satisfy the* Δ_2-*condition, if* $\sup_{n\geq 1}\rho(2f_n, D_k) \to 0$ *as* $k \to \infty$ *whenever* $\{D_k\}$ *decreases to* $\emptyset$ *and* $\sup_{n\geq 1}\rho(f_n, D_k) \to 0$ *as* $k \to \infty$.

Definition 2.8. *([20]) A function modular is said to satisfy the* Δ_2-*type condition, if there exists* $K > 0$ *such that for any* $f \in L_\rho$, *we have* $\rho(2f) \leq K\rho(f)$.

In general, Δ_2-condition and Δ_2-type condition are not equivalent, even though it is easy to see that Δ_2-type condition implies Δ_2-condition on the modular space L_ρ.

Definition 2.9. *([20]) Let* L_ρ *be a modular space. The sequence* $\{f_n\} \subset L_\rho$ *is called:*
(1) ρ*-convergent to* $f \in L_\rho$ *if* $\rho(f_n - f) \to 0$ *as* $n \to \infty$;
(2) ρ*-Cauchy, if* $\rho(f_n - f_m) \to 0$ *as* n *and* $m \to \infty$.
(3) The modular space L_ρ *is called* ρ*-complete if any* ρ*-Cauchy sequence is* ρ*-convergent.*
(4) A function $f \in L_\rho$ *is called a fixed point of* $T : L_\rho \to L_\rho$ *if* $f \in Tf$. *The set of all fixed points of* T *will be denoted by* $F_\rho(T)$.

Observe that $\rho-$convergence does not imply $\rho-$Cauchy since ρ does not satisfy the triangle inequality. In fact, one can easily show that this will happen if and only if ρ satisfies the Δ_2-condition.

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Definition 2.10. *([20]) A subset $D \subset L_\rho$ is called:*
(1) $\rho-$closed if the $\rho-$limit of a $\rho-$convergent sequence of D always belongs to D;
(2) $\rho-$a.e. closed if the $\rho-$a.e. limit of a $\rho-$a.e. convergent sequence of D always belongs to D;
(3) $\rho-$compact if every sequence in D has a $\rho-$convergent subsequence in D;
(4) $\rho-$a.e. compact if every sequence in D has a $\rho-$a.e. convergent subsequence in D;
(5) $\rho-$bounded if

$$diam_\rho(D) = \sup\{\rho(f-g) : f, g \in D\} < \infty.$$

A set $D \subset L_\rho$ is called $\rho-$proximinal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f-g) = \text{dist}_\rho(f, D)$. We shall denote the family of nonempty $\rho-$bounded $\rho-$proximinal subsets of D by $P_\rho(D)$, the family of nonempty $\rho-$closed $\rho-$bounded subsets of D by $C_\rho(D)$ and the family of $\rho-$compact subsets of D by $K_\rho(D)$.

A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \geq a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \leq b$ for every $n \in \mathbb{N}$.

The following contractive condition of integral type was defined by Olatinwo [13] following the results of Branciary [21] and Rhoades [22].

Definition 2.11. *([13]) For a selfmapping $T : E \to E$, there exist a real number $k \in [0,1)$ and monotone increasing functions $\nu, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\forall x, y \in E$, we have*

$$\int_0^{d(Tx,Ty)} \varphi(t)d\nu(t) \leq \psi\left(\int_0^{d(x,Tx)} \varphi(t)d\nu(t)\right) + k\int_0^{d(x,y)} \varphi(t)d\nu(t), \tag{2.1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each

$$\epsilon > 0, \quad \int_0^\epsilon \varphi(t)d\nu(t) > 0.$$

He showed that contractive condition (2.1) includes those defined by Imoru and Olatinwo [27] and some others in literature as special cases.

Mappings satisfying contractive condition (2.1) is of interest since it generalizes both the contraction maps and mappings satisfying contractive condition of integral type studied by Branciary [21]. Motivated by the results of Olatinwo [13] above, we now extend the definition above to modular function spaces as follows:

Definition 2.12. *Let ρ satisfy $(UUC1)$ and D a nonempty $\rho-$closed, $\rho-$bounded and convex subset of L_ρ. Let $T : D \to D$ be a selfmapping there exists a real number $k \in [0,1)$ and monotone increasing functions $\nu, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\forall x, y \in E$, we have*

$$\int_0^{\rho(Tx-Ty)} \varphi(t)d\nu(t) \leq \psi\left(\int_0^{\rho(x-Tx)} \varphi(t)d\nu(t)\right) + k\int_0^{\rho(x-y)} \varphi(t)d\nu(t), \tag{2.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each

$$\epsilon > 0, \quad \int_0^\epsilon \varphi(t)d\nu(t) > 0.$$

Next we define the concept of ρ-T-stablility of a fixed point iteration $\{f_n\}$ in modular function spaces.

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Let ρ satisfy $(UUC1)$ and D a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be a mapping with $F_\rho(T) \neq \emptyset$. Suppose that $\{f_n\}_{n=0}^{\infty}$ is a fixed point iterative process, i.e. a sequence $\{f_n\}_{n=0}^{\infty}$ defined by $f_0 \in D$ and

$$f_{n+1} = F(T, f_n), \qquad n = 0, 1, 2, 3, \cdots, \tag{2.3}$$

where F is a given function.

Several fixed point iterations exist in literature. For instance, Mann iteration, with $F(T, f_n) = (1 - \alpha_n)f_n + \alpha_n T f_n$, where $\{\alpha_n\} \subset [0, 1]$ such that $\{\alpha_n\}$ is bounded away from both 0 and 1. The Ishikawa iteration, with $F(T, f_n) = (1 - \alpha_n)f_n + \alpha_n T[(1 - \beta_n)f_n + \beta_n T f_n]$, such that $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ are both bounded away from both 0 and 1.

Let $\{f_n\}_{n=0}^{\infty}$ converge strongly to some $p \in F_\rho(T)$. In practice, we compute $\{f_n\}_{n=0}^{\infty}$ as follows:
(i) Choose the initial guess (approximation) $f_0 \in D$;
(ii) Compute $f_1 = F(T, f_0)$. However, as a result of various errors that occur during computations (numerical approximations of functions, rounding errors, derivatives, integration, etc), we do not obtain the exact value of f_1, but a different one, say h_1, which is close enough to f_1, this means that $h_1 \approx f_1$;
(iii) Therefore, during the computation of $f_2 = F(T, f_1)$ we have

$$f_2 = F(T, h_1). \tag{2.4}$$

This means that instead of the theoretical value of f_2, we expect another value h_2 will be obtained, and h_2 being close enough to f_2, i.e. $h_2 \approx f_2$, and so on.

Continuing this process, we see that instead of the theoretical sequence $\{f_n\}_{n=0}^{\infty}$ defined by the fixed point iteration (2.3), we obtain practically an approximate sequence $\{h_n\}_{n=0}^{\infty}$.

The fixed point iteration (2.3) is considered to be numerically stable if and only if for h_n close enough to f_n at each stage, we have that the approximate $\{h_n\}_{n=0}^{\infty}$ still converges to the fixed point p of $F_\rho(T)$.

Next, we give the following definition, which is the analogue of the concept of $T-$stability introduced by Harder and Hicks (see, [23], [24], [25]) in modular function spaces.

Definition 2.13. *Let ρ satisfy $(UUC1)$ and D a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be a mapping with $F_\rho(T) \neq \emptyset$. Suppose that the fixed point iterative process (2.3) converges to a fixed point p of T. Let $\{h_n\}_{n=0}^{\infty}$ be an arbitrary sequence in D and set*

$$\varepsilon_n = \rho(h_{n+1} - F(T, h_n)), \qquad n = 0, 1, 2, 3, \cdots \tag{2.5}$$

The fixed point iterative process (2.3) is said to be ρ-T-stable or ρ-stable or ρ-stable with respect to T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \implies \lim_{n\to\infty} h_n = p. \tag{2.6}$$

The following examples was presented by Razani *et al.* [26].

Example 2.14. *Let $(X, \|.\|)$ be a norm space, then $\|.\|$ is a modular. But the converse is not rue.*

Example 2.15. *Let $(X, \|.\|)$ be a norm space. For any $k \geq 1$, $\|.\|^k$ is a modular on X.*

The following example of a mapping which is ρ-nonexpansive but it is not $\|.\|_\rho$-nonexpansive can be found in [20].

Example 2.16. *([20]) Let $X = (0, \infty)$ and $\sum$ be the σ-algebra of all Lebesgue measurable subsets of X. Let $\mathcal{P}$ denote the δ-ring of subsets of finite measure. Define a function modular by*

$$\rho(f) = \frac{1}{e^2} \int_0^\infty |f(x)|^{x+1} dm(x). \tag{2.7}$$

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Let B be the set of all measurable functions $f:(0,\infty)\to\mathbb{R}$ such that $0\le f(x)\le\frac{1}{2}$. Consider the map

$$T(f)(x)=\begin{cases} f(x-1), & \text{for } x\ge 1,\\ 0, & \text{for } x\in[0,1].\end{cases} \tag{2.8}$$

Clearly, we have $T(B)\subset B$. For every $f,g\in B$ and $\lambda\le 1$, we have

$$\rho(\lambda(T(f)-T(g)))\le\lambda\rho(\lambda(f-g)), \tag{2.9}$$

which implies that T is ρ-nonexpansive. On the other hand, if we take $f=1_{[0,1]}$, then

$$\|T(f)\|_\rho>e\ge\|f\|_\rho,$$

which clearly implies that T is not $\|.\|_\rho$-nonexpansive. Note that T is linear.

The following lemma will be useful in this study.

Lemma 2.17. *([3]) If δ is a real number such that $0\le\delta<1$, and $\{\epsilon'_n\}_{n=0}^\infty$ is a sequence of positive numbers such that $\lim_{n\to\infty}\epsilon'_n=0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^\infty$ satisfying*

$$u_{n+1}\le\delta u_n+\epsilon'_n,\quad n=0,1,2,\cdots,$$

we have $\lim_{n\to\infty}u_n=0$.

3 Main Results

We begin this section with the following useful lemma, which is an analogue of ([13], Lemma 2.2) in modular function spaces.

Lemma 3.1. *Let ρ satisfy $(UUC1)$ and D be a nonempty $\rho-$closed, $\rho-$bounded and convex subset of a ρ-complete modular space L_ρ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each $\epsilon>0$, $\int_0^\epsilon\varphi(t)d\nu(t)>0$. Suppose that $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty\subset D$ and $\{a_n\}_{n=0}^\infty\subset(0,1)$ is bounded away both from 0 and 1 are sequences such that*

$$\left|\rho(u_n-v_n)-\int_0^{\rho(u_n-v_n)}\varphi(t)d\nu(t)\right|\le a_n, \tag{3.1}$$

with $\lim_{n\to\infty}a_n=0$. Then,

$$\rho(u_n-v_n)-a_n\le\int_0^{\rho(u_n-v_n)}\varphi(t)d\nu(t)\le\rho(u_n-v_n)+a_n. \tag{3.2}$$

Proof. Using (3.1), we have

$$\left|\rho(u_n-v_n)-\int_0^{\rho(u_n-v_n)}\varphi(t)d\nu(t)\right|\le a_n. \tag{3.3}$$

Relation (3.3) holds if and only if

$$-a_n\le\left|\rho(u_n-v_n)-\int_0^{\rho(u_n-v_n)}\varphi(t)d\nu(t)\right|\le a_n, \tag{3.4}$$

Relation (3.4) holds if and only if

$$\rho(u_n-v_n)-a_n\le\int_0^{\rho(u_n-v_n)}\varphi(t)d\nu(t)\le\rho(u_n-v_n)+a_n. \tag{3.5}$$

The proof of Lemma 3.1 is completed. □ □

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Next, we prove the following ρ-T-stability results for various fixed point iterative processes in modular function spaces.

Theorem 3.2. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space and $T : D \to D$ a selfmapping of D satisfying contractive condition (2.2). Suppose $p \in F_\rho(T) \neq \emptyset$. For arbitrary $f_0 \in D$, let $\{f_n\}_{n=0}^{\infty}$ be the Picard iteration process defined by*

$$f_{n+1} = Tf_n, \qquad n = 0, 1, 2, \cdots \tag{3.6}$$

Let $\nu, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be monotone increasing functions such that $\psi(0) = 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)d\nu(t) > 0$. Then the Picard iteration process is ρ-T-stable.

Proof. Let $\{h_n\}_{n=0}^{\infty} \subset D$ and $\varepsilon_n = \rho(h_{n+1} - Th_n)$. Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. We want to show that $\lim_{n\to\infty} h_n = p$. Using (2.2), (3.6), Lemma 3.1 and the convexity of ρ. Let $\{a_n\}_{n=0}^{\infty} \subset (0, 1)$ be bounded away from both 0 and 1. Then by Lemma 3.1, we have

$$\begin{array}{rcl} \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) & \leq & \rho(h_{n+1} - p) + a_n \\ & \leq & \psi\left(\int_0^{\rho(p-Tp)} \varphi(t)d\nu(t)\right) + k\int_0^{\rho(p-h_n)} \varphi(t)d\nu(t) + \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n \\ & = & k\int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n. \end{array} \tag{3.7}$$

Using Lemma 2.2 in (3.7), we see that $\delta = k \in [0, 1)$, $u_n = \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t)$ and $\epsilon'_n = \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n$, with

$$\lim_{n\to\infty} \epsilon'_n = \lim_{n\to\infty} \left(\int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n\right) = 0,$$

by Lemma 2.2 and the fact that $\int_0^\epsilon \varphi(t)d\nu(t) > 0$ for all $\epsilon > 0$, we obtain $\lim_{n\to\infty} \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) = 0$ which means that $\lim_{n\to\infty} \rho(h_n - p) = 0$, i.e.

$$\lim_{n\to\infty} h_n = p.$$

Conversely, suppose $\lim_{n\to\infty} h_n = p$. By contractive condition (2.2), Lemma 3.1 and the convexity of ρ, we have

$$\begin{array}{rcl} \int_0^{\varepsilon_n} \varphi(t)d\nu(t) & = & \int_0^{\rho(h_{n+1}-Th_n)} \varphi(t)d\nu(t) \\ & \leq & \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) + k\int_0^{\rho(p-h_n)} \varphi(t)d\nu(t) + 3a_n \longrightarrow 0 \text{ as } n \to \infty. \end{array} \tag{3.8}$$

Using the condition on φ, we have

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

Therefore, the Picard iteration $\{f_n\}_{n=0}^{\infty}$ is ρ-T-stable. The proof of Theorem 3.1 is completed. □ □

Theorem 3.3. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space and $T : D \to D$ a selfmapping of D satisfying contractive condition (2.2). Suppose $p \in F_\rho(T) \neq \emptyset$. For arbitrary $f_0 \in D$, let $\{f_n\}_{n=0}^{\infty}$ be the Mann iteration process defined by*

$$f_{n+1} = (1 - \alpha_n)f_n + \alpha_n Tf_n, \qquad n = 0, 1, 2, \cdots \tag{3.9}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$ is bounded away from both 0 and 1. Let $\nu, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be monotone increasing functions such that $\psi(0) = 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)d\nu(t) > 0$. Then the Mann iteration process is ρ-T-stable.

Proof. Let $\{h_n\}_{n=0}^{\infty} \subset D$, $\varepsilon_n = \rho(h_{n+1} - (1-\alpha_n)h_n - \alpha_n Th_n)$, $n = 0, 1, 2, \cdots$ Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. We want to show that $\lim_{n\to\infty} h_n = p$. Using contractive condition (2.2), relation (3.9), Lemma 3.1 and the convexity of ρ. Let $\{a_n\}_{n=0}^{\infty} \subset (0,1)$ be bounded away both from 0 and 1, then by Lemma 3.1, we have

$$\begin{aligned}
\int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) &\leq [\rho(h_{n+1} - (1-\alpha_n)h_n - \alpha_n Th_n) - a_n] + (1-\alpha_n)[\rho(h_n - p) - a_n] + \\
&\quad \alpha_n[\rho(Tp - Th_n) - a_n] + 3a_n \\
&\leq [1 - (1-k)\alpha_n] \int_0^{\rho(h_n - p)} \varphi(t)d\nu(t) + \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n \\
&\leq [1 - (1-k)\alpha] \int_0^{\rho(h_n - p)} \varphi(t)d\nu(t) + \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n.
\end{aligned} \tag{3.10}$$

Hence, we see that from Lemma 2.2, $0 \leq \delta = (1 - (1-k)\alpha) < 1$, $u_n = \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t)$ and $\epsilon'_n = \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n$ with

$$\lim_{n\to\infty} \epsilon'_n = \lim_{n\to\infty} \left(\int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n \right) = 0.$$

Therefore by Lemma 2.2 and the fact that $\int_0^{\epsilon} \varphi(t)d\nu(t) > 0$, for each $\epsilon > 0$, we have that

$$\lim_{n\to\infty} \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) = 0.$$

Hence, we have $\lim_{n\to\infty} \rho(h_n - p) = 0$, this means that

$$\lim_{n\to\infty} h_n = p.$$

Conversely, let $\lim_{n\to\infty} h_n = p$. Using contractive condition (2.2), Lemma 3.1 and the convexity of ρ, we obtain

$$\begin{aligned}
\int_0^{\varepsilon_n} \varphi(t)d\nu(t) &= \int_0^{\rho(h_{n+1}-(1-\alpha_n)h_n-\alpha_n Th_n)} \varphi(t)d\nu(t) \\
&\leq \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) + [1-(1-k)\alpha] \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + 3a_n \longrightarrow 0,
\end{aligned} \tag{3.11}$$

as $n \to \infty$. The proof of Theorem 3.2 is completed. □ □

Remark 3.4. *Theorem 3.2 unifies and extends several known results from metric and normed linear spaces to modular function spaces, including the results of [8], [7], [2], [9], [27].*

Theorem 3.5. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space and $T : D \to D$ a selfmapping of D satisfying contractive condition (2.2). Suppose $p \in F_\rho(T) \neq \emptyset$. For arbitrary $f_0 \in D$, let $\{f_n\}_{n=0}^{\infty}$ be the Kirk iterative sequence defined by*

$$f_{n+1} = \sum_{i=0}^{k} \alpha_i T^i f_n, \quad n \geq 0,\ \alpha_i \geq 0, \quad \alpha_1 > 0, \quad \sum_{i=0}^{k} \alpha_i = 1, \tag{3.12}$$

where $\{\alpha_i\}_{i=0}^{k}$ is bounded away from both 0 and 1. Let $\nu, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be monotone increasing functions such that $\psi(0) = 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)d\nu(t) > 0$. Then the iteration process (3.12) is ρ-T-stable.

Proof. Let $\{h_n\}_{n=0}^{\infty} \subset D$ and $\varepsilon_n = \rho(h_{n+1} - Th_n)$. Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. We want to show that $\lim_{n\to\infty} h_n = p$. Using (2.2), (3.12), Lemma 3.1 and the convexity of ρ. Let $\{a_n\}_{n=0}^{\infty} \subset (0,1)$ be bounded away from both 0 and 1. Then by Lemma 3.1, we have

$$\begin{aligned}\int_0^{\rho(h_{n+1}-p)}\varphi(t)d\nu(t) &\leq \rho(h_{n+1}-p)+a_n\\ &\leq \psi\left(\int_0^{\rho(p-Tp)}\varphi(t)d\nu(t)\right)+\left(\sum_{i=0}^k\alpha_ik^i\right)\int_0^{\rho(p-h_n)}\varphi(t)d\nu(t)+\\ &\quad \int_0^{\varepsilon_n}\varphi(t)d\nu(t)+3a_n\\ &= \left(\sum_{i=0}^k\alpha_ik^i\right)\int_0^{\rho(h_n-p)}\varphi(t)d\nu(t)+\int_0^{\varepsilon_n}\varphi(t)d\nu(t)+3a_n. \end{aligned}\tag{3.13}$$

Using Lemma 2.2 in (3.13), we see that $\delta=\left(\sum_{i=0}^k\alpha_ik^i\right)\in[0,1)$, $u_n=\int_0^{\rho(h_n-p)}\varphi(t)d\nu(t)$ and $\epsilon_n'=\int_0^{\varepsilon_n}\varphi(t)d\nu(t)+3a_n$, with

$$\lim_{n\to\infty}\epsilon_n'=\lim_{n\to\infty}\left(\int_0^{\varepsilon_n}\varphi(t)d\nu(t)+3a_n\right)=0,$$

by Lemma 2.2 and the fact that $\int_0^{\epsilon}\varphi(t)d\nu(t)>0$ for all $\epsilon>0$, we obtain $\lim_{n\to\infty}\int_0^{\rho(h_n-p)}\varphi(t)d\nu(t)=0$ which means that $\lim_{n\to\infty}\rho(h_n-p)=0$, i.e.

$$\lim_{n\to\infty}h_n=p.$$

Conversely, suppose $\lim_{n\to\infty}h_n=p$. By contractive condition (2.2), Lemma 3.1 and the convexity of ρ, we have

$$\begin{aligned}\int_0^{\varepsilon_n}\varphi(t)d\nu(t) &= \int_0^{\rho(h_{n+1}-\sum_{i=0}^{\infty}\alpha_iT^ih_n)}\varphi(t)d\nu(t)\\ &\leq \int_0^{\rho(h_{n+1}-p)}\varphi(t)d\nu(t)+\left(\sum_{i=0}^k\alpha_ik^i\right)\int_0^{\rho(p-h_n)}\varphi(t)d\nu(t)+3a_n\longrightarrow 0,\end{aligned}\tag{3.14}$$

as $n\to\infty$. Using the condition on φ, we have

$$\lim_{n\to\infty}\varepsilon_n=0.$$

Therefore, the fixed point iteration $\{f_n\}_{n=0}^{\infty}$ defined by (3.12) is ρ-T-stable. The proof of Theorem 3.3 is completed. □

□

Theorem 3.6. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space and $T:D\to D$ a selfmapping of D satisfying contractive condition (2.2). Suppose $p\in F_\rho(T)\neq\emptyset$. For arbitrary $f_0\in D$, let $\{f_n\}_{n=0}^{\infty}$ be the multistep iteration process defined by*

$$\begin{cases} f_{n+1}=(1-\alpha_n)f_n+\alpha_nTg_n^1\\ g_n^i=(1-\beta_n^i)f_n+\beta_n^iTg_n^{i+1}, \quad i=1,2,3,\cdots,j-2\\ g_n^{j-1}=(1-\beta_n^{j-1})f_n+\beta_n^{j-1}Tf_n, \quad j\geq 2,\end{cases}\tag{3.15}$$

where $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n^i\}_{n=0}^{\infty}$, $i=1,2,\cdots,j-1\subset[0,1]$ are appropriate sequences bounded away from both 0 and 1 such that $0<\beta^1<\beta_n^1$, $n=0,1,2,\cdots$. Let $\nu,\psi:\mathbb{R}^+\to\mathbb{R}^+$ be monotone increasing functions such that $\psi(0)=0$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\epsilon>0$, $\int_0^{\epsilon}\varphi(t)d\nu(t)>0$. Then the multistep iteration process is ρ-T-stable.

Proof. Let $\{h_n\}_{n=0}^{\infty}\subset D$ be an arbitrary sequence and

$$\varepsilon_n=\rho(h_{n+1}-(1-\alpha_n)h_n-\alpha_nTw_n^1),\quad n=0,1,2,\cdots$$

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where

$$\begin{cases} h_{n+1} = (1-\alpha_n)h_n + \alpha_n T w_n^1 \\ w_n^i = (1-\beta_n^i)h_n + \beta_n^i T w_n^{i+1}, \quad i = 1,2,3,\cdots, j-2 \\ w_n^{j-1} = (1-\beta_n^{j-1})h_n + \beta_n^{j-1} T h_n, \quad j \geq 2. \end{cases} \tag{3.16}$$

Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. We want to show that $\lim_{n\to\infty} h_n = p$. Using (2.2), (3.16), Lemma 3.1 and the convexity of ρ. Let $\{a_n\}_{n=0}^{\infty} \subset (0,1)$ be bounded away from both 0 and 1. Then by Lemma 3.1, we have

$$\begin{aligned} \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) &\leq \rho(h_{n+1}-p) + a_n \\ &\leq [\rho(h_{n+1} - (1-\alpha_n)h_n - \alpha_n T w_n^1) - a_n] + (1-\alpha_n)[\rho(h_n - p) - a_n] + \\ &\quad \alpha_n[\rho(Tp - Tw_n^1) - a_n] + 3a_n \\ &\leq \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + (1-\alpha_n)\int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + \psi\left(\int_0^{\rho(p-Tp)} \varphi(t)d\nu(t)\right) \\ &\quad + k\int_0^{\rho(p-w_n^1)} \varphi(t)d\nu(t) + 3a_n. \end{aligned} \tag{3.17}$$

We now obtain the following estimate:

$$\begin{aligned} \rho(w_n^1 - p) &\leq (1-\beta_n^1)\rho(h_n-p) + \beta_n^1\rho(Tw_n^2 - p) \\ &\leq [1-(1-k)\beta_n^1 - (1-k)k\beta_n^1\beta_n^2 - (1-k)k^2\beta_n^1\beta_n^2\beta_n^3 - \\ &\quad \cdots - (1-k)k^{j-2}\beta_n^1\beta_n^2\beta_n^3\cdots\beta_n^{j-1}]\rho(h_n-p) \\ &\leq (1-(1-k)\beta_n^1)\rho(h_n-p). \end{aligned} \tag{3.18}$$

Using (3.18) in (3.17), we have

$$\begin{aligned} \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) &\leq \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + (1-\alpha_n)\int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) \\ &\quad + k\alpha_n(1-(1-k)\beta_n^1)\int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + 3a_n. \end{aligned} \tag{3.19}$$

Hence,

$$\begin{aligned} \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) &\leq [(1-(1-k)\alpha_n - (1-k)k\alpha_n\beta_n^1]\int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + \\ &\quad \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n. \end{aligned} \tag{3.20}$$

Since $0 < \alpha < \alpha_n$ and $0 < \beta^1 < \beta_n^1$ for all $n \geq 1$, we have

$$\begin{aligned} \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) &\leq [1-(1-k)\alpha - (1-k)k\alpha\beta^1]\int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + \\ &\quad \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n. \end{aligned} \tag{3.21}$$

Using Lemma 2.2, we see that $0 \leq \delta = [1-(1-k)\alpha - (1-k)k\alpha\beta^1] < 1$, $u_n = \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t)$ and $\epsilon_n' = \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n$ with

$$\lim_{n\to\infty} \epsilon_n' = \lim_{n\to\infty}\left(\int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n\right) = 0.$$

Using the fact that $\int_0^{\epsilon} \varphi(t)d\nu(t) > 0$, for each $\epsilon > 0$. Then by Lemma 3.1, we have

$$\lim_{n\to\infty} \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) = 0.$$

This implies that $\lim_{n\to\infty} \rho(h_n - p) = 0$. This means that

$$\lim_{n\to\infty} h_n = p.$$

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Conversely, let $\lim_{n\to\infty} h_n = p$, then by contractive condition (2.2), Lemma 3.1 and the convexity of ρ, we have

$$\begin{aligned} \int_0^{\varepsilon_n} \varphi(t)d\nu(t) &= \int_0^{\rho(h_{n+1}-p+p-(1-\alpha_n)h_n-\alpha_n Tw_n^1)} \varphi(t)d\nu(t) \\ &\le \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) + \int_0^{\rho(p-(1-\alpha_n)h_n-\alpha_n Tw_n^1)} \varphi(t)d\nu(t) \\ &\le \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) + [1-(1-k)\alpha_n-(1-k)k\alpha_n\beta_n^1] \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) \\ &\quad +3a_n \\ &\le \int_0^{\rho(h_{n+1}-p)} \varphi(t)d\nu(t) + [1-(1-k)\alpha-(1-k)k\alpha\beta^1] \int_0^{\rho(h_n-p)} \varphi(t)d\nu(t) + \\ &\quad 3a_n \longrightarrow 0 \ \text{ as } n\to\infty. \end{aligned} \tag{3.22}$$

This means that

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

The proof of Theorem 3.4 is completed. □ □

Next, we state the following corollaries as a consequence of Theorem 3.4.

Corollary 3.7. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space and $T: D \to D$ a selfmapping of D satisfying contractive condition (2.2). Suppose $p \in F_\rho(T) \neq \emptyset$. For arbitrary $f_0 \in D$, let $\{f_n\}_{n=0}^\infty$ be the Noor iteration process defined by*

$$\begin{cases} f_{n+1} = (1-\alpha_n)f_n + \alpha_n T g_n \\ g_n = (1-\beta_n)f_n + \beta_n T z_n \\ z_n = (1-\gamma_n)f_n + \gamma_n T f_n, \qquad n \ge 0, \end{cases} \tag{3.23}$$

where $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty, \{\gamma_n\}_{n=0}^\infty \subset [0,1]$ are appropriate sequences bounded away from both 0 and 1. Let $\nu, \psi: \mathbb{R}^+ \to \mathbb{R}^+$ be monotone increasing functions such that $\psi(0) = 0$ and $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)d\nu(t) > 0$. Then the Noor iteration process is ρ-T-stable.

Corollary 3.8. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space and $T: D \to D$ a selfmapping of D satisfying contractive condition (2.2). Suppose $p \in F_\rho(T) \neq \emptyset$. For arbitrary $f_0 \in D$, let $\{f_n\}_{n=0}^\infty$ be the Ishikawa iteration process defined by*

$$\begin{cases} f_{n+1} = (1-\alpha_n)f_n + \alpha_n T g_n \\ g_n = (1-\beta_n)f_n + \beta_n T f_n \end{cases} \tag{3.24}$$

where $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty \subset [0,1]$ are appropriate sequences bounded away from both 0 and 1. Let $\nu, \psi: \mathbb{R}^+ \to \mathbb{R}^+$ be monotone increasing functions such that $\psi(0) = 0$ and $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)d\nu(t) > 0$. Then the Ishikawa iteration process is ρ-T-stable.

Next, we give the following nemerical example to support our results.

Example 3.9. *Let the real number system $\mathbb{R}$ be the space modulared as follows:*

$$\rho(f) = |f|^k, \qquad k \ge 1.$$

Let $D = [0,1]$. Define $T: [0,1] \to [0,1]$ by $Tf = \frac{f}{2}$. Let $\{f_n\}$ be the Picard-Mann hybrid iteration process defined by

$$\begin{cases} f_1 \in D, \\ f_{n+1} = T g_n \\ g_n = (1-\alpha_n)f_n + \alpha_n T f_n, \quad n \in \mathbb{N}. \end{cases} \tag{3.25}$$

Then $\{f_n\}$ is ρ-T-stable.

Clearly, D *is a nonempty* $\rho-$*compact,* $\rho-$*bounded and convex subset of* $L_\rho = \mathbb{R}$ *which satisfies* $UC1$ *condition. Moreover,* $\rho(f) = |f|^k$, $k \geq 1$ *is homogeneous and it is of degree* k*, hence by Proposition 2.1* $(UUC1)$ *hold. Clearly,* T *is a contraction mapping and* $F_\rho(T) = [0,1]$*. Suppose* $p = 0$, $\alpha_n = \frac{1}{2}$*, take* $h_n = \frac{1}{n}$*, for each* $n \geq 1$*. Hence,* $\lim_{n\to\infty} h_n = 0$*, we see that*

$$\begin{aligned}
\varepsilon_n &= \rho(h_{n+1} - F(T, h_n)) \\
&= \rho\left(h_{n+1} - \frac{h_n}{2}\right) \\
&= \left|h_{n+1} - \frac{h_n}{2}\right|^k \\
&= \left|h_{n+1} - \frac{1-\alpha_n}{2} h_n - \frac{\alpha_n}{2}\left(\frac{h_n}{2}\right)\right| \\
&= \left|\frac{1}{n+1} - \frac{1}{4n} - \frac{1}{8n}\right|.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

Clearly, $\lim_{n\to\infty} \varepsilon_n = 0$ *implies that* $\lim_{n\to\infty} h_n = 0$*. This means that the Picard-Mann hybrid iterative process (3.25) is* ρ*-*T*-stable.*

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of the paper.

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