International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

ON THE STABILITY OF MODIFIED CRANK-NICOLSON METHOD FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

OMOWO B.J[1*], ABHULIMEN C.E [2]

1. Department of Mathematics, Nasarawa State University, Keffi, Nigeria
2. Department of Mathematics, Ambrose Ali University, Ekpoma, Nigeria
*Corresponding author: johnsonomowo1@gmail.com

Article Info
Received: 17 December 2020 Revised: 27 January 2021
Accepted: 29 January 2021 Available online: 2 February 2021

Abstract

In this paper, we investigate the stability of Modified Crank-Nicolson method for solving one dimensional Parabolic equation knowing that finite difference solution of partial differential equations must satisfy the requirement of stability, if they are to be reasonably accurate. We examined it with the Von-Neumann scheme using the Fourier series methods and our result is confirmed with examples.

Keywords and Phrases: Partial differential equation, Finite difference method, Crank-Nicolson Method, Stability, Modified Crank-Nicolson Method, Parabolic Equations.
MSC2010:35A20, 35A35, 35B35, 35K05

1 Introduction

There are many boundary value problems which involve partial differential equations. Only a few of these equations can be solved by analytical methods. In most cases, we depend on the numerical solution of such partial differential equations. Finite difference method is the oldest and most direct approach to discretize partial differential equations; it is also the most commonly used method to solve Ordinary Differential equations and Partial differential equations in a bounded domain [1].
In this method, the derivative appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed to a difference equation which is solved by iterative procedures. The process is slow but gives good results for boundary value problems. [1] established an explicit finite difference scheme and applied it to solve one-dimensional heat equation using C program. Crank-Nicolson Method for solving parabolic partial differential equations is a prominent example finite difference method and it was developed by John Crank and Phyllis Nicolson in 1956. A practical method for numerical solution to partial differential equations of heat conduction type was considered by [2]. [3] modified the simple explicit scheme

http://ijmso.unilag.edu.ng/article/view/1151

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

and prove that it is much more stable than the simple explicit case, enabling larger time steps to be used. [4] considered the stability and accuracy of finite difference method for option pricing. Crank-Nicolson scheme, which is forward time central space (FTCS). According to Kreyszig (1993), the time derivative was replaced by forward difference in time because we have no information for negative t at the start. The freedom to experiment with any value of r (the gain parameter) is one of the reasons the Crank Nicolson scheme was chosen for this study, even though small values of r yield more accurate results. Because of this unconditional stability and ease of implementation in a computer no matter how small r becomes.
Recently, [5] established to approached methods to improve the θ-iterated Crank-Nicolson Method to second order accuracy.Also, [6] Modified the Crank-Nicolson scheme to get a 3-level Implicit finite difference scheme similar to the Crank-Nicolson scheme. The method utilizes one extra grid point at the lower level and the result is shown to be more accurate than the Crank-Nicolson scheme. [7] solved some parabolic differential equations using modified Crank-Nicolson scheme. They compared the results with the exact solutions. There are many exhaustive texts on this subject such as [8–11] to mention few.

2 FINITE DIFFERENCE METHODS

The general second order linear partial differential equation with two independent variables and one dependent variables is given by

$$A\frac{\partial^2 f}{\partial x^2} \;+\; B\frac{\partial^2 f}{\partial x \partial y} \;+\; C\frac{\partial^2 f}{\partial y^2} \;+\; D = 0 \tag{1}$$

Here, A, B, C, are functions of independent variables, and x, y and D can be a function of x, y, f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. It is important to note that for a partial differential equation to be parabolic, $B^2 - 4AC = 0$ is required. The one dimensional heat conduction equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \tag{2}$$

is a well known example of a parabolic partial differential equation. The solution of these equation is a temperature function $u(x,\ t)$ which is defined for values of x from 0 to l and for values of t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values satisfying the prescribed boundary conditions [12].

The equation (2) above together with initial condition

$$f(x, 0) = f(x), 0 < x < L$$

and the boundary condition

$$f(x, 0) = f(x, L) = 0$$

is an example of Parabolic partial differential equation and can be solved analytically but numerical method have proven exceedingly well for solving such or even more difficult equations. For this problem, different numerical methods such as finite difference method, finite element methods among others can be applied to solve the Partial differential equation. Here, we shall concentrate on finite difference methods for solving the equation (2) together with the initial and boundary conditions.

In this paper, we investigate the stability of modified Crank-Nicolson scheme for solving one dimensional parabolic partial differential equations and to verify our results, we solve problems on one dimensional heat equations. The method employ for the modification and derivation of the Modified Crank-Nicolson method is described in Section 2.2

2.1 CRANK-NICOLSON SCHEME

The classical Crank-Nicolson scheme is derived by finding the average of the $(j)^{th}$ and $(j+1)^{th}$ rows, as follows;

$$\frac{f_{i,\,j+1} - f_{i,\,j}}{k} = \frac{1}{2}\left[\frac{f_{i-1,\,j} - 2f_{i,\,j} + f_{i+1,\,j}}{h^2}\right] + \frac{1}{2}\left[\frac{f_{i-1,\,j+1} - 2f_{i,\,j+1} + f_{i+1,\,j+1}}{h^2}\right] \quad (3)$$

which gives

$$2(f_{i,\,j+1} - f_{i,\,j}) = \frac{k}{h^2}(f_{i-1,\,j} - 2f_{i,\,j} + f_{i+1,\,j} + f_{i-1,\,j+1} - 2f_{i,\,j+1} + f_{i+1,\,j+1})$$

let $\frac{k}{h^2} = r$ then

$$2(f_{i,\,j+1} - f_{i,\,j}) = r(f_{i-1,\,j} - 2f_{i,\,j} + f_{i+1,\,j} + f_{i-1,\,j+1} - 2f_{i,\,j+1} + f_{i+1,\,j+1})$$

$$rf_{i-1,\,j+1} + (2+2r)f_{i,\,j+1} - rf_{i+1,\,j+1} = rf_{i-1,\,j} + (2-2r)f_{i,\,j} + rf_{i+1,\,j}$$

which can be written as

$$2(1+r)f_{i,\,j+1} + r\,[f_{i-1,\,j+1} - f_{i+1,\,j+1}] = 2(1-r)f_{i,\,j} + r\,[f_{i-1,\,j} + f_{i+1,\,j}] \quad (4)$$

Equation (4) is the Crank-Nicolson method.

2.2 MODIFIED CRANK-NICOLSON METHOD

Here, we derive the Modified Crank-Nicolson scheme as follows; we replace the left hand sides of (3) by $\frac{f_{i,\,j} - f_{i,\,j-1}}{k}$ also, we replace $(j+1)^{th}$ row, on the second part of the right hand side (Implicit) with $(j-1)^{th}$ row. We then find the average of the $(j)^{th}$ and $(j-1)^{th}$ row. The finite difference approximation analogue to equation (2) is then given as

$$2f_{i,\,j} - 2f_{i,\,j-1} = rf_{i+1,\,j-1} - 2rf_{i,\,j-1} + rf_{i-1,\,j-1} + rf_{i+1,\,j} - 2rf_{i,\,j} + rf_{i-1,\,j}$$

where

$$-2rf_{i,\,j-1} + 2f_{i,\,j-1} + rf_{i+1,\,j-1} + rf_{i-1,\,j-1} = -rf_{i+1,\,j} + 2f_{i,\,j} + 2rf_{i,\,j} - rf_{i-1,\,j}$$

$$2(1+r)f_{i,\,j} - r(f_{i+1,\,j} + f_{i-1,\,j}) = 2(1-r)f_{i,\,j-1} + r(f_{i+1,\,j-1} + f_{i-1,\,j-1}) \quad (5)$$

here $r = \frac{k}{h^2}$ Equation(5) is the modified Crank-Nicolson scheme.
Equation (5) can be written in matrix form as $Uf = yb$, where the known concentrations are $b = f_{i,\,j-1}$ and the unknown concentrations are $f = f_{i,\,j}$ and U, y are tri-diagonal matrices of coefficients defined as

$$\begin{bmatrix} 2+2r & -r & 0 & \dots & 0 \\ -r & 2+2r & -r & \dots & 0 \\ 0 & -r & 2+2r & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -r \\ 0 & 0 & 0 & -r & 2+2r \end{bmatrix} \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \\ \vdots \\ f_{n,j} \end{bmatrix} = \begin{bmatrix} 2-2r & r & 0 & \dots & 0 \\ r & 2-2r & r & \dots & 0 \\ 0 & r & 2-2r & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & r \\ 0 & 0 & 0 & r & 2-2r \end{bmatrix} \begin{bmatrix} f_{1,j-1} \\ f_{2,j-1} \\ f_{3,j-1} \\ \vdots \\ f_{n,j-1} \end{bmatrix} \quad (6)$$

The convergence of this method follows from the condition (Tunner, 1994)
$r = \frac{\lambda \delta t}{\delta x^2} \leq \frac{1}{2}$, which implies $\delta t \leq \frac{(\delta x)^2}{2\lambda}$.
For sufficient accuracy we choose δx small, which makes δt very small by $\delta t \leq \frac{(\delta x)^2}{2\lambda}$. This will make the computation lengthy, as more time levels will be required to cover the region. A method that imposes no such restriction as $r = \frac{\lambda \delta t}{(\delta x)^2}$ was proposed by Crank and Nicolson in [2].

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

2.3 STABILITY ANALYSIS

The two fundamental sources of error are the truncation and round off error. The three fundamental factors that characterize a numerical scheme are consistency, stability and convergence.

Consistency: A finite difference of a partial differential equation is consistent, if the difference between the partial differential equation and finite difference equation vanishes as the interval and time step size approaches zero. Consistency deals with how well the finite difference equation approximates the partial differential equation and it is necessary condition for convergence.

Stability: For a stable numerical scheme, the errors from any source will not grow unboundedly with time.

Convergence: It means that the solution to a finite difference equation approaches the true solution to the partial differential equation as both grid and time step sizes are reduced. The necessary and sufficient conditions for convergent are consistency and stability.

These three factors that characterize a numerical scheme are linked together by Lax equivalence theorem [11] which state that given a well posed linear initial value problem and a consistent finite difference scheme, stability is a necessary and sufficient condition for convergence.
In general, a problem is said to be well posed if

1. A solution to the problem exists
2. The solution is unique when it exists
3. The solution depends continuously on the problem data

A finite difference approximation is said to be convergent if

$$\phi_{i,j} = ||\bar{f}_{i,j} - f_{i,j}|| \rightarrow 0, as\ h, k \rightarrow 0 \tag{7}$$

where $\bar{f}_{i,j}$ is the exact solution, $f_{i,j}$ is the numerical approximation and $\phi_{i,j}$ is the error. We shall test for this later, using one of the numerical examples.

Let $f(x, t)$ be the analytical solution of the partial differential equation and $f_{i,j}$ the solution of its finite difference approximation.
We define the error as

$$\phi_{i,j} = f(x_i, t_j) - f_{i,j}, \quad i, = 1, 2, 3, ...N, \ j = 1, 2, 3, ...M \tag{8}$$

we say $\phi_{i,j}$ satisfies the same difference equation $f_{i,j}$. Let the errors at the mesh point $(x_i, 0)$ at $t = 0$ be denoted by $\phi_i, \ i = 0, 1, 2, ...N$
We investigate the propagation of these errors as t increases by finding a solution of the finite difference equation in $\phi_{i,j}$ that reduces ϕ_i when $t = 0$. To this effect we apply the Von Newmann stability method. We use Fourier series method. We express the errors in terms of finite Fourier series, using the interval $[-L, L]$ (Interval of finite Fourier series function) given as

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n cos(\frac{n\pi}{l})x + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{l})x \tag{9}$$

the complex exponential form of the series is

$$f(x) = \sum_{n=0}^{\infty} A_n e^{z(\frac{n\pi}{l})x} \tag{10}$$

where $A_n, f(x), b_n, a_0$, and a_n are constants to be determined. Expressing ϕ_i in term of complex Fourier series we have

$$\phi_i = \sum_{n=0}^{\infty} A_n e^{z(\frac{n\pi}{l})x_i} \quad (z = \sqrt{-1}) \tag{11}$$

using the domain

$$D = \{(x,t)|0 \leq x \leq l,\ 0 \leq t \leq T\}$$

Let the mesh point (x_i, t_j) assume the form (ih, jk) where h and k are the mesh sizes, therefore;

$$\phi_i = \sum_{n=0}^{\infty} A_n e^{z\beta_n ih} \tag{12}$$

where $\beta_0 = \frac{n\pi}{l} = \frac{n\pi}{Nh}$, $i.e\ l = Nh, z = \sqrt{-1}$ from (12), because of linearity, we only consider one of the terms and thus only need $e^{zi\beta}$ where β is real. Let the solution of the finite difference approximation be given in separable form as

$$E(x,t) \approx\ e^{\gamma x} e^{zi\beta} \tag{13}$$

where $\gamma = \gamma(\beta)$ is complex. The solution at $x = 0$ equals the error introduced at $x = 0$. Observing from (13) that in order for the original error not to grow as x increases

$$| e^{\gamma x} | \leq 1\ \forall\ \gamma \tag{14}$$

Therefore, the Von Newmann condition for stability can be written as

$$| e^{\gamma h} | \leq 1 \tag{15}$$

If we define $\xi = e^{\gamma h}$ which is the amplification factor, then the stability constraint of (15) becomes

$$| \xi | \leq 1 \tag{16}$$

We shall use the above procedure to investigate the stability of modified Crank-Nicolson scheme as follows;

Illustration: Investigate the stability of the Parabolic Partial Differential equation (2) approximated using the modified Crank-Nicolson scheme as given below:

$$\frac{f_{i,\ j}\ -\ f_{i,\ j-1}}{k} = \frac{1}{2}[\frac{f_{i+1,\ j-1}\ -2f_{i,\ j-1}\ +\ f_{i-1,\ j-1}}{h^2} + \frac{f_{i+1,\ j}\ -\ 2f_{i,\ j}\ +\ f_{i-1,\ j}}{h^2}]$$

Solution:

The scheme above results into;

$$2(1+r)f_{i,\ j}\ -\ r(f_{i+1,\ j}\ +\ f_{i-1,\ j})\ =\ 2(1-r)f_{i,\ j-1}\ +\ r(f_{i+1,\ j-1}\ +\ f_{i-1,\ j-1}) \tag{17}$$

Let

$$E_{i,j} = e^{\gamma ih} e^{z\beta jk} = \xi^i e^{z\beta jk} \tag{18}$$

substituting (18) into (17) gives

$$2(1+r)\xi^i e^{z\beta jk} - r(\xi^i e^{z\beta(j+1)k} + e^{z\beta(j-1)k}\xi^i) = 2(1-r)\xi^{i-1} e^{z\beta jk} + r(\xi^{i-1} e^{z\beta(j+1)k} + \xi^{i-1} e^{z\beta(j-1)k}) \tag{19}$$

International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

factoring out $\xi e^{z\beta jk}$ we have

$$\xi^i e^{z\beta jk}[2(1+r)-r(e^{z\beta k}+e^{-z\beta k}) = \xi^i e^{z\beta jk}[2(1-r)\xi^{-1}+r(\xi^{-1}e^{z\beta k}+\xi^{-1}e^{-z\beta k})]$$

which gives

$$[2(1+r)-r(e^{z\beta k}+e^{-z\beta k})] = [2(1-r)+r(e^{z\beta k}+e^{-z\beta k})]\xi^{-1} \tag{20}$$

using Trigonometric identity, we define

$$e^{z\beta k}+e^{-z\beta k} = 2cos\beta k$$

and

$$1-cos\beta k = 2sin^2(\frac{\beta k}{2})$$

substituting these into (20) we have

$$2(1+r)-r(2cos\beta k) = \xi^{-1}[2(1-r)+r(2cos\beta k)]$$

$$2(1+r)(1-cos\beta k) = \xi^{-1}[2(1-r)(1-cos\beta k)]$$

$$\left[1+r\left(2sin^2\frac{\beta k}{2}\right)\right] = \xi^{-1}\left[1-r\left(2sin^2\frac{\beta k}{2}\right)\right]$$

$$\xi^{-1} = \frac{1+r\left(2sin^2\frac{\beta k}{2}\right)}{1-r\left(2sin^2\frac{\beta k}{2}\right)}$$

$$\xi^{-1} = \frac{1+2rsin^2\left(\frac{\beta k}{2}\right)}{1-2rsin^2\left(\frac{\beta k}{2}\right)}$$

from whence the amplification factor

$$\xi = \frac{1-2rsin^2\left(\frac{\beta k}{2}\right)}{1+2rsin^2\left(\frac{\beta k}{2}\right)}$$

for any value of r and βk, we have that $\mid \xi \mid < 1$ and thus the approximation is unconditional stable.

3 Numerical Examples

In this section, we present some numerical examples of one dimensional Parabolic partial differential equations solved using the modified Crank-Nicolson method, we also compare the results obtained with the analytical solution.

Example 1:
Solve the partial differential equation using modified Crank-Nicolson scheme:

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial f}{\partial t}, \quad 0 \le x \le 1 \\ with \quad f(x,\,0) &= 100sin\pi x \\ and \quad f(0,\,t) = 0 &= f(1,\,t) \end{aligned}\right\} \tag{21}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

Solution:
here, we use $h = 0.1$, and $k = 0.04$ then $r = \frac{k}{h^2} = 4$
using

$$2(1+r)f_{i,\,j} \;-\; r(f_{i+1,\,j} \;+\; f_{i-1,\,j}) \;=\; 2(1+r)f_{i,\,j-1} \;+\; r(f_{i+1,\,j-1} \;+\; f_{i-1,\,j-1})$$

at $i = 1,\; j = 1$ we have

$$2(1+4)f_{1,1} \;-\; 4(f_{2,1} \;+\; f_{0,1}) \;=\; 2(1-4)f_{1,0} \;+\; 4(f_{2,0} \;+\; f_{0,0})$$

$$10f_{1,1} \;-\; 4f_{2,1} \;=\; -6f_{1,0} \;+\; 4f_{2,0} \;+\; 4f_{0,0}$$

$$10f_{1,1} \;-\; 4f_{2,1} \;= 30.9017 \tag{22}$$

solving for $1 \leq i \leq 9$ at $j = 1$, we get a tridiagonal matrix which is represented below;

$$\begin{bmatrix} 10 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 10 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 10 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 10 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 10 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 10 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 10 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 10 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 10 \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{5,1} \\ f_{6,1} \\ f_{7,1} \\ f_{8,1} \\ f_{9,1} \end{bmatrix} = \begin{bmatrix} 49.7038 \\ 94.5426 \\ 130.1266 \\ 152.9726 \\ 160.8456 \\ 152.9726 \\ 130.1266 \\ 94.5426 \\ 49.7038 \end{bmatrix}$$

also, the results of the next steps $2 \leq j \leq 9$, and $1 \leq i \leq 10$ is given in the table1.

International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

Table 1: Results of Modified Crank-Nicolson Solution

j	t	$f_{1,\ j}$	$f_{2,\ j}$	$f_{3,\ j}$	$f_{4,\ j}$	$f_{5,\ j}$	$f_{6,\ j}$	$f_{7,\ j}$	$f_{8,\ j}$	$f_{9,\ j}$
1	0.04	20.7831	39.5319	54.4110	63.9640	67.2557	63.9640	54.4110	39.5319	20.7831
2	0.08	13.9779	26.5875	36.5945	43.0194	45.2333	43.0194	36.5945	26.5875	13.9779
3	0.12	9.4009	17.8816	24.6119	28.9330	28.9330	28.9330	24.6119	17.8816	9.4009
4	0.16	6.3227	12.0264	16.5529	19.4592	19.4592	19.4592	16.5529	12.0264	6.3227
5	0.20	4.2523	8.0885	11.1328	13.0873	13.7609	13.0873	11.1328	8.0885	4.2523
6	0.24	2.8600	5.4399	7.4874	8.8020	9.2549	8.8020	7.4874	5.4399	2.8600
7	0.28	1.9234	3.6587	5.0357	5.9198	6.2245	5.9198	5.0357	3.6587	1.9234
8	0.32	1.2937	2.4606	3.3868	3.9814	4.1863	3.9814	3.3868	2.4606	1.2937
9	0.36	0.8700	1.6550	2.2778	2.6777	2.8155	2.6777	2.2778	1.6550	0.8700

Table 2: Comparison of Modified Crank-Nicolson Solution and the Analytical Solution of table 1 above at $x = 0.5$ and $k = 0.04$

t	Modified Crank-Nicolson Solution	Analytical Solution	Error
0.24	9.2549	9.4052	0.1503
0.28	6.2245	6.3071	0.0826
0.32	4.1863	4.2499	0.0636

Example 2:
Solve the partial differential equation [13] using Modified Crank-Nicolson scheme:

$$\frac{1}{8}\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}, \quad 0 \le x \le 4 \tag{23}$$

with initial condition

$$f(x,\ 0) = 2000, \quad 0 \le x \le 4 \tag{24}$$

and boundary conditions

$$\frac{\partial f}{\partial x} = \begin{cases} 0.36f - 25.2, & x = 4 \\ 0 & x = 0 \end{cases} \tag{25}$$

Solution::
using (5) and $r = \frac{\lambda \delta t}{(\delta x)^2}$, where $\delta t = \delta x = 1$ which have $r = \frac{1}{8} < 1$
substituting r into (5) we have

$$2\left(1+\frac{1}{8}\right)f_{i,j} - \frac{1}{8}\left(f_{i+1,j} + f_{i-1,j}\right) = 2\left(1-\frac{1}{8}\right)f_{i,j-1} + \frac{1}{8}\left(f_{i+1,j-1} + f_{i-1,j-1}\right)$$

which gives

$$2.25f_{i,j} - 0.125\left(f_{i+1,j} + f_{i-1,j}\right) = 1.75f_{i,j-1} + 0.125\left(f_{i+1,j-1} + f_{i-1,j-1}\right) \tag{26}$$

at $i = 1,\ j = 1$

$$2.25f_{1,1} - 0.125(f_{2,1} + f_{0,1}) = 1.75f_{1,0} + 0.125(f_{2,0} + f_{0,0}) \tag{27}$$

solving for $f_{0,1}$ in terms of $f_{1,1}$ and $f_{2,1}$, using the boundary condition

$$\left.\frac{\partial f}{\partial x}\right|_0 = \frac{f_{i+1,j} - f_{i-1,j}}{2}$$

International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

Table 3: Results of Modified Crank-Nicolson Solution

j	t	$f_{1,\ j}$	$f_{2,\ j}$	$f_{3,\ j}$	$f_{4,\ j}$	$f_{5,\ j}$
1	1	2000.0	2000.0	2000.0	2000.0	2000.0
2	2.0	1850.6	1991.7	1999.5	2000.0	2000.0
3	3.0	1741.5	1970.7	1998.0	1999.9	2000.0
4	4.0	1659.0	1943.5	1995.3	1999.5	1999.9
5	5.0	1594.4	1913.7	1988.3	1998.7	1999.7
6	6.0	1542.1	1883.1	1979.4	1997.1	1999.3
7	7.0	1498.5	1852.9	1968.9	1994.7	1998.5
8	8.0	1461.2	1823.7	1957.1	1991.5	1997.3
9	9.0	1428.7	1795.7	1944.3	1987.5	1995.6
10	10	1399.8	1769.1	1930.8	1982.7	1993.3
11	11	1373.7	1743.8	1916.9	1977.2	1990.3

we get

$$2(0.36f_{1,1} - 25.2) = f_{2,1} - f_{0,1}$$

$$f_{0,1} = f_{2,1} - 0.72f_{1,1} + 50.4 \tag{28}$$

similarly, we solve for $f_{0,0}$ in terms of $f_{1,0}$ and $f_{2,0}$ so that

$$f_{0,0} = f_{2,0} - 0.72f_{1,0} + 50.4 \tag{29}$$

putting (28) and (29) into (27) gives

$$2.25f_{1,1} - 0.125[2f_{2,1} - 0.72f_{1,1} + 50.4] = 1.75f_{1,0} + 0.125[2f_{2,0} - 0.72f_{1,0} + 50.4]$$

on simplifying we get

$$2.34f_{1,1} - 0.25f_{2,1} = 1.66f_{1,0} + 0.25f_{2,0} + 12.6 \tag{30}$$

and for $2 \leq i \leq 4$ we use (26) also for $i = 5$ we use the fact that $\left.\frac{\partial f}{\partial x}\right|_{x=4} = 0$, therefore, $f_4 = f_6$ so we have

$$2.25f_{5,1} - 0.25f_{4,1} = 1.75f_{5,0} + 0.25f_{4,0} \tag{31}$$

now, equations (30), (26) for $2 \leq i \leq 4$ and (31) is expressed in a matrix form as

$$\begin{bmatrix} 2.34 & -0.25 & 0 & 0 & 0 \\ -0.125 & 2.25 & -0.125 & 0 & 0 \\ 0 & -0.125 & 2.35 & -0.125 & 0 \\ 0 & 0 & -0.125 & 2.25 & -0.125 \\ 0 & 0 & 0 & -0.25 & 2.25 \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{5,1} \end{bmatrix} = \begin{bmatrix} 3832.6 \\ 4000.0 \\ 4000.0 \\ 4000.0 \\ 4000.0 \end{bmatrix}$$

solving the tri-diagonal matrix for the various time steps (j=1-11) we have the following tabulated results in table 3.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

Table 4: Results of Modified Crank-Nicolson Solution

x	t	i	$f_{i,\ j=1}$	$f_{i,\ j=2}$	$f_{i,\ j=3}$	$f_{i,\ j=4}$	$f_{i,\ j=5}$	$f_{i,\ j=6}$	$f_{i,\ j=7}$	$f_{i,\ j=8}$
0.1	0.001	1	0.3060	0.3030	0.3001	0.2972	0.2943	0.2914	0.2886	0.2858
0.2	0.002	2	0.5821	0.5764	0.5708	0.5652	0.5597	0.5542	0.5488	0.5435
0.3	0.003	3	0.8011	0.7933	0.7856	0.7779	0.7703	0.7628	0.7554	0.7480
0.4	0.004	4	0.9418	0.9326	0.9235	0.9145	0.9056	0.8968	0.8880	0.8793
0.5	0.005	5	0.9903	0.9806	0.9710	0.9615	0.9521	0.9428	0.9336	0.9245
0.6	0.006	6	0.9418	0.9326	0.9235	0.9145	0.9056	0.8968	0.8880	0.8793
0.7	0.007	7	0.8011	0.7933	0.7856	0.7779	0.7703	0.7628	0.7554	0.7480
0.8	0.008	8	0.5821	0.5764	0.5708	0.5652	0.5597	0.5542	0.5488	0.5435
0.9	0.009	9	0.3060	0.3030	0.3001	0.2972	0.2943	0.2914	0.2886	0.2858

Table 5: Comparison of Modified Crank-Nicolson Solution and the Analytical Solution of table 4 above at $x = 0.5$ and $k = 0.001$

t	Modified Crank-Nicolson Solution	Analytical Solution	Error
0.005	0.9521	0.9518	0.0003
0.006	0.9428	0.9425	0.0003
0.007	0.9336	0.9332	0.0004

Example 3:
Solve the partial differential equations using modified Crank-Nicolson method

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial f}{\partial t}, \quad 0 \le x \le 1 \\ \text{with} \quad f(x,\ 0) &= \sin \pi x \\ \text{and} \quad f(0,\ t) = 0 &= f(1,\ t) \end{aligned}\right\} \tag{32}$$

given that the exact solution is $f(x,t) = exp(-\pi^2 t)\sin(\pi x)$ then from (5) we have at $i = 1,\ j = 1$ and using $r = \frac{k}{h^2}$, $k = 0.001,\ h = 0.1$

$$2.2 f_{1,1} - 0.1 f_{2,1} = 0.61498$$

solving for $1 \le i \le 9$ at $j = 1$, we get a tridiagonal matrix which is represented below;

$$\begin{bmatrix} 2.2 & -0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1 & 2.2 & -0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 2.2 & -0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & 2.2 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 2.2 & -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1 & 2.2 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 & 2.2 & -0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 2.2 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 2.2 \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{5,1} \\ f_{6,1} \\ f_{7,1} \\ f_{8,1} \\ f_{9,1} \end{bmatrix} = \begin{bmatrix} 0.61498 \\ 1.16984 \\ 1.61009 \\ 1.89288 \\ 1.99022 \\ 1.89288 \\ 1.61009 \\ 1.16984 \\ 0.61498 \end{bmatrix}$$

the results for the tridiagonal matrix above and the various next steps for $2 \le j \le 9$, and $1 \le i \le 10$ is given in the table 4:

4 Discussion of Results

Tables 1, 3 and 4 show that the Modified Crank-nicolson method is good for solving parabolic partial differential equation (Diffusion equation). Tables 2 and 5 also show that the Modified Crank-Nicolson method performs well, consistent and agree with the analytical solution. The method provides better accuracy and requires the solution of tridiagonal system at every time level.

5 Conclusion

From the results analysis, it is seen that our method provides approximate results and fast convergence compared to the classical Crank-Nicolson method. Since it is not possible to solve every partial differential equation analytically so numerical methods providing a good agreement in those cases where solutions do not exist or where Partial differential equations can not be solve analytically. The results of our method also agree with existing findings in literature that smaller time step produces more accurate results.This can be always be achieved when the value of $r = \frac{k}{h^2}$ is kept reasonably small for a close approximation to the solution of the partial differential equation as seen in the tables.

Acknowledgements

Authors are grateful to the reviewers for their valuable suggestions and contributions on improving the paper.

Competing financial interests

The author declares no competing financial interests".

References

[1] Olaiju,O.A., Hoe,Y.S. & Ogunbode,E.B. Finite difference Approximation to Heat Equation via C. *Journal of Applied Sciences and Sustainability* **3**, 188–200, (2017).

[2] Crank,J. & Philis,N. A practical method for Numerical Evaluation of solution of partial differential equation of heat conduction type. *Proc. camb. Phil. soc.* **1**, 50–57 (1996).

[3] Sanjaya,F. & Mungkasi,S. A simple but accurate explicit finite difference method for Advection-diffusion equation. *Journal of Phy.* **Conference series 909**, (2017).

[4] Fadugba,S.E., Adefolaju,F.H. & Akindutire,O.H. On stability and Accuracy of finite difference method for option pricing, International Institute of Science. *Technology and Education, Mathematical Theory and Modeling* **2**, 101–108, (2012).

[5] Tran,O. & Lin,J. Modified Iterated Crank-Nicolson method with improved Accuracy. *arXiv: 1608.01344 V1 [math.NA]* .

[6] Mariton,S.K. Modified Crank Nicholson Based Methods on the Solution of one dimensional Heat Equation. *Nonlinear Analysis and Differential Equations* **7**, 33–37, (2019).

[7] Abhulimen,C.E.& Omowo,B.J. Modified Crank-Nicolson Method for Solving One Dimensional Parabolic Equation. *international journal of scientific research* **15**, 60–66, (2019).

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 862 - 873.
doi.org/10.6084/m9.figshare.13664309.

[8] Cooper,J. Introduction to Partial differential Equation with Matlab. *Boston* , (1958).

[9] Mitchell,A.R.& Gridffiths,D.F. A Finite difference method in partial differential equations. *John Wiley and Sons* , (1980).

[10] Ames,W.F. Numerical methods for Partial differential Equations. *Academic press, Inc. Third Edition* , (1992).

[11] Smith,G.D. Numerical Solution of Partial Differential Equations: Finite Difference Methods. *Clarendon Press, Third Edition, Oxford* , (1985).

[12] Grewal,B.S. Higher Engineering Mathematics. *Khanna Publisher, Forty-second edition* , (2012).

[13] Fadugba,S.E., Edogbanya,O.H.& Zelibe,S.C. Crank-nicolson method for solving parabolic partial differential equations. *International journal of Applied mathematics and modelling* **1**, 8–23, (2013).