International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

Exponentiated Lomax-Weibull Distribution with Properties and Applications

Sunday A. Osagie[1*] and Joseph E. Osemwenkhae[2]

1*, 2, Department of Statistics, University of Benin, Benin City, Nigeria, P.M.B. 1154, Benin City, Edo State.
*Corresponding author: sunday.osagie@uniben.edu

Article Info
Received: 03 September 2020 Revised: 01 November 2020
Accepted: 06 November 2020 Available online: 30 November 2020

Abstract

In this paper, the exponentiated Lomax-Weibull distribution is constructed as a new lifetime model using combination of the competing risk and exponentiation methods. Some new and existing distributions are presented as submodels. Mathematical properties to define the distribution are presented in details. Statistical inference is presented for the exponentiated Lomax-Weibull distribution using the method of maximum likelihood estimation to estimate the parameters of proposed distribution. Two lifetime datasets are used to illustrate the usefulness and applicability of the proposed distribution in lifetime data analysis. The results of the analysisof the datasets show the superiority of the exponentiated Lomax-Weibull distribution over some compared distributions.

Keywords And Phrases: Exponentiation, Competing risk method, Exponentiated Lomax-Weibull distribution, Order statistics, Stress-strength, Residual lifetimes.
MSC2010: 60E05, 62-07

1 Introduction

Over the years, several new flexible univariate lifetime distributions have been generated and introduced as superior models to classical distributions. They fit complex data from reliability and survival systems adequately as well as predict their non-monotonic aging events more efficiently than the classical distributions. Several methods for constructing new univariate distributions are detailed in the works of [23] and [11]. Competing risk method generates new lifetime distribution by mixing two or more classical distributions to model data from series or parallel systems. The Lomax-Weibull ([22]), Lomax-Exponential ([20]), Burr XII-Weibull ([19]) and additive Weibull ([26]) distributions are some lifetime models in literature introduced using the competing risk method.

For a series system with i^{th} (i=1,2,..,n) independent component with i^{th} reliability function given as $R_i(y)$, the cdf of the distribution of the system is given as

$$G(y) = 1 - \prod_{i=1}^{n} R_i(y); y > 0.$$

For a system with two independent components connected in series, its distribution becomes

$$F(y) = 1 - R_1(y)R_2(y). \tag{1.1}$$

New distributions have been generated with this method over the last two decades by many researchers. Another method of generating new flexible distributions is the exponentiated method with its cdf given as

$$G_\lambda(y) = [F(y;\eta)]^\lambda, \tag{1.2}$$

where $\eta > 0$ is a parameter vector. [5] pioneered the exponentiated method with the cdf of his model given as

$$F_\lambda(y) = (1 - pe^{-\alpha y})^\lambda; p, \alpha, \lambda > 0, \tag{1.3}$$

with its corresponding pdf given as

$$f_\lambda(y) = \lambda\alpha p e^{-\alpha y}(1 - pe^{-\alpha y})^{\lambda-1}.$$

Equation (1.3) was used to analyse the mortality rates in lifespan of adult human. [14] generalized equation (1.3) by replacing $e^{-\alpha y}$ with $e^{-\alpha y^\theta}$ and $p = 1$, thereby obtaining the exponentiated Weibull family. Its application to lifetime data analysis was investigated by [15]. [6] introduced a submodel to equation (1.3) by substituting p=1. Since then, several new exponentiated distributions have been generated and some are found in the works of [4], [2], [13], [7], [15], [8], [5], [14] and [3].

The exponentiated Lomax-Weibull (ELW) distribution as new lifetime model is generated by the combination of equations (1.1) and (1.2) with expressions for $R_1(y)$ and $R_2(y)$. It is introduced to analyse lifetime data obtained from systems that exhibit variety of monotonic and non-monotonic failure patterns. Several submodels can be obtained as special cases. The essence of the proposed distribution is to present it as a flexible distribution that can be useful in lifetime analysis. Application of the proposed distribution is presented to show flexibility and usefulness in lifetime analysis.

The organization of the paper is presented as follows. Section 2 presents model definition of the ELW distribution. This section contains formulation of the new distribution, mathematical properties and estimation of parameters of the ELW distribution. Section 3 gives the results which arise from the application of the ELW distribution to lifetime data sets. In section 4, discussion on the results obtained is presented. Section 5 concludes the study on the ELW distribution.

2 Model definition

2.1 Formulation of the ELW distribution

[19] proposed the Lomax-Weibull distribution from equation (1.1) with its cdf given as

$$F(y) = 1 - (1 + \gamma y)^{-\beta} e^{-\alpha y^\theta}, y > 0, \gamma > 0, \beta > 0, \alpha > 0, \theta > 0. \tag{2.4}$$

Substituting equation (2.4) into equation (1.2), we obtain

$$G_\lambda(y) = \left(1 - (1 + \gamma y)^{-\beta} e^{-\alpha y^\theta}\right)^\lambda, \lambda > 0 \tag{2.5}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

Equation (2.5) defines the cdf of the ELW distribution which will be denoted by ELW$(\gamma,\beta,\alpha,\theta,\lambda)$, with corresponding pdf, survival and hazard functions given in equations (2.6), (2.7) and (2.8) respectively as

$$g_\lambda(y) = \lambda\left(\gamma\beta + \alpha\theta y^{\theta-1}(1+\gamma y)\right)(1+\gamma y)^{-\beta-1}e^{-\alpha y^\theta}\left(1-(1+\gamma y)^{-\beta}e^{-\alpha y^\theta}\right)^{\lambda-1}, \tag{2.6}$$

$$R_\lambda(y) = 1 - \left(1-(1+\gamma y)^{-\beta}e^{-\alpha y^\theta}\right)^{\lambda} \tag{2.7}$$

and

$$h_\lambda(y) = \frac{\lambda\left(\gamma\beta + \alpha\theta y^{\theta-1}(1+\gamma y)\right)(1+\gamma y)^{-\beta-1}e^{-\alpha y^\theta}\left(1-(1+\gamma y)^{-\beta}e^{-\alpha y^\theta}\right)^{\lambda-1}}{1-\left(1-(1+\gamma y)^{-\beta}e^{-\alpha y^\theta}\right)^{\lambda}}. \tag{2.8}$$

Plots for $g_\lambda(y)$ and $h_\lambda(y)$ are presented in Figures (1) and (2) showing the variety of shapes for some parameter values of ELW$(\gamma,\beta,\alpha,\theta,\lambda)$. It is observed that the ELW$(\gamma,\beta,\alpha,\theta,\lambda)$ can fit bimodal and other skewed lifetime data as shown in Figure 1. A bimodal data is a data set with two modes and can be split into two unimodal sets. The shapes of the hazard function in Figure 2 indicate the ability of the proposed distribution to model increasing, decreasing, unimodal, modified bathtub-shaped and bathtub-shaped failure events.

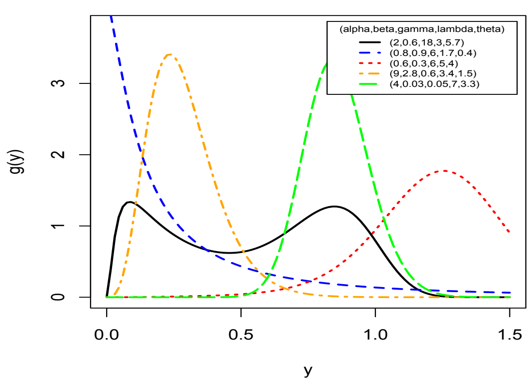

Figure 1: Plot for pdf of ELW distribution

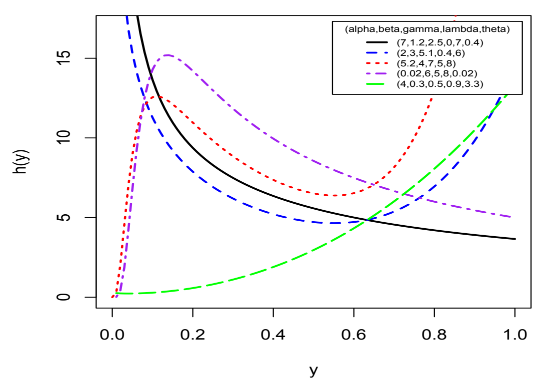

Figure 2: Plot for h(y) of ELW distribution

It is known that

$$(1-z)^n = \sum_{i=0}^{\infty}\binom{n}{i}(-1)^i z^i, z<1, (1+z)^{-n} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}(-1)^i z^i. \tag{2.9}$$

Employing the series expressions in equation (2.9), the expansion of $g_\lambda(y)$ in equation (2.6) is expressed as

$$\begin{aligned} g_\lambda(y) &= \lambda\sum_{i=0}^{\infty}\binom{\lambda-1}{i}(-1)^i\left(\gamma\beta+\alpha\theta y^{\theta-1}(1+\gamma y)\right)((1+\gamma y)^{-(i+1)\beta-1}e^{-(i+1)\alpha y^\theta} \\ &= \lambda\sum_{i,j=0}^{\infty}\binom{\lambda-1}{i}\binom{\beta(i+1)+j}{j}(-1)^{i+j}\gamma^j y^j\left(\gamma\beta+\alpha\theta y^{\theta-1}(1+\gamma y)\right)e^{-(i+1)\alpha y^\theta}. \end{aligned} \tag{2.10}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

From equation (2.5), some new and existing submodels can be generated which are listed as follows.
(1) If λ=1, the Lomax-Weibull distribution is obtained as a lifetime distribution [22].
(2) If θ=1, the exponentiated Lomax-exponential distribution is obtained [21].
(3) If θ=2, then a new lifetime distribution called the exponentiated Lomax-Rayleigh distribution is obtained.
(4) If λ=θ=1, the Lomax-exponential distribution is obtained [20].
(5) If λ=1 and θ=2, then the Lomax-Rayleigh distribution is obtained as a new lifetime distribution.
(6) The Lomax distribution [14] is obtained if λ=1 and α=0. If λ=θ=1 and β=0, then the exponential distribution is obtained.
(7) The Weibull distribution [25] is obtained if λ=1, β=0 and the Rayleigh distribution can be obtained if λ=1, θ=2 and β=0.

2.2 Mathematical properties of the ELW distribution

The section considers some mathematical properties that define the ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ as a lifetime distribution.

2.2.1 Quantile function

If $0 < q < 1$ and a random variable Y follows the ELW$(\gamma, \beta, \alpha, \theta, \lambda)$, then the quantile function can be derived from $y_q = F_\lambda^{-1}(q)$. This implies that ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ quantile function is the root of the equation given by

$$\beta log(1 + \gamma y_q) + \alpha y_q^\theta + log(1 - q^{\frac{1}{\lambda}}) = 0. \tag{2.11}$$

The root of equation (2.11) which is, y_q, gives the unique solution for every value of $q \in (0, 1)$ for a particular combination of parameter values of $\gamma, \beta, \alpha, \theta$ and λ, which is obtained by any known iterative method (such as the Newton-Raphson method). Alternatively, the uniroot package in the R software can be used to obtain the root of equation (2.11). If $q = 0.5$, the median of ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ denoted by $y_{0.5}$ can be obtained from equation (2.12) with its expression given as

$$\beta log(1 + \gamma y_{0.5}) + \alpha y_{0.5}^\theta + log(1 - 0.5^{\frac{1}{\lambda}}) = 0. \tag{2.12}$$

Table 1 presents different values of $q \in (0, 1)$ and sets of parameters for ELW$(\gamma, \beta, \alpha, \theta, \lambda)$.

Table 1: Values for quantile function of ELW distribution

q	(0.1,8,0.5,0.4,3)	(0.07,4,0.03,0.09,5)	(0.4,2,0.2,0.01,2.3)	(0.6,0.4,8,3,0.8)	(1,1,1,1,1)
0.1	0.3676	3.8883	0.3473	0.1452	0.0534
0.2	0.6051	5.2667	0.6898	0.2263	0.1146
0.3	0.8352	6.5325	1.0433	0.2863	0.1860
0.4	1.0787	7.8458	1.4447	0.3380	0.2710
0.5	1.3518	9.3134	1.9317	0.3865	0.3748
0.6	1.6759	11.0674	2.5640	0.4349	0.5065
0.7	2.0882	13.3400	3.4609	0.4863	0.6832
0.8	2.6723	16.6683	4.9273	0.5457	0.9444
0.9	3.7061	22.9362	8.1676	0.6262	1.4191

It is seen that for every value of $q \in (0, 1)$ and set of parameter values for ELW$(\gamma, \beta, \alpha, \theta, \lambda)$, the values obtained from equation (2.10) are monotonic increasing.

2.2.2 Moments and moment generating function

Suppose a random variable Y follows ELW$(\gamma,\beta,\alpha,\theta,\lambda)$, then the r^{th} raw moment (μ_r) for the ELW distribution are evaluated from the following expressions which are given as

$$\begin{aligned}\mu_r = E[Y^r] &= \int_0^\infty y^r g_\alpha(y)dy\\ &= \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\int_0^\infty y^{r+j}\left(\gamma\beta+\alpha\theta y^{\theta-1}(1+\gamma y)\right)e^{-(i+1)\alpha y^\theta}dy.\end{aligned}$$

Letting $m=(i+1)\alpha y^\theta$, the derivation of r^{th} raw moment is given as

$$\begin{aligned}\mu_r &= \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\Bigg\{\int_0^\infty\left[\gamma\beta\left(\frac{m}{(i+1)\alpha}\right)^{\frac{r+j}{\theta}}+\alpha\theta\left(\left(\frac{m}{(i+1)\alpha}\right)^{\frac{r+j-1}{\theta}+1}+\gamma\left(\frac{m}{(i+1)\alpha}\right)^{\frac{r+j}{\theta}+1}\right)\right]\\ &\times\frac{e^{-m}}{\theta m}\left(\frac{m}{(i+1)\alpha}\right)^{\frac{1}{\theta}}dm\Bigg\}\\ &= \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\Bigg\{\int_0^\infty\left[\frac{\gamma\beta}{\theta}\left(\frac{m^{\frac{r+j+1}{\theta}-1}}{((i+1)\alpha)^{\frac{r+j+1}{\theta}}}\right)+\alpha\left(\left(\frac{m^{\frac{r+j}{\theta}}}{((i+1)\alpha)^{\frac{r+j}{\theta}+1}}\right)+\gamma\left(\frac{m^{\frac{r+j+1}{\theta}}}{((i+1)\alpha)^{\frac{r+j+1}{\theta}+1}}\right)\right)\right]\\ &\times e^{-m}dm\Bigg\}.\end{aligned}$$

Employing the relationship between integral of mathematical expression and complete gamma function which is given as

$$\Gamma(\xi)=\int_0^\infty z^{\xi-1}e^{-z}dz,$$

we have

$$\mu_r=\lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma(\frac{r+j+1}{\theta})}{((i+1)\alpha)^{\frac{r+j+1}{\theta}}}+\alpha\left(\frac{\Gamma(\frac{r+j}{\theta}+1)}{((i+1)\alpha)^{\frac{r+j}{\theta}+1}}+\gamma\frac{\Gamma(\frac{r+j+1}{\theta}+1)}{((i+1)\alpha)^{\frac{r+j+1}{\theta}+1}}\right)\right]. \quad (2.13)$$

where $\eta_{i,j}=(-1)^{i+j}\binom{\lambda-1}{i}\binom{\beta(i+1)+j}{j}\gamma^j$.
The conditional moment for the ELW$(\gamma,\beta,\alpha,\theta,\lambda)$ is derived from the expression given as

$$\begin{aligned}\mu_r^* = E[Y^r/Y>t] &= \frac{1}{R_\lambda(t)}\int_t^\infty y^r g_\alpha(y)dy\\ &= \frac{\lambda}{1-\left(1-(1+\gamma t)^{-\beta}e^{-\alpha t^\theta}\right)^\lambda}\sum_{i,j=0}^{\infty}\eta_{i,j}\int_t^\infty y^{r+j}\left(\gamma\beta+\alpha\theta y^{\theta-1}(1+\gamma y)\right)e^{-(i+1)\alpha y^\theta}dy.\end{aligned} \quad (2.14)$$

Substituting $m=(i+1)\alpha y^\theta$ into equation (2.13), we have

$$\begin{aligned}\mu_r^* = &\frac{\lambda}{1-\left(1-(1+\gamma t)^{-\beta}e^{-\alpha t^\theta}\right)^\lambda}\sum_{i,j=0}^{\infty}\eta_{i,j}\Bigg\{\int_{(i+1)\alpha t^\theta}^\infty\left[\frac{\gamma\beta}{\theta}\left(\frac{m^{\frac{r+j+1}{\theta}-1}}{((i+1)\alpha)^{\frac{r+j+1}{\theta}}}\right)\right.\\ &\left.+\alpha\left(\left(\frac{m^{\frac{r+j}{\theta}}}{((i+1)\alpha)^{\frac{r+j}{\theta}+1}}\right)+\gamma\left(\frac{m^{\frac{r+j+1}{\theta}}}{((i+1)\alpha)^{\frac{r+j+1}{\theta}+1}}\right)\right)\right]\times e^{-m}dm\Bigg\}.\end{aligned}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

Hence, the conditional moment for the ELW$(\gamma,\beta,\alpha,\theta,\lambda)$ is given as

$$\mu_r^* = \frac{\lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma_u(\frac{r+j+1}{\theta},(i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j+1}{\theta}}}+\alpha\left(\frac{\Gamma_u(\frac{r+j}{\theta}+1,(i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j}{\theta}+1}}+\gamma\frac{\Gamma_u(\frac{r+j+1}{\theta}+1,(i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j+1}{\theta}+1}}\right)\right]}{1-\left(1-(1+\gamma t)^{-\beta}e^{-\alpha t^\theta}\right)^\lambda}, \tag{2.15}$$

respectively, where $\Gamma_u(\xi,\varkappa)=\int_{\varkappa}^{\infty} z^{\xi-1}e^{-z}dz$ is the incomplete upper gamma function. The expressions for the first four raw moments μ_1, μ_2, μ_3 and μ_4 are presented as

$$\mu_1 = \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma(\frac{2+j}{\theta})}{((i+1)\alpha)^{\frac{2+j}{\theta}}}+\alpha\left(\frac{\Gamma(\frac{1+j}{\theta}+1)}{((i+1)\alpha)^{\frac{1+j}{\theta}+1}}+\gamma\frac{\Gamma(\frac{2+j}{\theta}+1)}{((i+1)\alpha)^{\frac{2+j}{\theta}+1}}\right)\right],$$

$$\mu_2 = \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma(\frac{3+j}{\theta})}{((i+1)\alpha)^{\frac{3+j}{\theta}}}+\alpha\left(\frac{\Gamma(\frac{2+j}{\theta}+1)}{((i+1)\alpha)^{\frac{2+j}{\theta}+1}}+\gamma\frac{\Gamma(\frac{3+j}{\theta}+1)}{((i+1)\alpha)^{\frac{3+j}{\theta}+1}}\right)\right],$$

$$\mu_3 = \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma(\frac{4+j}{\theta})}{((i+1)\alpha)^{\frac{4+j}{\theta}}}+\alpha\left(\frac{\Gamma(\frac{3+j}{\theta}+1)}{((i+1)\alpha)^{\frac{3+j}{\theta}+1}}+\gamma\frac{\Gamma(\frac{4+j}{\theta}+1)}{((i+1)\alpha)^{\frac{4+j}{\theta}+1}}\right)\right]$$

and

$$\mu_4 = \lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma(\frac{5+j}{\theta})}{((i+1)\alpha)^{\frac{5+j}{\theta}}}+\alpha\left(\frac{\Gamma(\frac{4+j}{\theta}+1)}{((i+1)\alpha)^{\frac{4+j}{\theta}+1}}+\gamma\frac{\Gamma(\frac{5+j}{\theta}+1)}{((i+1)\alpha)^{\frac{5+j}{\theta}+1}}\right)\right].$$

where μ_1 represents mean of ELW$(\gamma,\beta,\alpha,\theta,\lambda)$. The values of μ_1, μ_2, μ_3 and μ_4 are obtained at different values for sets of parameters in ELW$(\gamma,\beta,\alpha,\theta,\lambda)$. The evaluation of standard deviation (SD), coefficients of variation (CV), skewness (CS) and kurtosis (CK) are obtained via moments based relations given by $SD=\sqrt{\mu_2-\mu_1^2}$, $CV=\frac{SD}{\mu_1}$, $CS=\frac{\left(\mu_3-3\mu_2\mu_1+2\mu_1^3\right)^2}{\left(\mu_2-\mu_1^2\right)^3}$ and $CK=\frac{\mu_4-4\mu_3\mu_1+6\mu_2\mu_1^2-3\mu_1^4}{\left(\mu_2-\mu_1^2\right)^2}$.

Table 2: Values of first four raw moments, standard deviation, coefficients of variation, skewness and kurtosis for ELW distribution

	(0.1,0.3,0.7,0.9,0.9)	(1,1,1,1,1)	(2.5,0.06,4,8,0.03)	(3.2,1.5,9,6,4.7)	(2.7,2.5,4,2.4,3.8)
μ_1	1.3915	0.5963	0.0706	0.5581	0.3456
μ_2	4.5735	0.8073	0.0453	0.3462	0.1538
μ_3	24.2134	1.7890	0.0325	0.2285	0.0822
μ_4	178.7238	5.6146	0.0245	0.1573	0.0504
SD	1.6239	0.6721	0.2009	0.1865	0.1856
CV	1.1670	1.1270	2.8460	0.3342	0.5370
CS	6.0225	6.4159	8.4721	0.2896	0.6778
CK	12.3423	13.1868	10.1792	2.5357	3.5848

It is evident from Table 2 that ELW$(\gamma,\beta,\alpha,\theta,\lambda)$ can be used to analyse fairly symmetrical, moderately and highly (postively) skewed lifetime data. Also, platykurtic and leptokurtic lifetime data can be analysie using the proposed distribution.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

The moment generating (mgf), $M_Y(t)$ of a random variable Y following ELW$(\gamma,\beta,\alpha,\theta,\lambda)$ is obtained as

$$M_Y(t)=E[e^{tY}]=\int_0^\infty e^{ty}g_\lambda(y)dy=\lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\int_0^\infty y^j\left(\gamma\beta+\alpha\theta y^{\theta-1}(1+\gamma y)\right)e^{ty-(i+1)\alpha y^\theta}dy. \tag{2.16}$$

Substituting $e^{ty}=\sum_{k=0}^{\infty}\frac{t^k}{k!}y^k$ and $m=(i+1)\alpha y^\theta$, equation (2.15) becomes

$$\begin{aligned}M_Y(t)&=\lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^\infty y^{k+j}\left(\gamma\beta+\alpha\theta y^{\theta-1}(1+\gamma y)\right)e^{-(i+1)\alpha y^\theta}dy\right]\\&=\lambda\sum_{i,j=0}^{\infty}\eta_{i,j}\left\{\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma(\frac{k+j+1}{\theta})}{((i+1)\alpha)^{\frac{k+j+1}{\theta}}}+\alpha\left(\frac{\Gamma(\frac{k+j}{\theta}+1)}{((i+1)\alpha)^{\frac{k+j}{\theta}+1}}\right.\right.\right.\\&\left.\left.\left.+\gamma\frac{\Gamma(\frac{k+j+1}{\theta}+1)}{((i+1)\alpha)^{\frac{k+j+1}{\theta}+1}}\right)\right]\right\}.\end{aligned} \tag{2.17}$$

which is the moment generating function of a random variable Y following ELW$(\gamma,\beta,\alpha,\theta,\lambda)$

2.2.3 Measures of entropy

The Rényi and Shannon entropies are two important measures in information theory to investigate the randomness related to random variable following a lifetime distribution. The applications of the two entropy measures are found in ecology, medicine, engineering, statistics and other sciencific areas. The Rényi $[\mathcal{I}_{\mathcal{R}}(\rho)]$ and Shannon $[\mathcal{H}_{\mathcal{S}}(g_\lambda)]$ entropies for ELW$(\gamma,\beta,\alpha,\theta,\lambda)$ are given respectively as

$$\begin{aligned}\mathcal{I}_{\mathcal{R}}(\rho)&=\frac{1}{(1-\rho)}log\int_0^\infty g_\lambda^\rho(y)dy=\frac{\rho}{(1-\rho)}log(\lambda)\\&+\frac{1}{(1-\rho)}log\left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\eta_{k,j,l}(\alpha\theta)^k(\gamma\beta)^{\rho-k}\gamma^l\frac{\Gamma\left(\frac{l+k(\theta-1)+1}{\theta}\right)}{\theta(\rho+j)^{\frac{l+k(\theta-1)+1}{\theta}}}\right];\rho>0,\rho\neq1\end{aligned} \tag{2.18}$$

where $\eta_{k,j,l}=\binom{\rho}{k}\binom{\rho(\lambda-1)}{j}\binom{\rho+\beta(\rho+j)-k+l-1}{l}$ and

$$\begin{aligned}\mathcal{H}_{\mathcal{S}}(g_\lambda)&=E[-log(g_\lambda(Y))]=-\int_0^\infty g_\lambda(y)log(g_\lambda(y))dy=-log(\lambda\gamma\beta)\\&-\sum_{k=1}^{\infty}\sum_{\omega=0}^{\infty}\binom{k}{\omega}\frac{(-1)^{k+1}}{k}\left(\frac{\alpha\theta}{\gamma\beta}\right)^k\gamma^\omega E\left(Y^{k(\theta-1)+\omega}\right)+(\beta+1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\gamma^kE\left(Y^k\right)\\&+\alpha E\left(Y^\theta\right)-(\lambda-1)\sum_{k=1}^{\infty}\sum_{\omega=0}^{\infty}\sum_{l=0}^{\infty}\binom{k\beta+l-1}{l}\frac{(-1)^{\omega+l+1}}{\omega!k}(k\alpha)^\omega\gamma^lE\left(Y^{\omega\theta+l}\right).\end{aligned} \tag{2.19}$$

where the mathematical expression for $E(Y^r)$ is given in equation (2.13).

2.2.4 Mean deviations about mean and median

Mean deviation (MD) is a measure of the aggregate variations in a data set from the mean and median, thereby presenting the amount of scatter in the data set. The mean deviations about

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

the mean (μ_1) and median ($\mathcal{M}$) for a random variable Y following ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ are given respectively as

$$
\begin{aligned}
MD(\mu_1) = E(|Y-\mu_1|) &= \int_0^\infty |y-\mu_1| g_\lambda(y)dy = 2\mu_1 G_\lambda(\mu_1) - 2\int_0^{\mu_1} y g_\lambda(y)dy \\
&= -2\left\{\lambda \sum_{i,j=0}^{\infty} \eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma_l(\frac{j+2}{\theta},(i+1)\alpha\mu_1^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}}} + \alpha\left(\frac{\Gamma_l(\frac{j+1}{\theta}+1,(i+1)\alpha\mu_1^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}+1}} + \gamma\frac{\Gamma_l(\frac{j+2}{\theta}+1,(i+1)\alpha\mu_1^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}+1}}\right)\right]\right\} \\
&+ 2\mu_1\left(1-(1+\gamma\mu_1)^{-\beta}e^{-\alpha\mu_1^\theta}\right)^\lambda
\end{aligned} \quad (2.20)
$$

and

$$
\begin{aligned}
MD(\mathcal{M}) = E(|Y-\mathcal{M}|) &= \int_0^\infty |y-\mathcal{M}| g_\lambda(y)dy = \mu_1 - 2\int_0^{\mathcal{M}} y g_\lambda(y)dy = \mu_1 \\
&- 2\left\{\lambda \sum_{i,j=0}^{\infty} \eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma_l(\frac{j+2}{\theta},(i+1)\alpha\mathcal{M}^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}}} + \alpha\left(\frac{\Gamma_l(\frac{j+1}{\theta}+1,(i+1)\alpha\mathcal{M}^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}+1}} + \gamma\frac{\Gamma_l(\frac{j+2}{\theta}+1,(i+1)\alpha\mathcal{M}^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}+1}}\right)\right]\right\}
\end{aligned} \quad (2.21)
$$

where $\Gamma_l(\xi, \varkappa) = \int_0^\varkappa z^{\xi-1}e^{-z}dz$ is the incomplete lower gamma function and the value of the median can be obtained by solving equation (2.12) if the values of $\gamma, \beta, \alpha, \theta$ and λ are known.

2.2.5 Lorenz and Bonferroni curves

The applications of the Lorenz and Bonferroni curves are found in poverty and income study in economics, medicine, lifetime analysis, demography and insurance. The Lorenz and Bonferroni curves for a random variable Y following ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ are defined respectively as

$$
\begin{aligned}
L(p) &= \frac{1}{E(Y)}\int_0^q y g_\lambda(y)dy = \frac{1}{\mu_1}\left(\mu_1 - \int_q^\infty y g_\lambda(y)dy\right) \\
&= 1 - \frac{\lambda \sum_{i,j=0}^{\infty} \eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma_u(\frac{j+2}{\theta},(i+1)\alpha q^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}}} + \alpha\left(\frac{\Gamma_u(\frac{j+1}{\theta}+1,(i+1)\alpha q^\theta)}{((i+1)\alpha)^{\frac{j+1}{\theta}+1}} + \gamma\frac{\Gamma_u(\frac{j+2}{\theta}+1,(i+1)\alpha q^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}+1}}\right)\right]}{\mu_1}
\end{aligned} \quad (2.22)
$$

and

$$
\begin{aligned}
B(p) &= \frac{1}{pE(Y)}\int_0^q y g_\lambda(y)dy = \frac{1}{p\mu_1}\left(\mu_1 - \int_q^\infty y g_\lambda(y)dy\right) \\
&= \frac{1}{p}\left(1 - \frac{\lambda \sum_{i,j=0}^{\infty} \eta_{i,j}\left[\left(\frac{\gamma\beta}{\theta}\right)\frac{\Gamma_u(\frac{j+2}{\theta},(i+1)\alpha q^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}}} + \alpha\left(\frac{\Gamma_u(\frac{j+1}{\theta}+1,(i+1)\alpha q^\theta)}{((i+1)\alpha)^{\frac{j+1}{\theta}+1}} + \gamma\frac{\Gamma_u(\frac{j+2}{\theta}+1,(i+1)\alpha q^\theta)}{((i+1)\alpha)^{\frac{j+2}{\theta}+1}}\right)\right]}{\mu_1}\right)
\end{aligned} \quad (2.23)
$$

where $q = G_\lambda^{-1}(p)$ and $E(Y) = \mu_1$ is the mean. It is obvious that $L(p) = pB(p)$.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

2.2.6 Order statistics, its cdf and raw moment

Let $Y_1, Y_2, ..., Y_\nu$ be ν continuous independent variables from ELW$(\gamma, \beta, \alpha, \theta, \lambda)$, then pdf of ω^{th} $(1 \leq \omega \leq \nu)$ order statistics is given by

$$\begin{aligned} g_{\omega;\nu}(y) &= \frac{\nu! g_\lambda(y)}{(\omega-1)!(\nu-\omega)!} \left[G_\lambda(y)\right]^{\omega-1} \left[1 - G_\lambda(y)\right]^{\nu-\omega} \\ &= \frac{\nu! g_\lambda(y)}{(\omega-1)!(\nu-\omega)!} \sum_{k=0}^{\omega-1} \binom{\omega-1}{k} (-1)^k \left[R_\lambda(y)\right]^{\nu+k-\omega} \\ &= \frac{\nu! \lambda}{(\omega-1)!(\nu-\omega)!} \sum_{k=0}^{\omega-1} \sum_{i=0}^{\infty} \binom{\omega-1}{k} \binom{\lambda-i}{i} (-1)^{k+i} \left(\gamma\beta + \alpha\theta y^{\theta-1}(1+\gamma y)\right) \\ &\quad (1+\gamma y)^{-\beta(\nu+k+i-\omega+1)-1} e^{-(\nu+k+i-\omega+1)\alpha y^\theta} \end{aligned} \tag{2.24}$$

The corresponding cdf for $g_{\omega;\nu}(y)$ of ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ is given as

$$\begin{aligned} G_{\omega;\nu}(y) &= \sum_{k=\omega}^{\nu} \binom{\nu}{k} \left[G_\lambda(y)\right]^k \left[1 - G_\lambda(y)\right]^{\nu-k} = \sum_{k=\omega}^{\nu} \sum_{j=0}^{k} \binom{\nu}{k} \binom{k}{j} (-1)^j \left[R_\lambda(y)\right]^{\nu+j-k} \\ &= \sum_{k=\omega}^{\nu} \sum_{j=0}^{k} \binom{\nu}{k} \binom{k}{j} (-1)^j (1+\gamma y)^{-\beta(\nu+j-k)} e^{-(\nu+j-k)\alpha y^\theta}. \end{aligned} \tag{2.25}$$

The r^{th} raw moments of $g_{\omega;\nu}(y)$ of ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ is evaluated from

$$\begin{aligned} E\left[Y_{\omega;\nu}^r\right] &= \int_0^\infty y^r g_{\omega;\nu}(y) dy = \frac{\nu! \lambda}{(\omega-1)!(\nu-\omega)!} \sum_{k=0}^{\omega-1} \sum_{i=0}^{\infty} \binom{\omega-1}{k} \binom{\lambda-i}{i} (-1)^{k+i} \\ &\quad \int_0^\infty y^r \left(\gamma\beta + \alpha\theta y^{\theta-1}(1+\gamma y)\right) (1+\gamma y)^{-\beta(\nu+k+i-\omega+1)-1} e^{-(\nu+k+i-\omega+1)\alpha y^\theta} dy \end{aligned}$$

which results as

$$\begin{aligned} E\left[Y_{\omega;\nu}^r\right] &= \frac{\nu! \lambda}{(\omega-1)!(\nu-\omega)!} \sum_{k=0}^{\omega-1} \sum_{i,j=0}^{\infty} \binom{\omega-1}{k} \binom{\lambda-i}{i} \binom{\beta(\nu+k+i-\omega+1)+j}{j} (-1)^{k+i+j} \\ &\left[\left(\frac{\gamma\beta}{\theta}\right) \frac{\Gamma(\frac{r+j+1}{\theta})}{((\nu+k+i-\omega+1)\alpha)^{\frac{r+j+1}{\theta}}} + \alpha \left(\frac{\Gamma(\frac{r+j}{\theta}+1)}{((\nu+k+i-\omega+1)\alpha)^{\frac{r+j}{\theta}+1}} + \gamma \frac{\Gamma(\frac{r+j+1}{\theta}+1)}{((\nu+k+i-\omega+1)\alpha)^{\frac{r+j+1}{\theta}+1}}\right)\right]. \end{aligned} \tag{2.26}$$

2.2.7 Residual and reversed residual lifetimes

Residual and reversed residual lifetime functions are important measures of reliability and life-testings for systems experiencing failures in diverse scientific fields. The residual lifetime function defines the lifetime remaining for a system beyond age $t \geq 0$ until failure time is known and it is the conditional random variable $Y_t = [(Y-t)/Y > t)]$. The reversed residual lifetime function (also known as the inactivity time) defines the elapsed lifetime for a system from its failure for which its lifetime is less than or equals to age $t \geq 0$. It is the conditional random variable $Y_t^* = [(t-Y)/Y \leq t)]$.
Let $m_r(t) = E\left[(Y-t)^r/Y > t)\right]$ and $m_r^*(t) = E\left[(t-Y)^r/Y \leq t)\right]$ denote the r^{th} moments for residual and reversed residual lifetime functions for a random variable Y following ELW$(\gamma, \beta, \alpha, \theta, \lambda)$,

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

then $m_r(t)$ and $m_r^*(t)$, $r = 1, 2, 3, ...$, are given as

$$m_r(t) = \frac{1}{S(t)} \int_t^{\infty} (y-t)^r g_\lambda(y) dy$$
$$= \frac{\lambda}{1 - \left(1 - (1+\gamma t)^{-\beta} e^{-\alpha t^\theta}\right)^\lambda} \sum_{k=0}^{r} \binom{r}{k} (-1)^k t^k \sum_{i,j=0}^{\infty} \binom{\lambda-1}{i} \binom{(i+1)\beta + j}{j} (-1)^{i+j} \gamma^i$$
$$\left[\left(\frac{\gamma\beta}{\theta}\right) \frac{\Gamma_u(\frac{r+j-k+1}{\theta}, (i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j-k+1}{\theta}}} + \alpha\left(\frac{\Gamma_u(\frac{r+j-k}{\theta}+1, (i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j-k}{\theta}+1}} + \gamma \frac{\Gamma_u(\frac{r+j-k+1}{\theta}+1, (i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j-k+1}{\theta}+1}}\right)\right]. \tag{2.27}$$

and

$$m_r^*(t) = \frac{1}{F(t)} \int_0^{t} (t-y)^r g_\lambda(y) dy$$
$$= \frac{\lambda}{\left(1 - (1+\gamma t)^{-\beta} e^{-\alpha t^\theta}\right)^\lambda} \sum_{k=0}^{r} \binom{r}{k} (-1)^{r+k} t^k \sum_{i,j=0}^{\infty} \binom{\lambda-1}{i} \binom{(i+1)\beta + j}{j} (-1)^{i+j} \gamma^i$$
$$\left[\left(\frac{\gamma\beta}{\theta}\right) \frac{\Gamma_l(\frac{r+j-k+1}{\theta}, (i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j-k+1}{\theta}}} + \alpha\left(\frac{\Gamma_l(\frac{r+j-k}{\theta}+1, (i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j-k}{\theta}+1}} + \gamma \frac{\Gamma_l(\frac{r+j-k+1}{\theta}+1, (i+1)\alpha t^\theta)}{((i+1)\alpha)^{\frac{r+j-k+1}{\theta}+1}}\right)\right] \tag{2.28}$$

respectively.

2.2.8 Statistical Inference for ELW($\gamma, \beta, \alpha, \theta, \lambda$)

The maximum likelihood estimation (MLE) is the most frequently used method among methods of estimating parameter(s) of distributions in statistical inference. The maximum likelihood estimates for parameters of ELW($\gamma, \beta, \alpha, \theta, \lambda$) are obtained using the procedures for the MLE method. Let a ν-sized random sample for a variable Y following ELW($\gamma, \beta, \alpha, \theta, \lambda$) be given as $y_1, y_2, ..., y_\nu$, then its log-likelihood function defined by $\psi_\nu = ln(l_\nu)$ becomes

$$\psi_\nu = \sum_{i=1}^{\nu} ln(g_\lambda(y_i)) = \nu ln\lambda + \sum_{i=1}^{\nu} ln(\gamma\beta + \alpha\theta y_i^{\theta-1}(1+\gamma y_i)) - (\beta+1)\sum_{i=1}^{\nu} ln(1+\gamma y_i) - \sum_{i=1}^{\nu} \alpha y_i^\theta$$
$$+ (\lambda-1)\sum_{i=1}^{\nu} ln(1-(1+\gamma y_i)^{-\beta} e^{-\alpha y_i^\theta}) \tag{2.29}$$

where l_ν is the likelihood function of the ν-sized random sample.
The first partial derivatives of equation (2.29) with respect to each of the parameters give the components of the score function, $\Psi = (\frac{\partial\psi_\nu}{\partial\gamma}, \frac{\partial\psi_\nu}{\partial\beta}, \frac{\partial\psi_\nu}{\partial\alpha}, \frac{\partial\psi_\nu}{\partial\theta}, \frac{\partial\psi_\nu}{\partial\lambda})$ which are equated to zero. The nonlinear equations are given as

$$\frac{\partial\psi_\nu}{\partial\gamma} = -(\beta+1)\sum_{i=1}^{\nu} \frac{y_i}{(1+\gamma y_i)} + (\lambda-1)\sum_{i=1}^{\nu}\left(\frac{\beta y_i(1+\gamma y_i)^{-(\beta+1)} e^{-\alpha y_i^\theta}}{1-(1+\gamma y_i)^{-\beta} e^{-\alpha y_i^\theta}}\right)$$
$$+ \sum_{i=1}^{\nu}\left(\frac{\beta + \alpha\theta y_i^{\theta-1}}{\gamma\beta + \alpha\theta y_i^{\theta-1}(1+\gamma y_i)}\right) = 0,$$

$$\frac{\partial\psi_\nu}{\partial\beta} = -\sum_{i=1}^{\nu} log(1+\gamma y_i) + (\lambda-1)\sum_{i=1}^{\nu}\left(\frac{(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}ln(1+\gamma y_i)}{1-(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}}\right) + \sum_{i=1}^{\nu}\left(\frac{\gamma}{\gamma\beta+\alpha\theta y_i^{\theta-1}(1+\gamma y_i)}\right) = 0,$$

$$\frac{\partial\psi_\nu}{\partial\alpha} = -\sum_{i=1}^{\nu} y_i^{\theta} + (\lambda-1)\sum_{i=1}^{\nu}\left(\frac{y_i^{\theta}(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}}{1-(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}}\right) + \sum_{i=1}^{\nu}\left(\frac{\theta y_i^{\theta-1}(1+\gamma y_i)}{\gamma\beta+\alpha\theta y_i^{\theta-1}(1+\gamma y_i)}\right) = 0,$$

$$\frac{\partial\psi_\nu}{\partial\theta} = -\sum_{i=1}^{\nu} y_i^{\theta}ln(y_i) + (\lambda-1)\sum_{i=1}^{\nu}\left(\frac{y_i^{\theta}(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}ln(y_i)}{1-(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}}\right) + \sum_{i=1}^{\nu}\left(\frac{(y_i^{\theta-1}(1+\gamma y_i))(1+\theta ln(y_i))}{\gamma\beta+\alpha\theta y_i^{\theta-1}(1+\gamma y_i)}\right) = 0,$$

and

$$\frac{\partial\psi_\nu}{\partial\lambda} = \frac{\nu}{\lambda} + \sum_{i=1}^{\nu} ln(1-(1+\gamma y_i)^{-\beta}e^{-\alpha y_i^{\theta}}) = 0.$$

The nonlinear equations are simultaneously solved by numerical method (Newton-Raphson or Nelder- Mead method) to obtain unique solutions for $\gamma, \beta, \alpha, \theta, \lambda$ which are the maximum likelihood estimates given as $\hat{\gamma}, \hat{\beta}, \hat{\alpha}, \hat{\theta}, \hat{\lambda}$. Also, the AdequacyModel package in R can be used to obtain $\hat{\gamma}, \hat{\beta}, \hat{\alpha}, \hat{\theta}, \hat{\lambda}$.
Given that $\hat{} = (\hat{\gamma}, \hat{\beta}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is the maximum likelihood estimate obtained for $= (\gamma, \beta, \alpha, \theta, \lambda)$, then the asymptotic distribution of $\sqrt{\nu}(\hat{} -) \xrightarrow{d} N_5(\underline{0}, I^{-1}())$ where $\underline{0} = (0,0,0,0,0)^T$ and $I() = [I_{\phi_k,\phi_j}]_{5\times 5} = \mathbb{E}\left(-\frac{\partial^2\psi_\nu}{\partial\gamma\partial\beta}\right), k,j = 1,2,3,4,5$ is the expected Fisher's information. If $\mathcal{J}(\hat{})$ replaces $I()$, then a valid asymptotic results exist with $\mathcal{J}(\hat{})$ as observed information matrix evaluated at $\hat{}$. The multivariate normal distribution given as $N_5(\underline{0}, \mathcal{J}^{-1}(\hat{}))$ is needed to construct approximate confidence regions and intervals for $\gamma, \beta, \alpha, \theta, \lambda$. The approximation of the total Fisher information matrix, $\nu I()$, is given as

$$\mathcal{J}_\nu() \approx -\begin{pmatrix} \mathfrak{J}_{\gamma\gamma} & \mathfrak{J}_{\gamma\beta} & \mathfrak{J}_{\gamma\alpha} & \mathfrak{J}_{\gamma\theta} & \mathfrak{J}_{\gamma\lambda} \\ \cdot & \mathfrak{J}_{\beta\beta} & \mathfrak{J}_{\beta\alpha} & \mathfrak{J}_{\beta\theta} & \mathfrak{J}_{\beta\lambda} \\ \cdot & \cdot & \mathfrak{J}_{\alpha\alpha} & \mathfrak{J}_{\alpha\theta} & \mathfrak{J}_{\alpha\lambda} \\ \cdot & \cdot & \cdot & \mathfrak{J}_{\theta\theta} & \Lambda_{\theta\lambda} \\ \cdot & \cdot & \cdot & \cdot & \mathfrak{J}_{\lambda\lambda} \end{pmatrix}$$

where the expressions for $\mathfrak{J}_{\gamma\gamma} = \frac{\partial^2\psi_\nu}{\partial\gamma^2}$, $\mathfrak{J}_{\gamma\beta} = \frac{\partial^2\psi_\nu}{\partial\gamma\partial\beta}, \ldots, \mathfrak{J}_{\lambda\lambda} = \frac{\partial^2\psi_\nu}{\partial\lambda^2}$ are presented in the appendix. For ω% level of significance, $100(1-\omega)$% aymptotic confidence intervals for $\gamma, \beta, \alpha, \theta$ and λ are given as $\hat{\gamma} \pm c\sqrt{\hat{\mathfrak{J}}_{\gamma\gamma}}$, $\hat{\beta} \pm c\sqrt{\hat{\mathfrak{J}}_{\beta\beta}}$, $\hat{\alpha} \pm c\sqrt{\hat{\mathfrak{J}}_{\alpha\alpha}}$, $\hat{\theta} \pm c\sqrt{\hat{\mathfrak{J}}_{\theta\theta}}$ and $\hat{\lambda} \pm c\sqrt{\hat{\mathfrak{J}}_{\lambda\lambda}}$, where $c = z_{\frac{\omega}{2}}$ is the standard normal critical value at $\frac{\omega}{2}$.

3 Applications of ELW distribution to real lifetime data

The flexibility and applicability of ELW$(\gamma, \beta, \alpha, \theta, \lambda)$ are presented in this study by fitting two lifetime data sets. Some nested and non-nested distributions are compared with the proposed distribution using well-known discrepancy criteria. The computations of parameter estimates (with standard errors in parentheses) and discrepancy criteria are achieved using AdequacyModel package in R software. The other non-nested competing distributions include exponentiated additive Weibull

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

(EAW)[4], exponentiated generalized Weibull Gompertz (EGWG)[5], exponentiated Weibull Weibull (EWW)[10], exponentiated exponentiated exponential-Weibull (EEEW)[1], new Weibull-Lomax (NWL)[24], Gompertz-Fréchet (GFr)[18] and Weibull inverse Lomax (WIL)[6]. The cdfs of the non-nested distributions are listed as follows.

$$F_{EWW}(y) = (1 - e^{-\gamma(e^{\alpha y^{\theta}}-1)^{\beta}})^{\lambda}, \gamma > 0, \beta > 0, \alpha > 0, \theta > 0, \lambda > 0.$$

$$F_{EEEW}(y) = (1 - e^{-\gamma(-ln(1-(1-e^{-\alpha y^{\theta}})^{\beta}))})^{\lambda}, \gamma > 0, \beta > 0, \alpha > 0, \theta > 0, \lambda > 0.$$

$$F_{EGWG}(y) = (1 - e^{-\alpha y^{\beta}(e^{\gamma y^{\theta}}-1)})^{\lambda}, \gamma > 0, \beta > 0, \alpha > 0, \theta > 0, \lambda > 0.$$

$$F_{EAW}(y) = (1 - e^{-\alpha y^{\theta}-\gamma y^{\beta}})^{\lambda}, \gamma > 0, \beta > 0, \alpha > 0, \theta > 0, \lambda > 0.$$

$$F_{NWL}(y) = 1 - e^{-\alpha(1+\gamma y)^{\lambda\beta}}, \gamma > 0, \beta > 0, \alpha > 0, \lambda > 0.$$

$$F_{GFr}(y) = 1 - e^{\frac{\gamma}{\lambda}(1-(1-e^{-(\frac{\alpha}{y})^{\beta}})^{-\lambda})}, \gamma > 0, \beta > 0, \alpha > 0, \lambda > 0.$$

$$F_{WIL}(y) = 1 - e^{-\alpha((1+\frac{\gamma}{y})^{\beta}-1)^{-\lambda}}, \gamma > 0, \beta > 0, \alpha > 0, \lambda > 0.$$

3.1 First data set: Exceedances of flood peaks from 1958–1984

The first data set represents 72 exceedances of flood peaks (in m^3/s) of Wheaton river, Canada for the years 1958–1984, rounded to one decimal place. The data set had been analysed by several researchers and recently by [3].
1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

The descriptive statistics of the first data set is summarized in the Table 4. Table 3 presents values for the parameter estimates and their standard errors (in parentheses), loglikelihood function, AIC, CAIC, W*, A* and KS test with p-value in parantheses for the competing distributions for data set 1.

Table 3: Parameter estimates, loglikelihood, information criteria and goodness-of-fit values of compared distributions for first data set

Model	γ (std. error)	β (std. error)	α (std. error)	θ (std. error)	λ (std. error)	-2loglik	AIC	CAIC	W*	A*	KS (p-value)
ELW	4.4325 (8.7984)	0.3556 (0.0944)	0.0054 (0.0041)	2.6203 (0.2068)	2.6203 (0.2068)	**495.7978**	**505.7978**	506.7069	**0.0421**	**0.2660**	**0.0682 (0.8908)**
LW	0.5957 (0.4511)	0.3000 (0.1452)	0.0046 (0.0032)	1.6884 (0.1845)	—	498.0936	506.0936	**506.6907**	0.0601	0.3762	0.0728 (0.8395)
ELE	20.8407 (87.6556)	0.2578 (0.1395)	0.0650 (0.0134)	—	3.8639 (7.0962)	499.5006	507.5007	508.0977	0.0992	0.5578	0.0823 (0.7139)
EL	0.0201 (0.0134)	4.6291 (2.4147)	—	—	0.9361 (0.1598)	505.4080	511.4081	511.7610	0.1717	0.9656	0.1005 (0.4615)
Weibull	—	—	0.1097 (0.0302)	0.9008 (0.0855)	—	502.9974	506.9973	507.1712	0.1380	0.7856	0.1051 (0.4043)
Lomax	0.0086 (0.0058)	10.4711 (6.6403)	—	—	—	504.6010	508.6010	508.6010	0.1504	0.8537	0.1269 (0.1963)
EWW	3.4518 (1.3134)	0.1419 (0.0833)	0.0414 (0.0654)	1.2430 (0.4165)	18.1352 (25.9622)	498.1832	508.1833	509.0924	0.0648	0.3910	0.0778 (0.7758)
EEEW	1.3145 (0.5161)	0.1275 (0.2067)	0.0078 (0.0078)	1.4501 (0.2596)	4.8491 (8.7516)	501.4988	511.4987	512.4078	0.1097	0.6454	0.1012 (0.4527)
EGWG	0.0259 (0.0796)	1.3517 (2.5353)	0.6350 (2.4729)	0.0211 (2.4238)	0.5261 (0.3247)	502.0509	512.0509	512.9600	0.1056	0.6444	0.1075 (0.3764)
EAW	0.0037 (0.0026	1.7125 (0.1928)	2.3080 (1.2167)	0.1650 (0.0721)	20.2357 (25.7797)	497.4311	507.4311	508.3402	0.0629	0.3753	0.0820 (0.7182)
WL	25.4448 (31.2078)	2.0638 (4.9081)	0.0807 (0.4791)	—	0.4417 (1.0502)	503.7300	511.7301	512.3271	0.1401	0.8067	0.1073 (0.3781)
GFr	0.6672 (0.9991)	0.3400 (0.1445)	9.0279 (22.7279)	—	3.6353 (2.9812)	500.7694	508.7694	509.3664	0.1158	0.6495	0.1020 (0.4424)
WIL	0.0616 (0.2136)	6.4419 (16.7628)	0.0577 (0.0763)	—	0.8544 (0.0941)	501.8020	509.8020	510.3990	0.1327	0.7353	0.0946 (0.5393)

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

Table 4: Summary of the exceedances of flood peaks

Statistics	Minimum	Maximum	Mean	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	Kurtosis	Skewness
1st data	0.1000	64.0000	12.2040	2.1250	9.5000	20.1250	5.8895	1.4725

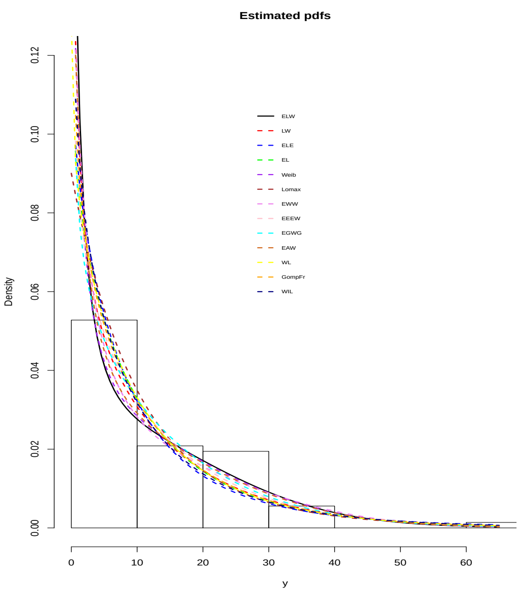

Figure 3: Histogram and estimated pdfs for flood peaks exceedances

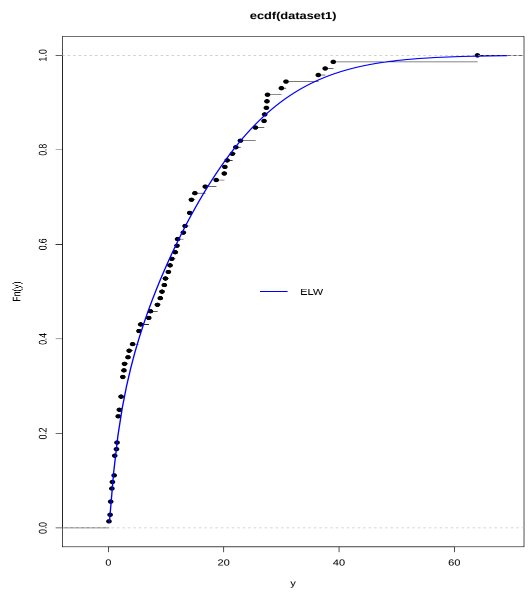

Figure 4: Ecdf vs ELW cdf for flood peaks exceedances

The approximate 95% confidence intervals for $\gamma, \beta, \alpha, \theta$ and λ are given by 6.0211 ± 32.9370, 0.3589 ± 0.1889, 0.0055 ± 0.0082, 1.6541 ± 0.4077 and 3.0143 ± 7.8633.

3.2 Second data set: Remission times of bladder cancer patients

The second data set represents a random sample of 128 bladder cancer patients with their remission times reported in months [10]. Recent analysis of the data set have been considered by [11]. Table 5 presents values for the parameter estimates and other discrecancy criteria for the competing distributions for data set 2. Also, descriptive statistics for the data sets are presented in the Table 6.

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 6.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 5: Parameter estimates, loglikelihood, information criteria and goodness-of-fit values of compared distributions for second data set

Model	γ (std. error)	β (std. error)	α (std. error)	θ (std. error)	λ (std. error)	-2loglik	AIC	CAIC	W*	A*	KS (p-value)
ELW	0.0706 (0.0986)	3.2982 (2.5767)	0.5850 (3.2316)	0.1893 (0.4987)	4.9315 (22.4444)	**813.5626**	823.5626	824.0544	**0.0153**	**0.0997**	**0.0301 (0.9998)**
LW	0.0235 (0.0263)	3.2651 (4.6489)	0.0278 (0.0394)	1.1716 (0.1968)	—	822.2730	830.2731	830.5983	0.1010	0.6620	0.0841 (0.3260)
ELE	0.1066 (0.0839)	1.2375 (1.1012)	0.0626 (0.0404)	—	1.5625 (0.3089)	816.5136	824.5135	824.8387	0.0505	0.3330	0.0492 (0.9156)
EL	0.0450 (0.0269)	4.3350 (1.8328)	—	—	1.6386 (0.2816)	815.0204	**821.0203**	**821.2139**	0.0311	02131	0.0408 (0.9835)
Weibull	—	—	0.0947 (0.0191)	1.0513 (0.0675)	—	823.7850	827.7849	827.8809	0.1383	0.8444	0.0720 (0.5201)
Lomax	0.0125 (0.0067)	9.6614 (4.7930)	—	—	—	823.4134	827.4135	827.4135	0.0721	0.4483	0.1043 (0.1237)
EWW	4.3603 (22.4011)	4.1407 (12.4123)	0.4818 (0.9370)	0.1085 (0.3231)	3.5453 (2.5429)	816.4802	826.4803	826.9721	0.0498	0.3317	0.0454 (0.9546)
EEEW	4.4423 (14.6999)	3.2084 (6.7209)	0.4072 (0.7651)	0.4333 (1.9585)	1.3663 (0.4667)	816.1418	826.1417	826.6335	0.0460	0.3069	0.0446 (0.9613)
EGWG	0.2632 (5.3650	0.5869 (1.3597)	1.5572 (36.0455)	0.0556 (1.1467)	2.8803 (1.4461)	816.4612	826.4612	826.9530	0.0496	0.3302	0.0454 (0.9544)
EAW	0.7623 (0.5018)	0.5483 (0.1405)	0.9397 (1.7240)	0.0177 (0.1500)	14.3402 (29.2956)	814.7862	824.7862	825.2780	0.0278	0.1844	0.0376 (0.9936)
WL	16.7811 (14.7253)	1.7275 (1.2112)	0.0439 (0.0926)	—	0.6160 (0.4311)	825.1196	833.1195	833.4447	0.1457	0.8868	0.0743 (0.4804)
GFr	0.0193 (0.0207)	1.0534 (0.4528)	0.1986 (0.1580)	—	0.9700 (0.4435)	822.8076	830.8077	831.1329	0.1310	0.8151	0.0733 (0.4970)
WIL	21.5206 (14.3133)	4.1645 (3.6316)	7.1872 (3.6600)	—	0.3742 (0.3381)	814.3908	822.3909	822.7161	0.0232	0.1620	0.0356 (0.9969)

International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

Table 6: Summary of the remission times of bladder cancer patients

Statistics	Minimum	Maximum	Mean	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	Kurtosis	Skewness
2nd data	0.0800	79.0500	9.2090	3.3480	6.2800	11.6780	19.3942	3.3987

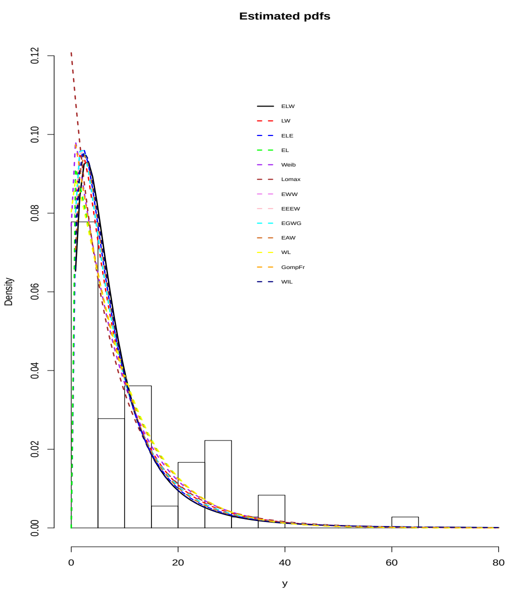

Figure 5: Histogram and estimated pdfs for remission times of bladder cancer patients

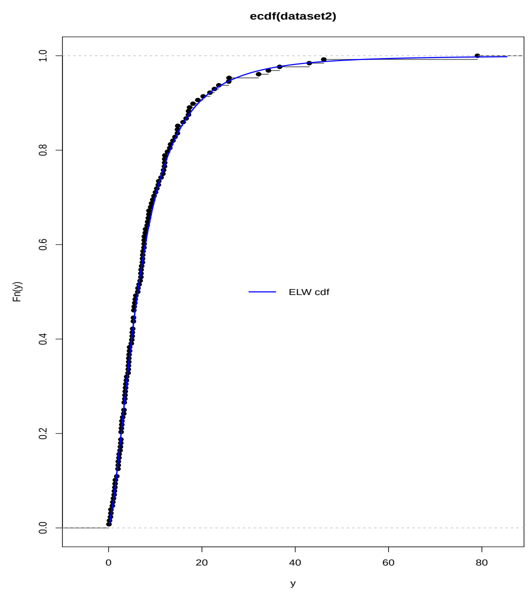

Figure 6: Ecdf vs ELW cdf for remission times of bladder cancer patients

The approximate 95% confidence intervals for $\gamma, \beta, \alpha, \theta$ and λ are given by 0.0706 ± 0.1932, 3.2982 ± 5.0503, 0.5850 ± 6.3339, 0.1893 ± 0.9774 and 4.9315 ± 43.9910.

4 Discussion of Results

From Tables 5 and 8, it is evident that the ELW distribution is a model that better fits the lfetime data sets used in the application than the other compared distributions. The preference of the proposed distribution over other compared distributions is presented in Figures 3 and 5. These figures show the histograms representing the data sets and fitted densities of the compared distributions obtained from their estimated parameter values. The ELW distribution exhibits the lowest value for three goodness-of fit tests (W*, A* and KS) amongst the compared distributions for the two data sets. The same finding can be presented from the p-values of the KS test. The p-value of the KS test of the ELW distribution is the highest among the p-values presented in the two tables, indicating that the ELW distribution better fits the two data sets than remaining competing models. Hence, its superiority over the remaining competing dstributions and applicability of the proposed distribution in lifetime analysis are presented.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

5 Conclusion

The paper introduces a new flexible five-parameter distribution called the exponentiated Lomax-Weibull (ELW) distribution using combination of two construction methods of generating lifetime distributions in literature. We generated some new and existing distributions as submodels. Explicit mathematical expressions are derived for several properties of the ELW distribution which include the moments, quantile function, mean deviations, residual and reversed residual liifetimes, order statistics and entropy measures. Estimates for the proposed distribution could be obtained with maximum likelihood estimation approach which is presented in the study. Applications of the ELW distribution to flood peaks and bladder cancer data sets to illustrate its flexiblity and usefulness are considered. The results of the analysis show the superiority of the ELW distribution over some compared distributions, since it provides better fits for the two lifetime datasets than the distributions used in the paper. The 95% confidence intervals for point estimates of the $\gamma, \beta, \alpha, \theta$ and λ are presented for the data sets employed. It is desired that the proposed five-parameter distribution will attract wider applications in scientific areas of research with lifetime analysis.

Competing financial interest

The authors wish to declare no competing financial interest.

References

[1] Abd El-Bar, A.M.T. and Dessoky, S.A. (2017). A new five parameter lifetime distribution: Properties and Application. *Journal of Advance in Mathematics*, vol.13, pp.7205-7218.

[2] Abdul-Moniem, I.B. and Abdel-Hameed, H.F. (2012). On exponentiated Lomax distribution. *International Journal of Mathematical Archive*, vol.3, pp.2144-2150.

[3] Afify, A.Z. and Alizadeh, M. (2020). The odd Dagum family of distributions: Properties and Applications. *Journal of Applied Probability and Statistics*, vol.15, pp.45-72.

[4] Ahmad, A.E-B.A. and Ghazal, M.G.M (2020). Exponentiated additive Weibull distribution. *Reliability Engineering and System Safety*, vol.193, issue c:s0951832019301656.

[5] El-Bassiouny, A.H., EL-Damcese, M.A., Mustafa, A. and Eliwa, M. S. (2017). Exponentiated generalized Weibull Gompertz distribution with Applications in Survival Analysis. *Journal of Statistics, Applocations and Probability*, vol.6, pp.7-16.

[6] Falgore, J.Y., Doguwa, S.I. and Isah, A. (2019). The Weibull-Inverse Lomax distribution with application on bladder cancer. *Biometrics and Biostatistics International Journal*, vol.8, pp.195-202.

[7] Gompertz, B. (1825). On the nature of the function expressiive of the law of human mortality and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society of London*, vol.115, pp.513-583.

[8] Gupta, R.C., Gupta, P. and Gupta, R.D. (1998). Modeling failure time data by Lehmann alternatives. *Communications in Statistics-Theory and Methods*, vol.27, pp.887-904.

[9] Hassan, A.S. and Abd-Allah, M. (2018). Exponentiated Weibull-Lomax distribution: Properties and Estimation. *Journal of Data Science*, vol.16, pp.227-298.

International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

[10] Hassan, A. and Elgarhy, M. (2019). Exponentiated Weibull Weibull distribution: Statistical Properties and Applications. *Gazi University Journal of Science*, vol.32, pp.616-635.

[11] Lee, C., Famoye, F. and Alzaatreh. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *WIREs Computational Statistics*, vol.5, pp. 219-238.

[12] Lee, E.T. and Wang, J.W. (2003). Statistical methods for survival data analysis. *John Wiley and Sons, Inc*, Hoboken, New Jersey, 3ed.

[13] Liu, Y., Ilyas, M., Khosa, S.K., Muhmoudi, E., Ahmad, Z., Khan, O.M. and Hamedani, G.G. (2020). A flexible reduced Logarithmic-X family of distributions with biomedical analysis. *Computational and Mathematical Methods in Medicine*, vol.2020.4373595. https://doi.org/10.1155/2020/4373595.

[14] Lomax, K.S. (1954). Business failures: Another example of the analysis of failure data. *Journal of the American Statistical Association*, vol.49, pp.847-852.

[15] Makubate, B., Dingalo, N., Oluyede, B.O. and Fagbamigbe, A.F. (2018). The exponentiated Burr XII Weibull distribution: Model, Properties and Applications. *Journal of Data Science*, vol.16, pp.431-462.

[16] Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, vol.42, pp.299-302.

[17] Mudholkar, G.S., Srivastava, D.K. and Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus motor failure data. *Technometric*, vol.37, pp.436-445.

[18] Oguntunde, P.E., Khaleel, M.A., Ahmed, M.T. and Okagbue, H.I. (2019). The Gompertz Frechet distribution: Properties and Applications. *Cogent Mathematics and Statistics*, vol.6, DOI:10.1080/25742558.2019.1568662.

[19] Oluyede, B.O, Makubate, B., Warahena-Liyanage, G. and Ayoku, S. (2018). A new model for lifetime data: The Burr XII-Weibull distribution: Properties and Applications. *Unpublished Manuscript*.

[20] Osagie, S.A. and Nosakhare, F.U. (2020). Properties and Application of a New Three-Parameter Lomax-Exponential Distribution. *Journal of the Nigerian Association of Mathematical Physics*, vol.57, pp.89-94.

[21] Osagie, S.A. and Osawe, N.L. (2020). A New Four-Parameter Distribution with Structural Properties And Application. *Journal of the Nigerian Association of Mathematical Physics*, vol.57, pp.105-112.

[22] Osagie, S.A. and Osemwenkhae, J.E. (2020). Lomax-Weibull Distribution with Properties and Applications in Lifetime Analysis. *International Journal of Mathematical Analysis and Optimization: Theory and Applications*, vol.2020, pp.718-732.

[23] Pham, H. and Lai, C-D. (2007). On recent generalizations of the Weibull distribution. *IEEE Transactions on Reliability*, vol.56, pp.454-458.

[24] Shabbir, M., Noor, N., Riaz, A. and Gull, H. (2014). The New Weibull Lomax distribution. *imperial Journal of Interdisciplinary Research*, vol.3, pp.1881-1885.

[25] Weibull, W. (1951). A Statistical Distribution Function with Wide Applicability. *Journal of Applied Mechanics*, vol.18, pp.293-297.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

[26] Xie, M. and Lai, C.D. (1995). Reliability analysis using an additive Weibull model with bathtub shaped failure rate function. *Reliability Engineering and System Safety*, vol.52, pp.87-93.

Appendix
If $K_i = (1+\gamma y_i)$, then the second partial derivatives of $\psi_\nu = ln(l_\nu)$ with respects to the parameters of interest are given as the elements of the approximation of the total Fisher information matrix, $\mathcal{J}_\nu()$. The mathematical expressions for the elements of $\mathcal{J}_\nu()$ are given as

$$\mathfrak{J}_{\gamma\gamma} = (\beta+1)\sum_{i=1}^{\nu}\frac{y_i^2}{K_i^2} - \sum_{i=1}^{\nu}\left(\frac{\beta+\alpha\theta y_i^\theta}{\gamma\beta+\alpha\theta y_i^{\theta-1}K_i}\right)^2 + (\lambda-1)$$
$$\left[\sum_{i=1}^{\nu}\frac{\beta {y_i}^2 {K_i}^{-(\beta+2)}e^{-\alpha y_i^\theta}(1+\beta-K_i^{-\beta}e^{-\alpha y_i^\theta})}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2}\right],$$

$$\mathfrak{J}_{\gamma\beta} = -\sum_{i=1}^{\nu}\frac{y_i}{K_i} + \sum_{i=1}^{\nu}\frac{1}{\gamma\beta+\alpha\theta y_i^{\theta-1}K_i} - \sum_{i=1}^{\nu}\frac{\gamma(\beta+\alpha\theta y_i^\theta)}{\gamma\beta+\alpha\theta y_i^{\theta-1}K_i^2} + (\lambda-1)$$
$$\left[\sum_{i=1}^{\nu}\frac{y_i {K_i}^{-(\beta+1)}e^{-\alpha y_i^\theta}(1-\beta-K_i^{-\beta}e^{-\alpha y_i^\theta})}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2}\right],$$

$$\mathfrak{J}_{\gamma\alpha} = -\sum_{i=1}^{\nu}\frac{\beta\theta y_i^{\theta-1}(K_i-\gamma y_i)}{(\gamma\beta+\alpha\theta y_i^{\theta-1}K_i)^2} + (\lambda-1)\left[\sum_{i=1}^{\nu}\frac{\beta y_i^{\theta-1}{K_i}^{-(\beta+1)}e^{-\alpha y_i^\theta}}{(1-K_i^{-\beta}e^{-\alpha y_i^\theta})^2}\right],$$

$$\mathfrak{J}_{\gamma\theta} = -\alpha\beta\sum_{i=1}^{\nu}\frac{y_i^{\theta-1}(1+\theta K_i ln y_i)(K_i-\gamma y_i)}{\gamma\beta+\alpha\theta y_i^{\theta-1}K_i} - \alpha\beta(\lambda-1)$$
$$\left[\sum_{i=1}^{\nu}\frac{y_i^{\theta+1}{K_i}^{-(\beta+1)}e^{-\alpha y_i^\theta}ln y_i}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2}\right],$$

$$\mathfrak{J}_{\gamma\lambda} = \beta\sum_{i=1}^{\nu}\frac{y_i {K_i}^{-(\beta+1)}e^{-\alpha y_i^\theta}}{1-K_i^{-\beta}e^{-\alpha y_i^\theta}}, \mathfrak{J}_{\beta\lambda} = \sum_{i=1}^{\nu}\frac{{K_i}^{-\beta}e^{-\alpha y_i^\theta}lnK_i}{1-K_i^{-\beta}e^{-\alpha y_i^\theta}}$$

$$\mathfrak{J}_{\beta\beta} = -\sum_{i=1}^{\nu}\frac{\gamma^2}{(\gamma\beta+\alpha\theta y_i^{\theta-1}K_i)^2} - (\lambda-1)\sum_{i=1}^{\nu}\frac{{K_i}^{-\beta}e^{-\alpha y_i^\theta}ln{K_i}^2}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2},$$

$$\mathfrak{J}_{\beta\alpha} = -\sum_{i=1}^{\nu}\frac{\gamma\theta y_i^{\theta-1}K_i}{(\gamma\beta+\alpha\theta y_i^{\theta-1}K_i)^2} - (\lambda-1\sum_{i=1}^{\nu}\frac{y_i^\theta {K_i}^{-\beta}e^{-\alpha y_i^\theta}lnK_i}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2},$$

$$\mathfrak{J}_{\beta\theta} = -\gamma\alpha\sum_{i=1}^{\nu}\frac{y_i^{\theta-1}K_i(1+\theta ln y_i)}{(\gamma\beta+\alpha\theta y_i^{\theta-1}K_i)^2} - (\lambda-1)\sum_{i=1}^{\nu}\frac{y_i^\theta {K_i}^{-\beta}e^{-\alpha y_i^\theta}(ln y_i)(lnK_i)}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2},$$

$$\mathfrak{J}_{\alpha\alpha} = -\theta^2\sum_{i=1}^{\nu}\frac{y_i^{2(\theta-1)}K_i^2}{(\gamma\beta+\alpha\theta y_i^{\theta-1}K_i)^2} - (\lambda-1)\sum_{i=1}^{\nu}\frac{y_i^{2\theta}{K_i}^{-\beta}e^{-\alpha y_i^\theta}}{\left(1-K_i^{-\beta}e^{-\alpha y_i^\theta}\right)^2},$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 814 - 833.
doi.org/10.6084/m9.figshare.13359584.

$$\mathfrak{I}_{\alpha\theta} = -\sum_{i=1}^{\nu} y_i^{\theta} lny_i + \sum_{i=1}^{\nu} \frac{\gamma\beta y_i^{\theta-1} K_i (1+\theta lny_i)}{(\gamma\beta + \alpha\theta y_i^{\theta-1} K_i)^2} + (\lambda-1)$$
$$\left[\sum_{i=1}^{\nu} \frac{y_i^{\theta} K_i{}^{-\beta} e^{-\alpha y_i^{\theta}} lny_i (1-\alpha y_i - K_i^{-\beta} e^{-\alpha y_i^{\theta}})}{\left(1-K_i^{-\beta} e^{-\alpha y_i^{\theta}}\right)^2}\right],$$

$$\mathfrak{I}_{\theta\theta} = -\alpha \sum_{i=1}^{\nu} y_i lny_i^2 + \sum_{i=1}^{\nu} \frac{\alpha y_i^{\theta-1} K_i \left[\alpha y_i^{\theta-1} K_i (\theta^2(1-(lny_i)^2)-1) + 2\gamma\beta lny_i(1+\theta)\right]}{(\gamma\beta + \alpha\theta y_i^{\theta-1} K_i)^2} + \alpha(\lambda-1)$$
$$\left[\sum_{i=1}^{\nu} \frac{y_i^{\theta} K_i{}^{-\beta} e^{-\alpha y_i^{\theta}} lny_i^2 (1-\alpha y_i^{\theta} - K_i^{-\beta} e^{-\alpha y_i^{\theta}})}{\left(1-K_i^{-\beta} e^{-\alpha y_i^{\theta}}\right)^2}\right],$$

$$\mathfrak{I}_{\alpha\lambda} = \sum_{i=1}^{\nu} \frac{y_i^{\theta} K_i{}^{-\beta} e^{-\alpha y_i^{\theta}}}{1-K_i^{-\beta} e^{-\alpha y_i^{\theta}}}, \mathfrak{I}_{\theta\lambda} = \alpha \sum_{i=1}^{\nu} \frac{y_i^{\theta} K_i{}^{-\beta} e^{-\alpha y_i^{\theta}} lny_i}{1-K_i^{-\beta} e^{-\alpha y_i^{\theta}}}, \mathfrak{I}_{\lambda\lambda} = \frac{-\nu}{\lambda^2}.$$