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An Introduction to Ω-Subgroup

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Abstract

In the language of Ω-groupoid we introduce Ω-subgroup, where a groupoid is an algebraic structure endow with one binary operation. Ω-subgroup is defined, as a generalization of the classical subgroup. In this case it was shown that the properties of Ω-groups are inherent in their Ω-subgroups. We then introduce and define the notions: center of an Ω-group, centralizers and normalizers of an Ω-subset of an Ω-group. Furthermore we investigate and prove some of the properties of these notions as in the case of classical group theory. .

Keywords: Ω-set; Ω-groupoid; Ω-group; Ω-subgroup; Ω-Equality; Complete lattice.
MSC2010: 08A72, 06D72.

1 Introduction

1.1 Historical remarks

Fuzzy set was introduced by Zadeh [1]. Since then numerous works by researchers have been ongoing in various areas of abstract algebra and related fields in the framework of fuzzy setting. Notably, this concept was quickly adopted by Goguen [2], who introduced the notion of L-fuzzy set, Sanchez [3] and Brown [4], generalized the concept of fuzzy set, in which case the unit interval on the real line as introduced by Zadeh [1] was replaced by a suitable partially ordered set (lattice) as the codomain of the fuzzy membership function.
Fuzzy groups and fuzzy semigroups as related concept were introduced and investigated early in the beginning of fuzzy era. The first appearance of fuzzy group was by Rosenfeld [5] . Thereafter, there were many other papers, see the monograph [6], and references cited there. For fuzzy semigroups, monograph [7] by Malik et al. gave a comprehensive overview of all the results up to 2003. Among numerous results, we can mention paper [8] where the structure of fuzzy subgroups of a group was investigated, paper [9] where it was shown that a fuzzy subgroupoid of a group need not be a fuzzy group.
In this research the topic investigated is some algebraic aspects of Ω-valued algebraic structures, with focus on groups. Ω is a complete lattice.

Our research originates in fuzzy structures and in Ω-sets. Ω-sets, as an intention for modeling intuitionistic logic, appeared 1979, in the paper [10] by Fourman and Scott. An Ω-set is a nonempty set A equipped with an Ω-valued equality E, with truth values in a complete Heyting algebra Ω. E is considered to be a symmetric and transitive function from A^2 to Ω. Both theories are well related in their basic ideals since they both deal with the notion of belonging in set-theoretic and logical sense. Quickly we want to mention that Ω-set in its interpretation is not totally equivalent to that of fuzzy. In fuzzy the notion of subset is generalized by a function, while in Ω-set approach we deal with the so-called "partial elements", where E is not reflexive. Ω-sets have been further applied to non-classical predicate logics, and also to theoretical foundations of the fuzzy set theory [11, 12].
Dealing with Ω-structures we use Ω-sets and in our research Ω is a complete lattice (not necessarily a Heyting algebra). The main reason for this membership values structure is that it allows the use of cut-sets as a tool appearing in the fuzzy set theory. In this setting, main algebraic and set-theoretic notions, and their properties can be generalized from their classical origin to the lattice-valued framework [13]. So we deal also with lattice-valued structures and Ω-sets as basic objects . These were developed within the fuzzy set theory
Since Goguen [2] replaced the unit interval with a complete lattice. This approach has been widely used for dealing with algebraic topics (see e.g., [14], then also [15–17]), and with the lattice-valued topology (starting with [18] and many others). In the recent decades, along with the development of the fuzzy logic, a complete lattice as a membership (truth values) structure is often replaced by a complete residuated lattice (see e.g., [19]). But then the cut structures do not keep algebraic properties satisfied on the basic fuzzy structure.
As a generalization of the classical equality we use lattice-valued equality. This was introduced into fuzzy mathematics by Höhle in his paper [20]. Afterwards several authors have used this notion in the investigation of fuzzy functions and fuzzy algebraic structures notably among them are: Demirci [21], Bělohlávek and V. Vychodil [22] and others.
Identities for lattice-valued structures with the fuzzy equality we use in this work were introduced in [23] and developed in [24–27]. Recently, several applications of it have appeared in publication [28–32]. Although similar notion first appeared in [22] . In this framework, an identity holds if the corresponding lattice-theoretic formula is fulfilled. What is new in this approach is that an identity may hold on a lattice-valued algebra, while the underlying classical algebra does not satisfy the analogue classical identity.
In [33] the authors introduced and investigated the notion of Ω-group in the language of Ω-groupiod. In this case the basic algebra is a groupoid with single binary operation. The authors proved that this study of Ω-groups can be equivalent to that of Ω-groups in the language of three operation as presented in [33].
In the present work we introduce and investigate Ω-subgroup arising from the notion of Ω-groups introduced [33]. We presented that the lattice-valued identities which holds in an Ω-group also hold in the Ω-subgroup. Then the notion of Ω-centralizer, Ω-normalize and Ω-center of an Ω- group was introduced and we proved that the sets of Ω-centralizer, Ω-normalize and Ω-center of an Ω- group form Ω-subgroup of the Ω-group , like in the classical group theory.

2 Preliminaries

In this section, we give some notations, definitions and propositions that would be needed in this paper. We refer to [34, 35] for general lattice theory.

2.1 Lattices and Algebras

We present a **complete lattice** as a complete poset. A poset (,) is complete if for every subset M of Ω both *sup* M and *inf* M exist (in Ω). The elements $supM$ and $infM$ will be denoted by $\bigvee M$ and $\bigwedge M$, respectively.
A complete lattice possesses the least and the greatest elements 0 and 1, respectively. A meet

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and a join of a two-element subset $\{a, b\}$ of Ω are binary operations, denoted by $a \wedge b$ and $a \vee b$, respectively.
A **language** (or a **type**) of algebras is a set $\mathcal{F}$ of function symbols such that a nonnegative integer n is assigned to each member f of $\mathcal{F}$. This integer is called the arity (or rank) of f, and f is said to be an n-ary function symbol.
Generally, an **algebra** of type $\mathcal{F}$ is a structure $\mathcal{A} = \langle A, F \rangle$, where A is a nonempty set and F is a set of (fundamental) operations on A. An n-ary operation in F corresponds to an n-ary symbol in the language. A **subalgebra** of $\mathcal{A}$ is an algebra of the same type, defined on a subset of A, closed under the operations in F. **Terms** in a language are regular expressions constructed by the variables and operational symbols see [36]. If $t(x_1, \ldots, x_n)$ is a term in the language of an algebra $\mathcal{A}$, then we denote in the same way the corresponding term-operation $A^n \to A$ on $\mathcal{A}$. An **identity** in a language is a formula $t_1 \approx t_2$, where t_1, t_2 are terms in the same language. An identity $t_1(x_1, \ldots, x_n) \approx t_2(x_1, \ldots, x_n)$ is said to be **valid** on an algebra $\mathcal{A} = \langle A, F \rangle$, or that $\mathcal{A}$ **satisfies** this identity, if for all $a_1, \ldots, a_n \in A$, the equality $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$ holds. An equivalence relation ρ on A which is compatible with respect to all fundamental operations, meaning that $x_i \rho y_i, i = 1, \ldots, n$ imply $f(x_1, \ldots, x_n) \rho f(y_1, \ldots, y_n)$, is a **congruence** relation on $\mathcal{A}$.

Suppose ρ is a congruence on $\mathcal{A}$, then for $a \in A$, the **congruence class** of a, $[a]_\rho$, is defined by

$$[a]_\rho := \{x \in A \mid (a, x) \in \rho\}$$

and the quotient algebra $\mathcal{A}/\rho$ is defined by

$$\mathcal{A}/\rho := (A/\rho, F),$$

where $A/\rho = \{[a]_\rho \mid a \in A\}$, and the operation on classes are defined by representatives.

An algebra $\mathcal{G} = (G, *)$ with a single binary operation is called a **groupoid**.
An element e in a groupoid $\mathcal{G} = (G, *)$ is said to be a **neutral element** if for every $x \in G$, $x * e = e * x = x$.
A **semigroup** is a groupoid $\mathcal{G} = (G, *)$ that fulfills the **associative identity**:

$$x * (y * z) \approx (x * y) * z.$$

A **monoid**, G is a semigroup with a neutral element.
For $x \in G$ its **inverse** denoted by $x^{-1} \in G$ is referred to as **inverse element**, of x in a monoid G, such that

$$x * x^{-1} = x^{-1} * x = e.$$

The inverse of every element $x \in G$ is unique.
A **group**, $\mathcal{G} = (G, *)$ is a monoid in which every element possesses an (unique) inverse. The following formula defines a neutral element and inverses:

$$(\exists z)(\forall x)(x * z = z * x = x \wedge (\exists y)(x * y = y * x = z)) : \tag{2.1}$$

in addition to the associative identity.
In universal algebra a group can equivalently be defined as an algebra with three operations, $(G; *; x^{-1}; e)$: binary operation $*$, unary operation x^{-1} and a nullary operation, constant, e, so that the following identities hold:

$$x * (y * z) \approx (x * y) * z; \tag{2.2}$$

$$x * e \approx x \approx e * x; \tag{2.3}$$

$$x * x^{-1} \approx e \approx x^{-1} * x. \tag{2.4}$$

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It is clear that the only identity in the definition of a group as a groupoid, $\mathcal{G} = (G, *)$ is the associativity property.
In the language with one operation properties of the neutral element and inverses, as a formula (2.1), contain existential quantifiers. In the language with three operations, all three formulas, (2.2, 2.3, 2.4), defining a group are identities. Still, these notions describing the structure of a group in different languages are equivalent, as follows. If $\mathcal{G} = (G, *)$ is a group as a groupoid with the neutral element e and inverse x^{-1} for $x \in G$, then $(G, *, ^{-1}, e)$ is a group in the language with three operations. Conversely, if $(G, *, ^{-1}, e)$ is a group in the corresponding language, then also the groupoid $\mathcal{G} = (G, *)$ is a group.

2.2 Ω-valued functions and relations

An **Ω-valued function** μ on a nonempty set A is a mapping $\mu : A \rightarrow$, where $(,)$ is a complete lattice. This notion can be related to fuzzy set on A. If μ and ν are Ω-valued functions on A, then ν is said to be a **fuzzy subset** of μ, if for all $x \in A$ $\nu(x)\mu(x)$.
For $p \in$, a **cut set** or a p-**cut** of an Ω-valued function $\mu : A \rightarrow$ is a subset μ_p of A which is the inverse image of the principal filter in Ω, generated by p:

$$\mu_p = \mu^{-1}(\uparrow(p)) = \{x \in X \mid \mu(x)p\}.$$

An **Ω-valued** (binary) **relation** R **on** A is an Ω-valued function on A^2, i.e., it is a mapping $R : A^2 \rightarrow$.
R is **symmetric** if

$$R(x, y) = R(y, x) \text{ for all } x, y \in A; \tag{2.5}$$

R is **transitive** if

$$R(x, y)R(x, z) \wedge R(z, y) \text{ for all } x, y, z \in A. \tag{2.6}$$

Observe that an Ω-valued symmetric and transitive relation R on A fulfills the **strictness** property (see [12]):

$$R(x, y)R(x, x) \wedge R(y, y). \tag{2.7}$$

Likewise we say an Ω-valued symmetric and transitive relation R on A satisfies the **separation property** if the following holds:

$$R(x, y) = R(x, x) = R(y, x) \neq 0 \text{ implies } x = y. \tag{2.8}$$

Therefore, R is said to be **separated** if 2.8 holds.
We now consider the connection of the above notion with Ω-valued relations on Ω-set.
Suppose $\mu : A \rightarrow$ is an Ω-valued function on A and $R : A^2 \rightarrow$ an Ω-valued relation on A. If for all $x, y \in A$ the following holds:

$$R(x, y)\mu(x) \wedge \mu(y). \tag{2.9}$$

then we say that R is an **Ω-valued relation** on μ.
Next, we consider the notion of (weak) reflexiveness of R. An Ω-valued relation R on $\mu : A \rightarrow$ is said to be **reflexive** on μ or μ-**reflexive** if

$$R(x, x) = \mu(x) \text{ for every } x \in A. \tag{2.10}$$

A symmetric and transitive Ω-valued relation R on A, which is reflexive on $\mu : A \rightarrow$ is an **Ω-valued equivalence** on μ.

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Clearly, an Ω-valued equivalence R on μ fulfills the strictness property (2.7).

Furthermore, if R is an Ω-valued equivalence on μ, which is separated according to (2.8), then we say that R is an **Ω-valued equality** on μ.

2.3 Ω-set

The following is defined in [10], and then adopted to a fuzzy framework in [27].
An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, which may fulfill the separation property (2.8) if indicated. But we note that in this work the separation property is needed for most of the result.
Consequently, for an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x). \tag{2.11}$$

Then μ is said to be *determined* by E. This enable us in generalizing the notion of subset of an Ω-set (A, E).
Clearly, by the strictness property (2.7), E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ. That is why we say that in an Ω-set (A, E), E is an **Ω-valued equality**.

If (A, E) is an Ω-set and $p \in$, then the cut E_p is an equivalence relation on the corresponding cut μ_p of μ.

2.4 Ω-algebra and Ω-subalgebra

For reference on the results in this section see ([29], [27]).

Let $A = (A, F)$ be an algebra and $E : A^2 \to$ an Ω-valued equality on A, which is compatible with the operations in F. Then we say that $(\mathcal{A}, E)$ is an **Ω-algebra**. Algebra A is the **underlying algebra** of $(\mathcal{A}, E)$.

[27] *Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold:*

$(i\ )$ The function $\mu : A \to$ determined by E ($\mu(x) = E(x, x)$ for all $x \in A$), is compatible over A.

$(ii\ )$ For every $p \in$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$, and

$(iii\ )$ For every $p \in$, the cut E_p of E is a congruence relation on μ_p. Let $(\mathcal{A}, E)$ be an Ω-algebra, and (A, E_1) an Ω-subset of (A, E). Then E_1 is a symmetric and transitive Ω-relation on A, fulfilling for all $x, y \in A$

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y).$$

Let also E_1 be compatible with the operations in $\mathcal{A}$. Obviously, $(\mathcal{A}, E_1)$ is an Ω-algebra and we say that it is an **Ω-subalgebra** of the Ω-algebra $(\mathcal{A}, E)$.

2.5 Ω-group: $(G, *, ^{-1}, e)$

For reference on the results in this section see ([27], [23]).

[27] Let $\overline{\mathcal{G}} = (G, *, ^{-1}, e)$ be an Ω-group, and E and Ω-equality on $\mathcal{G}$. Then $\overline{\mathcal{G}} = (\mathcal{G}, E)$ is an Ω-group
Consider $\nu : G \to$ to be an Ω-valued subset of the Ω-valued set $\mu : G \to$, E an Ω-valued relation on μ and $E^1 : G^2 \to$ an Ω-valued relation on G. Then E^1 is a restriction of E to ν if

$$E^1(x, y) = \nu(x) \wedge \nu(y) \wedge E(x, y) \tag{2.12}$$

[27] Let $(\mathcal{A}, E)$ be an Ω-algebra on the algebra $\mathcal{A}$ and ν is an Ω-valued subset of the Ω-valued set μ and Ω-subalgebra of $\mathcal{A}$. Then $(\mathcal{A}, E^1)$, is also Ω-algebra on the algebra $\mathcal{A}$ where $\nu(x) = E^1(x, x)$ and the restriction of E to ν.

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Observe that if $\overline{\mathcal{G}} = (\mathcal{G}, E)$ and $\overline{\mathcal{G}}^1 = (\mathcal{G}, E^1)$ are Ω-groups over the same algebra $\mathcal{G} = (G, *, ^{-1}, e)$, then $\overline{\mathcal{G}}^1$ is an Ω-**subgroup** of the Ω-group $\overline{\mathcal{G}}$ if E^1 is a restriction of E to the Ω-subalgebra ν of $\mathcal{G}$ determined by E^1.

Theorem 2.1. *[27] Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-group and E^1 an Ω-valued relation on G satisfying the formula*

$$E^1(x, y) = E^1(x, x) \wedge E^1(y, y) \wedge E(x, y) \tag{2.13}$$

Then the Ω-algebra $\overline{\mathcal{G}}^1 = (\mathcal{G}, E^1)$ is an Ω-subgroup of the Ω-group $\overline{\mathcal{G}}$ if and only if the following hold:

$$E^1(x * y, x * y) E^1(x, x) \wedge E^1(y, y) \tag{2.14}$$

$$E^1(x^1, x^1) E^1(x, x) \tag{2.15}$$

$$E^1(e, e) = 1 \tag{2.16}$$

2.6 Ω-group: as Ω-groupoid

For reference on the results in this section see ([33]). In both languages, the associative property is equivalent to the fulfillment of the corresponding identity in the framework of Ω-algebras:

$$\mu(x) \wedge \mu(y) \wedge \mu(z) E(x * (y * z), (x * y) * z), \tag{2.17}$$

Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid, where the underlying algebra is a groupoid $\mathcal{G} = (G, *)$. Then $\overline{\mathcal{G}}$ is a strict Ω-**group** if it is associative in the sense of (2.17) and the following hold:
There is $e \in G$ such that for every $x \in G$

$$\mu(x) \mu(e) \wedge E(e * x, x) \wedge E(x * e, x) \tag{2.18}$$

and

$$\text{there is } x' \in G \text{ such that } \mu(x) \mu(x') \wedge E(x * x', e) \wedge E(x' * x, e) \tag{2.19}$$

[33] Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid which is a strict Ω-group, then the following hold: $(i\ )$ if $e \in G$ is a neutral element in $\overline{\mathcal{G}}$, then for all $x \in G$ $\mu(e) \mu(x)$.

$(ii\ )$ A neutral element e is unique and this is also the neutral element in the underlying groupoid $(G, *)$.

[33] Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid which is a strict Ω-group, then the following hold for every $x \in G$: $(i\ )$ $(x')' = x$. $(ii\ )$ $\mu(x) = \mu(x')$. $(iii\ )$ Inverse element x' is unique.

Theorem 2.2. *[33] Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid with a neutral element e. If every nonempty quotient μ_i / E_i, $i \in$ is a group whose neutral element class is $[e]_{E_i}$, then $\overline{\mathcal{G}}$ is an Ω-group.*

3 Result

In subsection (2.5), Ω-subgroup of an Ω-group (as presented in ([31])) was investigated in the case where the underlying algebra is an algebra with three operations, namely: a binary operation, a unary operation and a nullary operation. But our investigation here is to deal with Ω-subgroup of an Ω-group in the language of one operation, as was introduced in ([33]).
The following result follows directly from theorem (2.1).

Theorem 3.1. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group, where $\mathcal{G} = (G, *)$ the underlying algebra is a groupoid and E^1 an Ω-valued relation on G satisfying the formula*

$$E^1(x, y) = E^1(x, x) \wedge E^1(y, y) \wedge E(x, y) \tag{3.1}$$

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Then the Ω*-groupoid* $\overline{\mathcal{G}}^1 = (\mathcal{G}, E^1)$ *is an* Ω*-subgroup of the* Ω*-group* $\overline{\mathcal{G}}$ *if and only if the following hold:*

$$E^1(x,x) \wedge E^1(y,y) E^1(x*y, x*y) \tag{3.2}$$

$$E^1(x,x) E^1(x',x') (\text{for the inverse element } x' \text{ of } x \in \overline{\mathcal{G}}) \tag{3.3}$$

$$E^1(x,x) E(x,x) (\ x \in \overline{\mathcal{G}}) \tag{3.4}$$

$$E^1(e,e) = E(e,e) \text{for the neutral element } e \in \overline{\mathcal{G}} \tag{3.5}$$

The following proposition is considered a special case of theorem (**??**) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G, *)$ is a classical groupoid and $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ an Ω-subgroupoid of $\overline{\mathcal{G}}$. Therefore, if $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ is an Ω-subgroup of $\overline{\mathcal{G}}$, then the following holds: There is $e \in \overline{\mathcal{G}}$ such that for all $x \in \overline{\mathcal{G}}$

$$\hat{\mu}(x)\hat{\mu}(e) \wedge \hat{E}(x*e, x) \wedge \hat{E}(e*x, x) \tag{3.6}$$

there is $x' \in G$

$$\hat{\mu}(x)\hat{\mu}(x') \wedge \hat{E}(x*x', e) \wedge \hat{E}(x'*x, e) \tag{3.7}$$

Proof. Assuming $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$. Since $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ is an Ω-subgroupoid of $\overline{\mathcal{G}}$ we have that for $x, y \in G$

$$\hat{E}(x,y) = \hat{\mu}(x) \wedge \hat{\mu}(y) \wedge E(x,y) \tag{3.8}$$

and also for any $x \in G$

$$\hat{\mu}(x)\mu(x)$$

therefore, it follows that

$$\hat{\mu}(x)\mu(e) \wedge E(x*e, x) \wedge E(e*x, x) \tag{3.9}$$

Where $\hat{\mu}(x)\hat{\mu}(x) = \hat{\mu}(x) \wedge \hat{\mu}(x)$.
Thus from equation 3.9, we have

$$\hat{\mu}(x)\mu(e) \wedge E(x*e, x) \wedge E(e*x, x) \wedge \hat{\mu}(x) \wedge \hat{\mu}(x) \tag{3.10}$$

Clearly, form 3.10

$$\hat{\mu}(x) \wedge \hat{\mu}(e*x) \wedge \hat{\mu}(x*e)\mu(e) \wedge E(x*e, x) \wedge E(e*x, x) \wedge \hat{\mu}(x) \wedge \hat{\mu}(x) \wedge \hat{\mu}(e*x) \wedge \hat{\mu}(x*e) \tag{3.11}$$

From equation 3.8 we have,

$$\hat{E}(e*x, x) = \hat{\mu}(e*x) \wedge \hat{\mu}(x) \wedge E(e*x, x) \tag{3.12}$$

$$\hat{E}(x*e, x) = \hat{\mu}(x*e) \wedge \hat{\mu}(x) \wedge E(x*e, x) \tag{3.13}$$

combine equations 3.12 and 3.13 with equation 3.11, to obtain

$$\hat{\mu}(x) \wedge \hat{\mu}(e*x) \wedge \hat{\mu}(x*e)\mu(e) \wedge \hat{E}(e*x, x) \wedge \hat{E}(x*e, x) \tag{3.14}$$

since $\hat{\mu}(x)\hat{\mu}(e*x)$ and $\hat{\mu}(x)\hat{\mu}(x*e)$, then equation 3.14, becomes

$$\hat{\mu}(x)\mu(e) \wedge \hat{E}(e*x, x) \wedge \hat{E}(x*e, x) \tag{3.15}$$

since $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ is an Ω-subgroup, and e is the neutral element, then $\mu(e) = \hat{\mu}(e)$. Therefore, 3.15 becomes

$$\hat{\mu}(x)\hat{\mu}(e) \wedge \hat{E}(e*x, x) \wedge \hat{E}(x*e, x) \tag{3.16}$$

The second assertion follows similarly. Therefore rewriting equation 3.15, we have

$$\hat{\mu}(x)\mu(x') \wedge \hat{E}(x'*x, e) \wedge \hat{E}(x*x', e) \tag{3.17}$$

But since $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$ is an Ω-subgroup of the Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$, then

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$$\hat{\mu}(x)\hat{\mu}(x')$$

Thus equation 3.17 becomes

$$\hat{\mu}(x)\mu(x') \wedge \hat{\mu}(x') \wedge \hat{E}(x' * x, e) \wedge \hat{E}(x * x', e) \tag{3.18}$$

but

$$\hat{\mu}(x') = \hat{\mu}(x') \wedge \mu(x')$$

Hence equation 3.18 becomes

$$\hat{\mu}(x)\hat{\mu}(x') \wedge \hat{E}(x' * x, e) \wedge \hat{E}(x * x', e) \tag{3.19}$$

as required. □

The above result shows that every identity satisfied on an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$ is also satisfied on its Ω-subgroup $\hat{\mathcal{G}} = (\mathcal{G}, \hat{E})$.

We let $(\overline{\mathcal{G}}, E)$ be the set of all Ω-subgroups of the Ω-group $(\overline{\mathcal{G}}, E)$.

3.1 some definitions

Let $A = (G, E^1)$ to be an Ω-subset of $\overline{\mathcal{G}}$. Then we have the following definitions:

1) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G, *)$ is a classical groupoid and $A = (G, E^1)$ an Ω-subset of $\overline{\mathcal{G}}$. An element $x \in \overline{\mathcal{G}}$ is an Ω-centralizer of A if for every $a \in A$ the following holds

$$\mu^1(a) \wedge \mu(x)E(xax', a). \tag{3.20}$$

2) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G, *)$ is a classical groupoid and $A = (G, E^1)$ be an Ω-subset of $\overline{\mathcal{G}}$. An element $x \in \overline{\mathcal{G}}$ is an Ω-normalizer of A if for all $a \in A$ there exist $b \in A$ such that

$$\mu^1(a) \wedge \mu^1(b) \wedge \mu(x)E(xax', b). \tag{3.21}$$

3) Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a strict Ω-group , where $\mathcal{G} = (G, *)$ is a classical groupoid. An element $x \in \overline{\mathcal{G}}$ is in the Ω-center of $\overline{\mathcal{G}}$ if for all $a \in G$ the following holds

$$\mu(a) \wedge \mu(x)E(xa, ax). \tag{3.22}$$

Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$. Then the following hold
(i) the neutral element of $\overline{\mathcal{G}}$ is an Ω-centralizer A.
(ii) if $x \in \overline{\mathcal{G}}$ is an Ω-centralizer of A, then the inverse element of $x \in \overline{\mathcal{G}}$ is also an Ω-centralizer of A.
(iii) if $x, y \in \overline{\mathcal{G}}$ are Ω-centralizers of A, then $xy \in \overline{\mathcal{G}}$ is also an Ω-centralizer of A.

Proof. (i) Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$, then for the neutral element $e \in \overline{\mathcal{G}}$ we have

$$\mu^1(a) \wedge \mu(e)\mu(a) = E(a, a) = E(eae, a) \Rightarrow \mu^1(a) \wedge \mu(e)E(eae, a)$$

so the neutral element e in $\overline{\mathcal{G}}$ is an Ω-centralizer of $A = (G, E^1)$.
(ii) Assume that x is an Ω-centralizer of A in $\overline{\mathcal{G}}$ therefore by definition

$$\mu^1(a) \wedge \mu(x)E(xax', a)$$

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hence

$$
\begin{aligned}
&\mu^1(a)\wedge\mu(x)\le E(xax',a)\wedge\mu(x)\\
&\Rightarrow \mu^1(a)\wedge\mu(x)\wedge\mu(x')\le E(xax',a)\wedge\mu(x)\wedge\mu(x') \quad (x' \text{ is the inverse of } x \text{ in } \overline{\mathcal{G}})\\
&\Rightarrow \mu^1(a)\wedge\mu(x)\wedge\mu(x')\le E(x',x')\wedge E(xax',a)\wedge E(x,x) \quad (\mu \text{ reflexivity})\\
&\Rightarrow \mu^1(a)\wedge\mu(x)\wedge\mu(x')\le E(x',x')\wedge E(xax'x,ax) \quad (\text{compatibility})\\
&\Rightarrow \mu^1(a)\wedge\mu(x)\wedge\mu(x')\le E(x'xax'x,x'ax) \quad (\text{compatibility}).
\end{aligned}
$$

Let $p=\mu^1(a)\wedge\mu(x)\wedge\mu(x')$, then $E(x'xax'x,x'ax)p$. Therefore $(x'xax'x,x'ax)\in E_p$ implying

$$
\begin{aligned}
&[x'xax'x]_{E_p}=[x'ax]_{E_p}\\
&\Rightarrow [x'x]_{E_p}[a]_{E_p}[x'x]_{E_p}=[x'ax]_{E_p}\\
&\Rightarrow [e]_{E_p}[a]_{E_p}[e]_{E_p}=[x'ax]_{E_p}\\
&\Rightarrow [a]_{E_p}=[x'ax]_{E_p}\\
&\Rightarrow (a,x'ax)\in E_p\\
&\Rightarrow E(a,x'ax)p.
\end{aligned}
$$

Thus $\mu^1(a)\wedge\mu(x)\wedge\mu(x')E(a,x'ax)$ implying $\mu^1(a)\wedge\mu(x')E(a,x'ax)$.
Proving that for any $x\in G$ that is an Ω-centralizer of $A=(G,E^1)$, then its inverse is also is an Ω-centralizer of $A=(G,E^1)$.
(iii) Now let $x,y\in\overline{\mathcal{G}}$ be Ω-centralizers of A and let $p=\mu^1(a)\wedge\mu(xy)$

$$
\begin{aligned}
&\mu^1(a)\wedge\mu(xy)\wedge\mu((xy)')E(a,a)\wedge E(xy,xy)\wedge E((xy)',(xy)')\\
&\le E(axy,axy)\wedge E((xy)',(xy)') \quad (\text{compatibility})\\
&= E((xy)',(xy)')\wedge E(axy,axy) \quad (\text{commutativity})\\
&\le E((xy)'axy,(xy)'axy) \quad (\text{compatibility}).
\end{aligned}
$$

Thus $\mu^1(a)\wedge\mu(xy)\le E((xy)'axy,(xy)'axy)$, where $\mu(xy)\le\mu((xy)')$.
Therefore, $E((xy)'axy,(xy)'axy)p$. It follows that $((xy)'axy,(xy)'axy)\in E_p$ implying

$$
\begin{aligned}
&[(xy)'axy]_{E_p}=[y'x'axy]_{E_p}\\
&=[y']_{E_p}[x'ax]_{E_p}[y]_{E_p}=[y']_{E_p}a[y]_{E_p}\\
&=[y'ay]_{E_p}=[a]_{E_p}\\
&\Rightarrow [a]_{E_p}=[(yx)'a(yx)]_{E_p}\\
&\Rightarrow (a,(yx)'a(yx))\in E_p\\
&\Rightarrow E(a,(yx)'a(yx))p.
\end{aligned}
$$

Thus $\mu^1(a)\wedge\mu(xy)E(a,(yx)'a(yx))$.
Proving that for $x,y\in\overline{\mathcal{G}}$ that are Ω-centralizers of $A=(G,E^1)$, then their product $xy\in\overline{\mathcal{G}}$ is also an Ω-centralizer of $A=(G,E^1)$. □

Let $A=(G,E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}}=(\mathcal{G},E)$. Then the following hold:
(i) the neutral element of $\overline{\mathcal{G}}$ is an Ω-normalizer of A.
(ii) if $x\in\overline{\mathcal{G}}$ is an Ω-normalizer of A, then the inverse element of $x\in\overline{\mathcal{G}}$ is an Ω-normalizer of A.
(iii) if $x,y\in\overline{\mathcal{G}}$ are Ω-normalizers of A, then $xy\in\overline{\mathcal{G}}$ is an Ω-normalizer of A.

Proof. (i) It is easily proved.
(ii) Assume that x is an Ω-normalizer of A in $\overline{\mathcal{G}}$ therefore by definition

$$\mu^1(a)\wedge\mu^1(b)\wedge\mu(x)E(xax',b)$$

hence

$$\begin{aligned}
&\mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \leq E(xax', b) \\
&\Rightarrow \mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \wedge \mu(x') \leq E(xax', b) \wedge \mu(x) \wedge \mu(x') \\
&\ (x' \text{ inverse of } x \text{ in } \overline{\mathcal{G}}) \\
&\Rightarrow \mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \wedge \mu(x') \leq E(xax', b) \wedge E(x', x') \wedge E(x, x) \\
&(\mu \text{ reflexivity}) \\
&\Rightarrow \mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \wedge \mu(x') \leq E(x'xax', x'b) \wedge E(x, x) \text{ (compatibility)} \\
&\Rightarrow \mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \wedge \mu(x') \leq E(x'xax'x, x'bx) \quad \text{(compatibility)}.
\end{aligned}$$

Let $p = \mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \wedge \mu(x')$, then $E(x'xax'x, x'bx)p$. Therefore $(x'xax'x, x'bx) \in E_p$ implying

$$\begin{aligned}
&[x'xax'x]_{E_p} = [x'bx]_{E_p} \\
&\Rightarrow [x'x]_{E_p}[a]_{E_p}[x'x]_{E_p} = [x'bx]_{E_p} \\
&\Rightarrow [e]_{E_p}[a]_{E_p}[e]_{E_p} = [x'bx]_{E_p} \\
&\Rightarrow [a]_{E_p} = [x'bx]_{E_p} \\
&\Rightarrow (a, x'ax) \in E_p \\
&\Rightarrow E(a, x'bx)p.
\end{aligned}$$

Thus $\mu^1(a) \wedge \mu^1(b) \wedge \mu(x) \wedge \mu(x')E(a, x'bx)$ implying $\mu^1(a) \wedge \mu^1(b) \wedge \mu(x')E(a, x'bx)$.
Proving that for any $x \in G$ that is an Ω-normalizer of $A = (G, E^1)$, then its inverse also is an Ω-normalizer of $A = (G, E^1)$.
(iii) Now let $x, y \in \overline{\mathcal{G}}$ be Ω-normalizers of A and let $p = \mu^1(a) \wedge \mu^1(b) \wedge \mu(xy)$

$$\begin{aligned}
&\mu^1(a) \wedge \mu^1(b) \wedge \mu(xy) \wedge \mu((xy)')E(a, a) \wedge E(b, b) \wedge E(xy, xy) \wedge E((xy)', (xy)') \\
&\leq E(axy, axy) \wedge E((xy)', (xy)') \quad \text{(compatibility)} \\
&= E((xy)', (xy)') \wedge E(axy, axy) \quad \text{(commutativity)} \\
&\leq E((xy)'axy, (xy)'axy) \quad \text{(compatibility)}.
\end{aligned}$$

Thus $\mu^1(a) \wedge \mu^1(b) \wedge \mu(xy) \leq E((xy)'axy, (xy)'axy)$, where $\mu(xy) \leq \mu((xy)')$.
Therefore, $E((xy)'axy, (xy)'axy)p$. It follows that $((xy)'axy, (xy)'axy) \in E_p$ implying

$$\begin{aligned}
&[(xy)'axy]_{E_p} = [y'x'axy]_{E_p} \\
&= [y']_{E_p}[x'ax]_{E_p}[y]_{E_p} = [y']_{E_p}[x'ax]_{E_p}[y]_{E_p} \\
&\Rightarrow [y'by]_{E_p} = [a]_{E_p} \\
&\Rightarrow (y'by, a) \in E_p \\
&\Rightarrow E(a, y'by)p.
\end{aligned}$$

Thus $\mu^1(a) \wedge \mu^1(b) \wedge \mu(xy)E(a, y'by)$.
since $E(a, y'by)E(a, b)$, then
$\mu^1(a) \wedge \mu^1(b) \wedge \mu(xy) \leq E(a, y'by) \wedge E(a, b) \leq E(b, y'by)$ Therefore, $E(b, y'by)p$. It follows that $(b, y'by) \in E_p$ implying

$$\begin{aligned}
&[y'by]_{E_p} = [y']_{E_p}[b]_{E_p}[y]_{E_p} \\
&= [y']_{E_p}[x'ax]_{E_p}[y]_{E_p} = [y']_{E_p}[x']_{E_p}[a]_{E_p}[x]_{E_p}[y]_{E_p} \\
&= [(xy)']_{E_p}[a]_{E_p}[xy]_{E_p} \\
&\Rightarrow ((xy)'a(xy), b) \in E_p \\
&\Rightarrow E((xy)'a(xy), b)p.
\end{aligned}$$

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Hence $\mu^1(a) \wedge \mu^1(b) \wedge \mu(xy) \leq E((xy)'a(xy), b)$

Proving that for $x, y \in \overline{\mathcal{G}}$ that are Ω-normalizers of $A = (G, E^1)$, then their product $xy \in \overline{\mathcal{G}}$ is also an Ω-normalizer of $A = (G, E^1)$. □

Let $A = (G, E^1)$ be an Ω-subset of the (strict) Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$. Then the following hold:
(i) the neutral element of $\overline{\mathcal{G}}$ is an element of the Ω-center of $\overline{\mathcal{G}}$.
(ii) if $x \in \overline{\mathcal{G}}$ is an element of the Ω-center of $\overline{\mathcal{G}}$, then the inverse element of $x \in \overline{\mathcal{G}}$ is also an element of the Ω-center of $\overline{\mathcal{G}}$.
(iii) if $x, y \in \overline{\mathcal{G}}$ are elements of the Ω-center of $\overline{\mathcal{G}}$, then $xy \in \overline{\mathcal{G}}$ is also an element of the Ω-center $\overline{\mathcal{G}}$.

Proof. The proofs follow similarly as above. □

3.2 Concrete example

In this section we present an example.
Given that $(\mathcal{G}, E)$ is an Ω-groupoid. Let $G = \{e, a, b, c, d, f, g\}$, and Ω be a membership values lattice given in Figure 1.

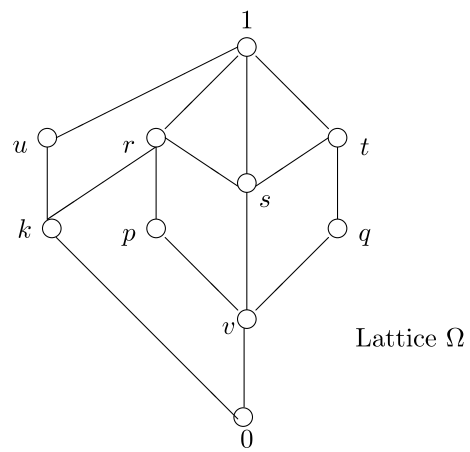

Figure 1: Membership values lattice

An Ω-valued, equality $E : G^2 \to$ is given in Table 2, and the operation $*$ is given in Table 1 .

$*$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	c	b	f	d	a
b	b	c	e	a	f	d	b
c	c	b	a	e	a	e	c
d	d	f	d	d	a	e	d
f	f	d	f	f	e	a	e
g	g	a	b	c	d	e	a

Table 1: Operation $*$

E	e	a	b	c	d	f	g
e	1	p	v	q	0	0	k
a	p	r	v	v	0	0	0
b	v	v	s	v	0	0	0
c	q	v	v	t	0	0	0
d	0	0	0	0	u	0	0
f	0	0	0	0	0	u	0
g	k	0	0	0	0	0	k

Table 2: Ω-valued equality E

The membership Ω-function $\mu : G \to$, defined by $\mu(x) := E(x, x)$, is given as

$$\mu = \begin{pmatrix} e & a & b & c & d & f & g \\ 1 & r & s & t & u & u & k \end{pmatrix}.$$

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The cut sets of μ and E represented by partitions are:

$$\begin{array}{ll}
\mu_0 = G \,; & E_0 = \{\{e,a,b,c,d,f\}\}; \\
\mu_u = \{e,d,f\} \,; & E_u = \{\{e\},\{d\},\{f\}\}; \\
\mu_r = \{e,a\} = \mu_p \,; & E_r = \{\{e\},\{a\}\}; \\
\mu_s = \{e,a,b,c\} = \mu_v \,; & E_s = \{\{e\},\{a\},\{b\},\{c\}\}; \\
\mu_t = \{e,c\} = \mu_q \,; & E_t = \{\{e\},\{c\}\}; \\
\mu_k = \{e,a,d,f,g\} \,; & E_k = \{\{e,g\},\{a\},\{d\},\{f\}\}; \\
\mu_1 = \{e\}; & E_1 = \{\{e\}\}; \\
E_p = \{\{e,a\}\} \,; & E_q = \{\{e,c\}\}; \\
 & E_v = \{\{e,a,b,c\}\}.
\end{array}$$

$(\mathcal{G}, E)$ is an Ω-group by theorem (2.2). This fact is shown in the analysis above. For all $i \in$, the quotient cut subgroupoids $(\mu_i/E_i, *)$ are groups. Observe that μ_0/E_0 and μ_v/E_v are one-element quotient cut subgroupoids, and the other quotient cut subgroupoids are presented in Tables 3 - 7.

$*$	e	a	d	f
e	e	a	d	f
a	a	e	f	d
d	d	f	a	e
f	f	d	e	a

Table 3: Quotient cut $(\mu_k/E_k, *)$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 4: Quotient cut $(\mu_s/E_s, *)$

$*$	e	d	f
e	e	d	f
d	d	f	e
f	f	e	d

Table 5: $(\mu_u/E_u, *)$

$*$	e	a
e	e	a
a	a	e

Table 6: $(\mu_r/E_r, *)$

$*$	e	c
e	e	c
c	c	e

Table 7: $(\mu_t/E_t, *)$

Note in Table 4 element $g \in G$ could be used in place of element $e \in G$ since e and g belong to the equivalence class in cut set E_s.

Finally, observe that for the Ω-group $(\mathcal{G}, E)$ with $G = \{e,a,b,c,d,f,g\}$, and Ω in Figure 1. we see that definition 3.1 holds. Let $A = (G, E^1)$ an Ω-subset of $\overline{\mathcal{G}}$, for $a \in A = (G, E^1)$ and $b \in \overline{\mathcal{G}}$, we have by equation 3.20

$$\mu^1(a) \wedge \mu(b) E(bab', a) \Rightarrow p \wedge sE(a,a) \Rightarrow vr. \tag{3.23}$$

Therefore, b is an Ω-centralizer of A.
Next, for $c \in \overline{\mathcal{G}}$ and $a, b \in A = (G, E^1)$, we have by equation 3.21

$$\mu^1(a) \wedge \mu^1(b) \wedge \mu(c) E(cac', b) \Rightarrow p \wedge s \wedge tE(a,b) \Rightarrow vv. \tag{3.24}$$

Therefore, c is an Ω-normalizer of A.
Lastly, for $a \in \overline{\mathcal{G}}$ and for $d \in \overline{\mathcal{G}}$, we have by equation 3.22

$$\mu(d) \wedge \mu(a) E(da, ad) \Rightarrow u \wedge rE(f,f) \Rightarrow ku. \tag{3.25}$$

Therefore, d is an element of the Ω-center of $\overline{\mathcal{G}}$.
Obviously, in each of the quotient cut to which the above elements belong they also satisfy these notions classically.

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4 Conclusion

In this work we have introduced and investigated the notion of Ω-subgroup of an Ω-group with one binary operation as introduced in [33]. As in the classical case it was shown that an Ω-subgroup of an Ω-group fulfill the same properties, with regards to the neutral element and inverses. We relied on cut structures which are the connections to the classical structures. Furthermore we presented and investigated some particular notions as in the case of classical group theory: Ω-centralizer, Ω-center and Ω-normalizer in an Ω-group. In this framework our next task is to further our investigation on these notions and consider Ω-permutation groups with focus on the class of symmetric group presented in [37]. related concepts.

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References

[1] Zadeh, L.A.: *Fuzzy Sets*, Information and Control, 8, 338-353, (1965)

[2] Goguen, J.A.:*L-fuzzy Sets*, J. Math. Anal. Appl. 18, 145-174, (1967)

[3] Sanchez, E. : *Resolution of Composite Fuzzy Relation Equations*, Information and Control, 30, 38-48 , (1976)

[4] Brown, J. G.: *A Note on Fuzzy Sets*, Information and Control, 18, 32-39, (1971)

[5] Rosenfeld, A.: *Fuzzy groups*, J. Math. Anal, 35, 512–517, (1971)

[6] Mordeson , J.N., Bhutani, K.R., Rosenfeld, A.: *Fuzzy Group Theory* Studies in Fuzziness and Soft Computing, vol. 182, Springer-Verlag, Berlin, Heidelberg, (2005)

[7] Mordeson , J.N., Malik, D.S., Kuroki, N.: *Fuzzy Semigroups* Studies in Fuzziness and Soft Computing, vol. 131, Springer-Verlag, Berlin, Heidelberg, (2003)

[8] Filep, L.: *Structure and Constructing of Fuzzy Subgroups of a Group*, Fuzzy Sets Syst. 51, 105-109, (1992)

[9] Wen-Xiang, G. , De-Gang, C. : *A Fuzzy Subgroupoid which is not a Fuzzy Group*, Fuzzy Sets Syst. 62 , 115–116, (1994)

[10] . Fourman, M.P, Scott, D.S.: *Sheaves and Logic*, in: M.P. Fourman, C.J. Mulvey D.S. Scott (Eds.), Applications of Sheaves, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, pp. 302—401, (1979)

[11] Gottwald, S. :*Universes of Fuzzy Sets and Axiomatizations of Fuzzy Set Theory, Part II: Category theoretic approaches*, Studia Logica, 84(1), 23-50, 1143-1174, (2006)

[12] Höhle, U.: *Fuzzy Sets and Sheaves. Part I: Basic Concepts*, Fuzzy Sets and Systems, 158(11), (2007).

[13] Klir, G., Yuan, B.: *Fuzzy Sets and Fuzzy Logic*, Prentice Hall P T R, New Jersey, (1995)

[14] Di Nola, A., Gerla, G.: *Lattice Valued Algebras*, Stochastica 11, 137—150, (1987)

[15] Šešelja, B., Tepavčević, A.: *Partially Ordered and Relational Valued Algebras and Congruences*, Review of Research, Faculty of Science, Mathematical Series 23, 273—287, (1993)

[16] Šešelja, B., Tepavčević, A.: *On Generalizations of Fuzzy Algebras and Congruences*, Fuzzy Sets and Systems 65, 85—94, (1994)

[17] Oyem, A., Olaleru, J. O., Jaiyeola, T. G., Akewe, H .:*Some algebraic properties of soft quasi-groups*, International Journal of Mathematical Sciences and Optimization: Theory and Applications, 6(2), 834 – 846, (2021) https://doi.org/10.6084/m9.figshare.13524392

[18] Höhle, U., Šostak, A.P.: *Axiomatic Foundations of Fixed-basis Fuzzy Topology.* Springer US, 123-272, (1999)

[19] Bělohlávek, R.:*Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/-Plenum Publishers, New York, (2002)

[20] Höhle, U.: *Quotients with respect to Similarity Relations*, Fuzzy Sets and Systems 27, 31–44, (1988)

[21] Demirci, M.: *Foundations of fuzzy functions and vague algebra based on many-valued equivalence relations part I: fuzzy functions and their applications, part II: vague algebraic notions, part III: constructions of vague algebraic notions and vague arithmetic operations*, Int. J. General Systems 32 (3) , 123—155, 157—175, 177—201, (2003)

[22] Bělohlávek, R., Vychodil, V.: *Algebras with fuzzy equalities*, Fuzzy Sets and Systems 157, 161—201, (2006)

[23] Šešelja, B., Tepavčević, A.: *Fuzzy Identities*, Proc. of the IEEE International Conference on Fuzzy Systems 1660—1664, (2009)

[24] Budimirović, B., Budimirović, V., Šešelja, B., Tepavčević, A.: *Fuzzy Identities with Application to Fuzzy Semigroups*, Information Sciences, 266, 148-159, (2014)

[25] Budimirović, B., Budimirović, V., Šešelja, B., Tepavčević, A.: *Fuzzy Equational Classes are Fuzzy Varieties*, Iranian Journal of Fuzzy Systems 10, 1-18, (2013)

[26] Budimirović, B., Budimirović, V., Šešelja, B., Tepavčević, A.: *Fuzzy equational classes*, Fuzzy Systems (FUZZ-IEEE), IEEE International Conference, pp. 1-6, (2012)

[27] Budimirović, B., Budimirović, V., Šešelja, B., Tepavčević, A.: *E-Fuzzy Groups*, Fuzzy Sets and Systems 289, 94-112, (2016)

[28] Edeghagba, E. E. , Šešelja, B., Tepavčević, A.: *Omega-Lattices*, Fuzzy Sets and Systems 311 , 53-69, (2017)

[29] Edeghagba, E. E. , Šešelja, B., Tepavčević, A.: *Congruences and Homomorphisms on Ω-Algebras* , Kybernetika, 53,(5), 892-910, (2017)

[30] Edeghagba, E. E. , Šešelja, B., Tepavčević, A.: *Representation Theory for Complete L-Lattices* , J. of Mult.-Valued Logic and Soft Computing, 33 , (6):593-617, (2019)

[31] Bleblou, O.S. A., Šešelja, B., Tepavčević, A.: *Normal Ω-subgroup* , Faculty of Sciences and Mathematics, University of Nis, Serbia, 32 , (19):6699-6711, (2018)

[32] Krapeža, A., Šešelja, B., Tepavčević, A.: *Solving Linear Equations in Fuzzy Quasigroups* , Information Sciences, 491,179-189, (2019)

[33] Šešelja, B., Tepavčević, A.: *Ω-group in the Language of Ω-groupiod*, Fuzzy Sets and Systems, 397, 152–167 (2019)

[34] Birkhoff, G.: *Lattice Theory*, 3^{rd} edition, Colloq. Publ. Vol. 25, Amer, Math. soc., Providence, (1967)

[35] Davey, B.A., Priestley, H.A.: *Introduction to Lattices and Order*, Cambridge University Press, (1992)

[36] Burris, S. Sankappanavar, H.P.: *A Course in UUniversal Algebra*, Springer-Verlage, (1981)

[37] Edeghagha, E., Rabiu, K. I., Tajudeen, G. I.:*Characterization of a class of Symmetric Group*, International Journal of Mathematical Sciences and Optimization: Theory and Applications, (1), 455-467, (2019) http://ijmso.unilag.edu.ng/article/view/450