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Order Conditions of a Class of Three-step Hybrid Methods for $y'' = f(x, y)$

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Abstract

A theoretical approach of B-series is used to analyze the convergence and order of convergence of a newly introduced class of three-step hybrid methods for integrating systems of special second order ordinary differential equations (ODEs). A straightforward technique that generates algebraic order conditions, which is easier to handle than the well known Taylor series technique, is presented. The validity of the order conditions is tested by deriving a fourth order method and compared with existing methods in the literature.

Keywords: Hybrid Method, B-series, Order Conditions, Three-step Method, Convergence.
MSC2010: 65L05.

1 Introduction

The problem of interest in this paper is

$$y''(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{1.1}$$

It occurs in a number of areas in applied sciences including engineering, celestial mechanics, astrophysics, chemistry, physics, to mention a few. See [1,2]. Its importance generated a lot of interests on its solution techniques in the last couple of decades, which resulted to a substantial number of literature on the schemes/methods for obtaining its solutions directly or otherwise [1–23] and the references therein. One of the schemes that received the most attention lately is the class of hybrid methods [5]. Earliest of the methods, as noted by Coleman [5], are those due to De Vogelaere [24], Cash [25], Chawla [26], Coleman [27], Chawla et al. [28], Simos [29] and Tsitouras [30]. Order conditions for such methods are usually derived by ad hoc expansions in Taylor series which are specific to the particular method or class of methods under study. The narrative has changed by the pioneering work of Coleman [5] that offered an alternative approach that is more comprehensive and general for deriving the algebraic order conditions of two-step hybrid methods for solving (1.1) directly. This approach is very much similar to those of Runge-Kutta (RK) and Runge-Kutta

Nytröm (RKN) methods. With this approach in place, two-step hybrid methods of higher order could be obtained with relative ease, as the ambiguity associated with derivation of order conditions of higher order methods using Taylor series approach is no longer there. It is important to mention that two-step hybrid method is similar to the famous RKN method, with a little difference. For instance, while RKN method is one-step method that is applied in a block fashion of two equations to obtain approximate solution of the problem and its derivative at each step of integration, two-step hybrid method uses two previous information of a solution to obtain the solution without its derivative at each step of integration. This feature of the method is identified to be the main reason for its improved efficiency and accuracy [31]. Hence, the motivation to develop a class of three-step hybrid methods recently by [32]. The new method relies solely on the Taylor series approach for its order conditions, which has limitations as noted above. It is on this ground a theoretical approach to its order conditions like that obtainable in [5] is proposed in this paper.

The method under consideration here has the following form [32]:

$$
\begin{aligned}
y_{n+1} &= \frac{3}{2}y_n - \frac{1}{2}y_{n-2} + h^2\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i),\\
Y_i &= \frac{1}{2}(2+c_i)y_n - \frac{1}{2}c_i y_{n-2} + h^2\sum_{i=j}^{s} a_{i,j} f(x_n + c_j h, Y_j),
\end{aligned}
\tag{1.2}
$$

where y_{n+1} and y_{n-2} are approximations for $y(x_{n+1})$ and $y(x_{n-2})$, respectively. $a_{i,j}$, b_i and c_i are coefficients of the method and they are real.

The remaining part of the paper is organized as follows: in section 2, basic theory related to the method is presented. Order of convergence of the method is analyzed via local truncation error in section 3. Order conditions of the method are derived and presented together with simplifying assumption in section 4. In section 5, fourth order method is derived and analyzed. And conclusion is given in section 6.

2 Basic Theory

In this section, rooted trees related to second order ODEs and the associated B-series theory are presented as analyzed in [5] exactly to help in the analysis of the proposed approach to order conditions in this paper.

Definition 2.1. *(Set T_2) Suppose τ and τ_1 are empty tree (vertex-less tree) and tree of order one. Let T_2 be set of trees associated with second order ODEs. T_2 is defined as follows:*

(i) τ and τ_1 are elements of T_2 i.e τ, $\tau_1 \in T_2$;

(ii) $t = [t_1, ..., t_m]_2 \in T_2$ if $t_i \in T_2$ $\forall\, i = 1, ..., m$.

Remark 2.2. *Definition 2.1 gives the membership of the set T_2, starting from tree with order zero. The next definition would shed light on what is meant by order of a tree and some other coefficients of the terms in B-series expression. Order of a tree t, denoted by $\rho(t)$, is simply the number of its vertices.*

Definition 2.3. *The order, $\rho : T_2 \to N$, of a tree t is defined recursively as follows:*

(i) $\rho(\tau) = 0$, $\rho(\tau_1) = 1$, $\rho(\tau_2) = 2$;

(ii) for $t = [t_1^{\mu_1}, ..., t_m^{\mu_m}]_2 \in T_2$, $\rho(t) = 2 + \sum_{i=1} \mu_i \rho(t_i)$. The set of all T_2 order q is denoted by T_{2q}.

(iii) $\alpha(\tau) = \alpha(\tau_1) = \alpha(\tau_2) = 1$;

(iv) if $t = [t_1^{\mu_1}, ..., t_m^{\mu_m}]_2 \in T_2$*, with* t_i *distinct for different* i *and different from* τ*, then*

$$\alpha(t) = (\rho(t) - 2)! \prod_{i=1}^{m} \frac{1}{\mu_i!} \left(\frac{\alpha(t_i)}{\rho(t_i)!} \right)^{\mu_i}.$$

It is important to associate each tree, t, to a corresponding elementary differential, $F(t)$. The next definition does that.

Definition 2.4. *The function* F *on* T_2 *is defined recursively by*

(i) $F(\tau)(y, y') = y$, $F(\tau_1)(y, y') = y'$, *and* $F(\tau_2)(y, y') = f(y)$;

(ii) if $t = [t_1, ..., t_m]_2 \in T_2$*, then*

$$F(t)(y, y') = f^{(m)}(y)(F(t_1)(y, y'), ..., F(t_m)(y, y')).$$

B-series can now be defined as follows:

Definition 2.5. *See [5]. Let* β *be a mapping from* T_2 *to set of real numbers* R*, with* $\beta(\tau) = 1$*. The* B_2*-series with coefficient function* β *is a formal series of the form*

$$B(\beta, y) = y + h\alpha(\tau_1)\beta(\tau_1)y' + ... = \sum_{t \in T_2} \frac{h^{\rho(t)}}{\rho(t)!} \alpha(t)\beta(t)F(t)(y, y').$$

At this point it is important to state the lemma that says $h^2 f(.)$ applied to a B-series generates a B-series. It is adopted from [5] and stated here without proof. The proof can be found in the same paper.

Lemma 2.6. *Let* $B(\beta, y)$ *be a* B_2*-series with coefficient function* β*. Then* $h^2 f(B(\beta, y))$ *is also a* B_2*-series, i.e*

$$h^2 f(B(\beta, y)) = B(\beta'', y),$$

with

$$\beta''(\tau) = \beta''(\tau_1) = 0, \ \beta''(\tau_2) = 2,$$

and for all other tree $t = [t_1, ..., t_m]_2 \in T_2$,

$$\beta''(t) = \rho(t)(\rho(t) - 1) \prod_{i=1}^{m} \beta(t_i).$$

3 Order of Convergence of the Method

Order of convergence of a numerical method for ODE is assessed via its local truncation error. The proposed method, being a three-step method, is transformed to a one-step method for simplicity to obtaining its local truncation error while generality is not lost. Suppose the one-step method is

$$G_n = G_{n-1} + h\phi(G_{n-1}, h), \tag{3.1}$$

where G_n is a well defined numerical solution whose initial point G_0 is generated by some starting procedure. The first part of eqn (1.2) is written as a system of three equations as follows: let $F_n = y_{n+1} - y_n$, so that

$$y_n = y_{n-1} + F_{n-1}. \tag{3.2}$$

That means

$$F_n = \frac{1}{2}(F_{n-1} + F_{n-2}) + h^2 \sum_{i=1} b_i f(Y_i).$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 1, pp. 9 - 21
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Suppose $H_n = \frac{F_{n+1}-F_n}{h}$, so that

$$F_n = F_{n-1} + hH_{n-1}. \tag{3.3}$$

Hence, the final update equation becomes

$$H_n + \frac{1}{2}H_{n-1} = h\sum_{i=1} b_i f(Y_i). \tag{3.4}$$

Eqns (3.2)-(3.4) can be written as (3.1) with

$$G_n = \begin{pmatrix} y_n \\ F_n \\ H_n \end{pmatrix} \quad \text{and} \quad \Phi(G_{n-1}, h) = \begin{pmatrix} \frac{1}{h}F_{n-1} \\ H_{n-1} \\ \sum b_i f(Y_i) \end{pmatrix}.$$

Let w be an exact-value function, so that G_n is an approximate form of $w_n = w(x_n, h)$, then

$$w(x,h) = \begin{pmatrix} y(x) \\ y(x+h) - y(x) \\ \frac{F(x+h)-F(x)}{h} \end{pmatrix}. \tag{3.5}$$

The local truncation error of the method at point x_n is therefore given as

$$d_n = w_n - w_{n-1} - h\Phi(w_{n-1}, h), \tag{3.6}$$

where

$$\Phi(w_{n-1}, h) = \begin{pmatrix} \frac{y(x_n)-y(x_{n-1})}{h} \\ \frac{F(x_n)-F(x_{n-1})}{h} \\ \sum b_i f(Y_i) \end{pmatrix}. \tag{3.7}$$

Theorem 3.1. *For exact starting values, the three-step hybrid method is said to be convergent of order p if and only if $\forall$ trees $t \in T_2$,*

$$\sum_{i=1}^{s} b_i \psi_i''(t) = 1 - (-2)^{\rho(t)-1}, \tag{3.8}$$

for $\rho(t) \leq p+1$ but not for some trees of order $p+2$.

Proof. Expand the components of the update stage in eqn (1.2) as B_2-series.

$$B(1, y(x_n)) = \frac{3}{2}y(x_n) - \frac{1}{2}B\left((-2)^{\rho(t)}, y(x_n)\right) + \sum_{i=1}^{s} b_i B(\psi''(t), y(x_n)). \tag{3.9}$$

From (3.5)–(3.7), local truncation error of the method becomes

$$d_n = \begin{pmatrix} y(x_n) \\ y(x_{n+1}) - y(x_n) \\ \frac{F(x_{n+1})-F(x_n)}{h} \end{pmatrix} - \begin{pmatrix} y(x_{n-1}) \\ y(x_n) - y(x_{n-1}) \\ \frac{F(x_n)-F(x_{n-1})}{h} \end{pmatrix} - h\begin{pmatrix} \frac{y(x_n)-y(x_{n-1})}{h} \\ \frac{F(x_n)-F(x_{n-1})}{h} \\ \sum b_i f(Y_i) \end{pmatrix}. \tag{3.10}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 1, pp. 9 - 21
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First Component of eqn (3.10):

$$y(x_n) - y(x_{n-1}) - y(x_n) + y(x_{n-1}) = 0;$$

Second Component of eqn (3.10):

$$y(x_{n+1}) - y(x_n) - y(x_n) + y(x_{n-1}) - h\frac{F(x_n) - F(x_{n-1})}{h} = 0;$$

Third Component of eqn (3.10):

$$d_n = H(x_n) + \frac{1}{2}H(x_{n-1}) - h\sum b_i f(Y_i). \tag{3.11}$$

It is obvious that eqn (3.11) takes the one-step form of the update stage of the method, as in eqn (3.4). Replacing this with the B_2-series form of the update stage equation, as in eqn (3.9), the local truncation error becomes

$$\frac{1}{h}\left(\ B(1, y(x_n)) - \tfrac{3}{2}y(x_n) + \tfrac{1}{2}B\left((-2)^{\rho(t)}, y(x_n)\right) - \textstyle\sum_{i=1}^{s} b_i B(\psi''(t), y(x_n)) \ \right).$$

The method is convergent of order p if p is the largest integer such that

$$d_n = O\left(h^{p+1}\right), \tag{3.12}$$

$\forall\ n \geq 0$. This is possible only if the B_2-series coefficients sum up to zero, i.e

$$1 - (-2)^{\rho(t)-1} - \sum_{i=1}^{s} b_i \psi''(t) = 0, \quad \text{for } \rho(t) \leq p+1,$$

or

$$\sum_{i=1}^{s} b_i \psi''(t) = 1 - (-2)^{\rho(t)-1} \quad \text{for } \rho(t) \leq p+1.$$

Hence, the proof. □

Theorem 3.2. *The B_2-series coefficients ψ of three-step hybrid method $\forall\ t \in T_2$ with $\rho(t) \geq 2$ is*

$$\psi_i(t) = c_i\,(-2)^{\rho(t)-1} + \sum_{i,j} a_{i,j}\psi_j'', \tag{3.13}$$

where

$$\psi_i''(t) = \rho(t)\,(\rho(t)-1)\prod_{i=1}^{m}\psi_i(t_i), \tag{3.14}$$

and for trees with $\rho(t) = 0, 1,$ the coefficients are respectively given as

$$\psi_i(\tau) = 1, \quad \psi_i(\tau_1) = c_i. \tag{3.15}$$

Proof. Apply B_2-series to the components of the internal stage equation of eqn (1.2):

$$B(\psi_i, y(x_n)) = \frac{1}{2}(2 + c_i)\,y(x_n) - \frac{1}{2}B\left((-2)^{\rho(t)}, y(x_n)\right) + h^2\sum_{j=1}^{s} a_{i,j} f(Y_j).$$

Applying Lemma 2.6 gives

$$B(\psi_i, y(x_n)) = \frac{1}{2}(2 + c_i)\,y(x_n) - \frac{1}{2}B\left((-2)^{\rho(t)}, y(x_n)\right) + \sum_{j=1}^{s} a_{i,j} B\left(\psi_j'', y(x_n)\right),$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 1, pp. 9 - 21
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or

$$B\left(\psi_i, y(x_n)\right) = y(x_n) + c_i y'(x_n) - \frac{1}{2} B\left((-2)^{\rho(t)}, y(x_n)\right) + \sum_{j=1}^{s} a_{i,j} B\left(\psi_j'', y(x_n)\right).$$

Now, equating coefficients of the series for each tree gives:

for τ i.e tree with $\rho(t) = 0 : \psi_i(\tau) = 1$;

for τ_1 i.e tree with $\rho(t) = 1 : \psi_i(\tau_1) = c_i$;

for t i.e tree with $\rho(t) \geq 2 : \psi_i(t) = c_i(-2)^{\rho(t)-1} + a_{i,j}\psi_j''$,

where

$$\psi_j''(t) = \rho(t)(\rho(t)-1)\prod_{i=1}^{m}\psi_j(t_i), \quad \forall t \in T_2,$$

see Lemma 2.6. □

4 The Order Conditions

With the aid of eqns (3.8) and (3.13)–(3.15), derivation of algebraic order conditions of three-step hybrid method is demonstrated in this section.

4.1 Condition for second order tree

The only tree with this order in T_2 is $\tau_2 = [\tau]_2$: [grow'=up] (0,0)–(0,0.4); [fill=black](0,0.4)circle(0.05); .

From eqn (3.8):

$$\sum_{i=1}^{s} b_i \psi_i''(\tau_2) = 3;$$

from eqn (3.14):

$$\psi_i''(\tau_2) = 2 \times 1 \times \psi_i(\tau),$$

and from (3.15):

$$\psi_i(\tau) = 1.$$

∴ the condition is

$$\sum_{i=1}^{s} b_i = \frac{3}{2}, \tag{4.1}$$

and eqn (3.13) gives

$$\psi_i(\tau_2) = -2c_i + 2\sum_{j=1}^{s} a_{i,j}. \tag{4.2}$$

4.2 Condition for third order tree

The tree in T_2 with order 3 is $t_{3,1} = [\tau_1]_2$: [grow'=up] (0,0)–(0,0.4); [fill=black](0,0.4)circle(0.05); (0,0.4)–(0.4,0.6); . Following the same procedure, the corresponding order condition is obtained as

$$\sum_{i=1}^{s} b_i c_i = -\frac{1}{2}, \tag{4.3}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 1, pp. 9 - 21
https://doi.org/10.6084/m9.figshare.20758075

and eqn (3.13) gives

$$\psi_i(t_{31}) = 4c_i + 6\sum_{j=1}^{s} a_{i,j}c_j. \tag{4.4}$$

4.3 Condition for fourth order trees

Two of such trees exist in T_2: $t_{41} = [\tau_1, \tau_1]_2$:[grow'=up] (0,0)–(0,0.4); [fill=black](0,0.4)circle(0.05); (0,0.4)–(0.4,0.8); (0,0.4)–(-0.4,0.8); and $t_{4,2} = [\tau_2]_2$:[grow'=up] (0,0)–(0,0.4); [fill=black](0,0.4)circle(0.05); (0,0.4)–(0.4,0.8); (0.4,0.8)–(0,1); [fill=black](0,1)circle(0.05); . The corresponding order conditions are:

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{3}{4}, \tag{4.5}$$

where

$$\psi_i(t_{41}) = -8c_i + 12\sum_{j=1}^{s} a_{i,j}c_j^2. \tag{4.6}$$

For the second tree we have

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{3}{4}, \tag{4.7}$$

provided the condition of third order tree (4.3) is satisfied. Eqn (3.13) gives

$$\psi_i(t_{42}) = -\left(8c_i + 24\left(\sum_{j=1}^{s} a_{i,j}c_j - \sum_{j,k}^{s} a_{i,j}a_{j,k}\right)\right). \tag{4.8}$$

Hence, the algebraic order conditions of the method up to trees of order six are derived and presented in Table1 using the second order ODEs rooted trees presented in p. 205 of [5].

4.4 Simplifying Assumptions

As it is with two-step hybrid methods and other multistage methods, order conditions are a collection of equations that show the relationship between coefficients of the methods. Some of the equations are repetition of the others provided some conditions that are genuinely set up are satisfied. In this subsection, such conditions are formulated to reduce the number of independent order conditions of three-step hybrid method.

Theorem 4.1. *For a three-step hybrid method (see eqn (1.2)) with stage order q, the simplifying condition that reduces independent order conditions is*

$$\sum a_{i,j}c_j^{\kappa} = \frac{c_i^{\kappa+2} - (-2)^{\kappa+1}c_i}{(\kappa+1)(\kappa+2)}. \tag{4.9}$$

Proof. From eqn (1.2), the approximation of the solution, $y(x)$, of problem (1.1) at point $x_n + c_i h$ is $Y_i(x_n)$. That is to say the exact value of the solution at this point is $y(x_n + c_i h)$.

Suppose the difference of these values is taken, that is

$$Y_i(x_n) - y(x_n + c_i h) = O\left(h^{q+1}\right). \tag{4.10}$$

Expressing eqn (4.10) in B_2-series gives

$$B(\psi_i, y(x_n)) - B(c_i^{\rho(t)}, y(x_n)) = O\left(h^{q+1}\right),$$

where q is the stage order of the method. Therefore,

$$\psi_i(t) = c_i^{\rho(t)}. \tag{4.11}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 1, pp. 9 - 21
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Table 1: Order Conditions

t	$\rho(t)$	Order condition
τ	0	-
τ_1	1	-
τ_2	2	$\sum b_i = \frac{3}{2}$
$t_{3,1}$	3	$\sum b_i c_i = -\frac{1}{2}$
$t_{4,1}$	4	$\sum b_i c_i^2 = \frac{3}{4}$
$t_{4,2}$		$\sum b_i a_{i,j} = -\frac{1}{8}$
$t_{5,1}$	5	$\sum b_i c_i^3 = -\frac{3}{4}$
$t_{5,2}$		$\sum b_i c_i a_{i,j} = \frac{3}{8}$
$t_{5,3}$		$\sum b_i a_{i,j} c_j = \frac{5}{24}$
$t_{6,1}$	6	$\sum b_i c_i^4 = \frac{11}{10}$
$t_{6,2}$		$\sum b_i c_i^2 a_{i,j} = \frac{11}{20}$
$t_{6,3}$		$\sum b_i c_i a_{i,j} c_j = \frac{41}{60}$
$t_{6,4}$		$\sum b_i a_{i,j} a_{i,k} = \frac{3}{16}$
$t_{6,5}$		$\sum b_i a_{i,j} c_j^2 = -\frac{87}{360}$
$t_{6,6}$		$\sum b_i a_{i,j} a_{j,k} = \frac{21}{240}$

Twice differentiation of eqn (4.11) gives

$$\psi_i''(t) = \rho(t)(\rho(t)-1)c_i^{\rho(t)-2}. \tag{4.12}$$

Now, substituting eqns (4.11) and (4.12) in (3.13) gives

$$\sum a_{i,j} c_j^{\rho(t)-2} = \frac{c_i^{\rho(t)} - (-2)^{\rho(t)-1} c_i}{\rho(t)(\rho(t)-1)}.$$

Let $\rho(t) - 2 = \kappa$, then

$$\sum a_{i,j} c_j^{\kappa} = \frac{c_i^{\kappa+2} - (-2)^{\kappa+1} c_i}{(\kappa+1)(\kappa+2)}.$$

Hence, the proof. □

5 Fourth Order Method

To obtain a method of order four, a set of order conditions of trees up to order five is required according to Theorem 3.1. This will involve a system of seven equations, see Table 1. As such, at

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least seven parameters are required to solve the system. Suppose the stage of the method is to be four, that is, $s = 4$. It means the system would be solved in eleven unknown parameters, which is in order as the number of unknowns is greater than the number of the equations. The specific coefficients are summarized in Butcher-like tableau as follows:

Table 2: General coefficients of four stage fourth order three-step hybrid method

-2	0	0	0	0
0	0	0	0	0
c_3	$a_{3,1}$	$a_{3,2}$	0	0
c_4	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	0
	b_1	b_2	b_3	b_4

Seven equations against eleven unknowns gives four free parameters. The choice of optimal values of these parameters might be a bit tasking. This leads to imposition of simplifying assumption (4.9) to reduce the number of independent equations leaving only the equations of order conditions generated by trees τ_2 and t_{N1}, $N = 3, ...$ and to increase the number of the overall equations, so that the number of free parameters reduces. Putting $\alpha = 0, 1$ reduces the equations to the four equations associated to τ_2 and $t_{3,1}$, $t_{4,1}$ and $t_{4,1}$. The resulting equations are solved to obtain the method as follows:

Table 3: Specific coefficients of four stage fourth order three-step hybrid method

-2	0	0	0	0
0	0	0	0	0
$-\frac{19}{21}$	$-\frac{26657}{111132}$	$-\frac{28405}{111132}$	0	0
$\frac{117}{220}$	$\frac{99085054731}{215515520000}$	$\frac{154111151571}{178034560000}$	$-\frac{1335209777811}{2047397440000}$	0
	$\frac{4245}{102488}$	$\frac{10093}{17784}$	$\frac{7195797}{11601476}$	$\frac{117128000}{432526653}$

Note that the aim of deriving the fourth order method in this section is to validate the order conditions derived, as stated earlier. No detail analysis of the method as regards to numerical properties like stability etc. is given in the paper.

5.1 Numerical Test

The aim of this subsection is to verify numerically the convergence order of the method as established in Theorem 3.1 using the derived method as a specific example.

- **ThHM4(4):** The three-step four stage fourth order hybrid method derived in this section.
- **FHM4(3):** The two-step three-stage fourth order hybrid method derived in Franco [31].
- **FHM5(4):** The two-step four-stage fifth order hybrid method derived in Franco [31].

5.2 Results and Discussion

Maximum error is the biggest of the errors recorded when a problem is solved in a given interval of solution with a given step-size h.

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Problem 1 (Homogeneous Problem)

$$\frac{d^2y(x)}{dx^2} = -y(x), \quad y(0) = 0, \quad y'(0) = 1,$$

Exact solution: $y(x) = \sin(x)$,

Source: [17].$x \in [0, 100]$.

Problem 2 (Inhomogeneous Problem)

$$\frac{d^2y(x)}{dx^2} = -y(x) + x, \quad y(0) = 1, \quad y^{'}(0) = 2.$$

Exact solution: $y(x) = \sin(x) + \cos(x) + x$.

Source: [1].$x \in [0, 100]$

Problem 3 (Duffing Problem)

$y'' + y + y^3 = F\cos(vx), \quad y(0) = 0.200426728067,$
$y'(0) = 0.$ where $F = 0.002$ and $v = 1.01$.

Exact solution: $y(x) = \sum_{i=0}^{4} \upsilon_{2i+1}\cos[(2i+1)vx],$

where $\upsilon_1 = 0.200179477536$, $\upsilon_3 = 0.246946143 \times 10^{-3}$,
$\upsilon_5 = 0.304014 \times 10^{-6}$, $\upsilon_7 = 0.374 \times 10^{-9}$, and
$\upsilon_9 < 10^{-12}$ $\alpha = 1$.

Source: [20].$x \in [0, 100]$

Table 4: Maximum Error for Problem 1

h	**FHM4(3)**	**ThHM4(4)**	**FHM5(4)**
0.25	5.300900×10^{-04}	2.716900×10^{-04}	4.780000×10^{-06}
0.125	3.310000×10^{-05}	4.250000×10^{-06}	1.495546×10^{-07}
0.0625	2.060000×10^{-06}	6.637301×10^{-08}	4.676539×10^{-09}
0.03125	1.290024×10^{-07}	1.037274×10^{-09}	1.463207×10^{-10}
0.015625	8.062982×10^{-09}	1.552958×10^{-11}	5.215000×10^{-12}

Table 5: Maximum Error for Problem 2

h	**FHM4(3)**	**ThHM4(4)**	**FHM5(4)**
0.25	7.753200×10^{-04}	3.942300×10^{-04}	6.670000×10^{-06}
0.125	4.815000×10^{-05}	6.180000×10^{-06}	2.086749×10^{-07}
0.0625	3.010000×10^{-06}	9.656097×10^{-08}	6.536830×10^{-09}
0.03125	1.878102×10^{-07}	1.520130×10^{-09}	2.096200×10^{-10}
0.015625	1.185461×10^{-08}	2.265000×10^{-11}	1.182300×10^{-10}

The accuracy of the proposed method alongside those of existing hybrid methods (FHM4(3), FHM5(4)) is tested on different test problems listed to confirm the order of convergence of the method numerically, as postulated in Theorem 3.1. The errors recorded are shown in Tables 4–6. It is clear that the errors of the proposed method correspond to those of FHM4(3) for all the problems, which shows that the method has fourth order accuracy. Although it appears to be more accurate than the FHM4(3) as step-size decreases to the extent that it approaches the accuracy of FHM5(4), which is a fifth order method.

Table 6: Maximum Error for Problem 3

h	**FHM4(3)**	**ThHM4(4)**	**FHM5(4)**
0.25	9.770000×10^{-05}	1.764500×10^{-04}	9.260000×10^{-06}
0.125	6.240000×10^{-06}	4.360000×10^{-06}	5.916277×10^{-07}
0.0625	3.948210×10^{-07}	1.205372×10^{-07}	2.265073×10^{-08}
0.03125	2.482881×10^{-08}	3.548960×10^{-09}	7.523360×10^{-10}
0.015625	1.562831×10^{-09}	1.133479×10^{-10}	1.866150×10^{-11}

6 Conclusion

A new approach to algebraic order conditions of a class of three-step hybrid methods, which is more robust and reliable than the ah hoc Taylor series approach is proposed. The order conditions are derived and presented using the proposed technique. In the process, condition for convergence of the three-step hybrid method is provided and proved. This is further confirmed numerically by a specific example of a fourth order method.

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