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Some approximate fixed point theorems for partial Chatterjea convex contractive maps in metric spaces

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Abstract

In this paper, a class of partial Chatterjea convex contraction mappings of type 2 is introduced. The approximate fixed point theorems for partial Chatterjea convex contractive mappings in metric spaces is proved. Examples are provided to validate our main theorem. The results obtained in this study extend and generalize some works in the literature.

Keywords: Partial Chatterjea Convex Contractive Maps of Type 2, Approximate Fixed Point, Metric Space.
MSC2010: 47H10, 47H09.

1 Introduction

Fixed point theory is an essential tool for solving problems in mathematics and applied mathematics. Due to numerous practical problems in applied mathematics which require approximate solutions, "nearly" (approximate fixed point) is introduced. As a result, there is need to relax the conditions in fixed point theorems that guarantee the existence of the solution to these problems. Thus, Tijs et al. [1] introduced approximate fixed point theorems by weakening the conditions for the existence of approximate fixed point for such problems. By approximate fixed point of a function T, we mean that $T(X)$ is "near to X" where X is the nonempty set of the space. Banach [2] proved that contraction mappings from a complete metric space to itself possesses a unique fixed point. Tijs et al. [1] established the existence of the approximate fixed point of contraction mappings in metric spaces without the completeness condition. However, Berinde [3] proved approximate fixed point theorems for certain contractive mappings in metric spaces by adapting the asymptotic regularity of the operator. The approximate fixed point for Reich contractive mappings which generalize the result of Berinde [3] is proved in the context of metric space without the completeness by Dey and Saha [4].
The fixed point theorem for convex contraction mapping in metric space is initiated by Istratescu [5]. Many researchers proved interesting results in this area (see; [6-15] and the references therein). Reecently, Eke et al. [16] proved some fixed point theorems for Chatterjea two-sided convex contraction of type 2 in metric spaces. Miandargh et al. [17] established approximate fixed point theorems for

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generalized convex contraction mappings in metric spaces. Latif et al. [18] generalized the result of Miandargh et al. [17] by establishing the approximate fixed point theorems for partial generalized convex contraction mappings in metric spaces.
In this work, partial Chatterjea convex contraction mappings is introduced and the approximate fixed point theorem for these maps is proved in metric spaces. Our result is independent of the work carried by Latif et al. [18].

2 Preliminaries

In this section, we recall some preliminary definitions and theorems that led to the development of our main results.

Definition 2.1 [19]: Let X be a non-empty set. Suppose that $d : X \times X \to R$ is a mapping that satisfies the following axioms for all $x, y, z \in X$;
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.
Then d is called a metric on X and (X, d) is called a metric space.

Definition 2.2 [20]: Let (X, d) be a metric space and $T : X \to X$ be a mapping with $\epsilon > 0$ given. A point $x_0 \in X$ is said to be an approximate fixed point of T if $d(x_0, Tx_0) < \epsilon$.

Definition 2.3 [21]: A self mapping T on a metric space (X, d) is said to be asymptotically regular at a point $x \in X$ if
$d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$.

Definition 2.4 [22]: Let (X, d) be a metric space and $T : X \to X$ be a mapping . T is said to have approximate fixed point property if for all $\epsilon > 0$ there is an approximate fixed point, that is, for all $\epsilon > 0$,
$\inf_{x \in X} d(x, Tx) = 0$.

Lemma 2.5 [18] : Let (X, d) be a metric space and $T : X \to X$ be asymptotically regular at a point $z \in X$, then T has the approximate fixed point property. The proof of this Lemma is found in [18].

Definition 2.6 [23]: Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. T is said to be α- admissible if
$x, y \in X$, $\alpha(x, y) \geq 1 \longrightarrow \alpha(Tx, Ty) \geq 1$.

Definition 2.7 [24]: Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. The metric space X is said to be α - complete if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$, converges in X.

Definition 2.8 [24]: Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be two mappings. T is said to be α- continuous mapping on (X, d) if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.9 [18]: Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a mapping. X is said to have property (H) whenever for each $x, y \in X$, there is $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

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3 MAIN RESULTS

The definition of partial Chatterjea two-sided convex contraction mappings of type 2 with the existence and uniqueness of their fixed points in a metric space is discuss as follows.
Definition 3.1: Let (X, d) be a metric space. The mapping $T : X \to X$ is called partial Chatterjea two-sided convex contraction mappings if there exist a mapping $\alpha : X \times X \to [0, \infty)$ and positive numbers $a_1, a_2, b_1, b_2 \in (0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ satisfying the following condition:
for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies

$$\begin{aligned} d(T^2(x), T^2(y)) &\leq a_1 d(x, Ty) + a_2 d(Ty, T^2y) \\ &\quad + b_1 d(y, Tx) + b_2 d(Tx, T^2x). \qquad (1) \end{aligned}$$

Remarks 3.2: If $\alpha(x, y) = 1$ for all $x, y \in X$ then the mapping T is called Chatterjea two- sided convex contraction mapping.
Example 3.3: Let $X = [1, 2]$ and $d : X \times X \to R$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \frac{x^2+3}{4}$ and $\alpha(x, y) = \frac{e^x+e^y+1}{|e^x-e^y|+1}$.
Now we show that T is a partial Chatterjea convex contractive map with $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{2}$, $b_1 = \frac{1}{8}$ and $b_2 = \frac{1}{3}$. For $\alpha(x, y) \geq 1$, and $x, y \in X$ with $x \neq y$ we obtain
$|Tx - Ty| = \frac{1}{6}|x^2 - y^2| \leq |x - y|$
and

$$\begin{aligned} |T^2x - T^2y| &= \frac{1}{64}(|x^4 + 6x^2 - y^4 - 6y^2|) \\ &\leq \frac{1}{4}(|Tx - y| + |x - Ty|) \\ &\leq \frac{1}{4}|x - Ty| + \frac{1}{6}|Tx - y| \\ &= \frac{1}{4}d(x, Ty) + \frac{1}{6}d(y, Tx) \\ &\leq \frac{1}{4}d(x, Ty) + \frac{1}{6}d(y, Tx) + \frac{1}{8}d(Tx, T^2x) + \frac{1}{3}d(Ty, T^2y). \end{aligned}$$

Theorem 3.4: Let (X, d) be a metric space and $T : X \to X$ be a partial Chatterjea convex contraction mapping of type 2 with based mapping $\alpha : X \times X \to [0, \infty)$. Assume that T is α-admissable and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has approximate fixed point property. In addition, if T is α- continuous and (X, d) is an α- complete metric space then T has a fixed point. T has a unique fixed point if it has property (H).
Proof: Let $x_0 \in X$ and since T is α-admissible then we have $\alpha(x, y) \geq 1$. Develop a sequence $\{x_n\}$ in X as $x_{n+1} = T^{n+1}x_0$ for all $n \in N \cup \{0\}$. If there exists $n \in N \cup \{0\}$ such that $x_n = x_{n+1}$, then x_n is the fixed point of T and the proof is complete. So we assume $x_n \neq x_{n+1}$ for all $n \in N \cup \{0\}$. Since $\alpha(x, y) \geq 1$ and T is α- admissable, we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. Let $k = max\{d(x_0, Tx_0), d(Tx_0, T^2x_0)\}$. Since $\alpha(x_0, Tx_0) \geq 1$ and using (1) we have

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$$\begin{aligned} d(T^2x_0,T^3x_0) &\leq a_1d(x_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_1d(Tx_0,Tx_0) \\ &+b_2d(Tx_0,T^2x_0) \\ &= a_1d(x_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_2d(Tx_0,T^2x_0) \\ &\leq a_1d(x_0,Tx_0)+a_1d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0) \\ &+b_2d(Tx_0,T^2x_0) \\ &= a_1d(x_0,Tx_0)+(a_1+b_2)d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0) \\ &\leq (2a_1+b_2)max\{d(x_0,Tx_0),d(Tx_0,T^2x_0)\}+a_2d(T^2x_0,T^3x_0) \\ &\leq \frac{(2a_1+b_2)k}{1-a_2} \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} d(T^3x_0,T^4x_0) &\leq a_1d(Tx_0,T^3x_0)+a_2d(T^3x_0,T^4x_0)+b_1d(T^2x_0,T^2x_0) \\ &+b_2d(T^2x_0,T^3x_0) \\ &= a_1d(Tx_0,T^3x_0)+a_2d(T^3x_0,T^4x_0)+b_2d(T^2x_0,T^3x_0) \\ &\leq a_1d(Tx_0,T^2x_0)+a_1d(T^2x_0,T^3x_0)+a_2d(T^3x_0,T^4x_0) \\ &+b_2d(T^2x_0,T^3x_0) \\ &= a_1d(Tx_0,T^2x_0)+(a_1+b_2)d(T^2x_0,T^3x_0)+a_2d(T^3x_0,T^4x_0) \\ &\leq (2a_1+b_2)max\{d(Tx_0,T^2x_0),d(T^2x_0,T^3x_0)\}+a_2d(T^3x_0,T^4x_0)) \\ &\leq \frac{(2a_1+b_2)k}{1-a_2} \end{aligned} \tag{3}$$

Continuing the process we have for $n>2$,

$$\begin{aligned} d(T^nx_0,T^{n+1}x_0) &\leq (\frac{(2a_1+b_2)}{1-a_2})^{n-2}k \\ &\leq \gamma^{n-2}k \quad where \quad \gamma=\frac{(2a_1+b_2)}{1-a_2}. \end{aligned}$$

Thus $d(T^nx_0,T^{n+1}x_0)\to 0$ as $n\to\infty$. This shows that T is asymptotically regular at point $x_0\in X$. By using Lemma 2.4 [18], T has approximate fixed point property.
Next, we show that T has a fixed point provided T is α- continuous and (X,d) is α- complete metric space. First we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m,n\in N$ such that $m>n$, then we have

$$\begin{aligned} d(T^nx_0,T^mx_0) &\leq d(T^nx_0,T^{n+1}x_0)+d(T^{n+1}x_0,T^{n+2}x_0) \\ &+d(T^{n+2}x_0,T^{n+3}x_0)+\cdots+d(T^{m-1}x_0,T^mx_0) \\ &\leq \gamma^{n-2}k+\gamma^{n-1}k+\gamma^nk+\cdots+\gamma^{m+n-3}k \\ &\leq (\gamma^{n-2}+\gamma^{n-1}+\gamma^n+\cdots+\gamma^{m+n-3})k \\ &\leq \gamma^{n-2}(1+\gamma+\gamma^2+\cdots)k \\ &\leq \frac{\gamma^{n-2}}{1-\gamma}k \end{aligned}$$

As $n\to\infty$, $\frac{\gamma^{n-2}}{1-\gamma}k\to 0$. Thus $\{x_n\}$ is a Cauchy sequence in X. Since $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in N$, by α- completeness of X, there is $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Since T is α- continuous, we get $x_{n+1}=Tx_n\to Tx^*$ as $n\to\infty$. By uniqueness of the limit of $\{x_n\}$, we obtain $Tx^*=x^*$.

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Thus T has a fixed point .
To prove the uniqueness of the fixed point of T, we assume x^* and y^* to be different fixed points of T. By property (H), we can choose $z \in X$ such that $\alpha(x^*, z) \geq 1$ and $\alpha(y^*, z) \geq 1$. Since T is α- admissable, we get $\alpha(x^*, T^m z) \geq 1$ and $\alpha(y^*, T^m z) \geq 1$ for all $m \in N$. Since $\alpha(x^*, Tz) \geq 1$ and utilizing (1) we have

$$\begin{aligned}
d(x^*, T^3 z) &= d(T^2 x^*, T^2(Tz)) \\
&\leq a_1 d(x^*, T^2 z) + a_2 d(T^2 z, T^3 z) + b_1 d(Tz, Tx^*) \\
&+ b_2 d(Tx^*, T^2 x^*) \\
&= a_1 d(x^*, T^2 z) + b_1 d(Tz, Tx^*) \\
&\leq a_1 d(x^*, T^2 z) + b_1 d(Tz, x^*) + b_1 d(x^*, Tx^*) \\
&\leq a_1 d(x^*, T^2 z) + b_1 d(Tz, x^*) \\
&\leq (a_1 + b_1) max\{d(x^*, T^2 z), d(Tz, x^*)\} \\
&\leq wk, \quad where \quad w = (a_1 b_1 + b_1).
\end{aligned}$$

For $\alpha(x^*, T^2 z) \geq 1$ we have

$$\begin{aligned}
d(x^*, T^4 z) &= d(T^2 x^*, T^2(T^2 z)) \\
&\leq a_1 d(x^*, T^3 z) + a_2 d(T^3 z, T^4 z) + b_1 d(T^2 z, T^2 x^*) \\
&+ b_2 d(Tx^*, T^3 x^*) \\
&\leq a_1 d(x^*, T^3 z) + a_2 d(T^3 z, T^4 z) + b_1 d(T^2 z, T^2 x^*) + b_2 d(Tx^*, T^2 x^*) \\
&+ b_2 d(T^2 x^*, T^3 x^*) \\
&= a_1 d(x^*, T^3 z) + b_1 d(T^2 z, T^2 x^*) \\
&\leq a_1 (a_1 d(x^*, T^2 z) + b_1 d(Tz, x^*)) + b_1 d(T^2 z, x^*) + b_1 d(x^*, T^2 x^*) \\
&\leq a_1^2 d(x^*, T^2 z) + a_1 b_1 d(Tz, x^*) + b_1 d(T^2 z, x^*) \\
&\leq (a_1^2 + a_1 b_1 + b_1) max\{d(x^*, T^2 z), d(Tz, x^*)\} \\
&\leq w^2 k.
\end{aligned}$$

Continuing the process we have $d(x^*, T^m z) \leq w^{m-2k}$. Since $w < 1$ then $T^m z \to x^*$ as $m \to \infty$. Similarly, we can prove that $T^m z \to y^*$ as $m \to \infty$. By the uniqueness of limit $x^* = y^*$. Thus T has a unique fixed point.

Remarks 3.5 : Theorem 3.4 is a generalization of the result of ([16] Theorem 2.5) because if $\alpha(x, y) = 1$ then partial Chatterjea convex contraction mappings of type 2 reduces to Chatterjea convex contraction mappings of type 2 and the fixed point theorem is proved. If we replace our map with partial Kannan convex contraction mapping of order 2 in Theorem 3.4 then we have the work of ([18] Theorem 3.10). If $\alpha(x, y) = 1$ and $a_2 = b_2$ with $T^2 = T$ in Theorem 3.4 then we have the result of ([3] Theorem 2.3).

Example 3.6: Let $X = [0, \infty)$ be equipped with the usual metric. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be defined by

$$Tx = \begin{cases} \frac{x+5}{6} & if \quad x \in [1, 3), \\ \frac{x^2}{4} + \frac{1}{8} & if \quad x = 3 \end{cases}$$

$$\alpha(x, y) = \begin{cases} 2 & if \quad x, y \in [1, 3), \\ 0 & otherwise. \end{cases}$$

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T is not continuous at $x = 1$ but α- continuous. Let $\{x_n\}$ be a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. For $n \in N$, we have $x_n \in [1, 3]$ and $Tx_n = \frac{x_n+5}{6}$. If $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have
$Tx_n = \frac{x_n+5}{6} \to \frac{x+5}{6} = Tx$ as $n \to \infty$.
Therefore T is α-continuous. T is α-admissible if there is $x_0 = 2 \in X$ such that $\alpha(x_0, Tx_0) = \alpha(2, T(2)) = \alpha(2, 1.16) \geq 1$.

Corollary 3.7: Let (X, d) be a metric space and $T : X \to X$ be a Chatterjea convex contraction mapping of type 2 with based mapping $\alpha : X \times X \to [0, \infty)$ such that

$$\begin{aligned} \alpha(x,y)d(T^2(x), T^2(y)) \quad &\leq \quad a_1 d(x, Ty) + a_2 d(Ty, T^2 y) \\ &\quad\; +b_1 d(y, Tx) + b_2 d(Tx, T^2 x), \end{aligned}$$

for all $x, y \in X$, where $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$. Assume that T is α-admissable and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has approximate fixed point property. In addition, if T is α- continuous and (X, d) is an α- complete metric space then T has a fixed point. T has a unique fixed point if it has property (H),

Corollary 3.8: Let (X, d) be a metric space and $T : X \to X$ be a Chatterjea convex contraction mapping of type 2 such that

$$\begin{aligned} (d(T^2(x), T^2(y)) + r)^{\alpha(x,y)} \quad &\leq \quad a_1 d(x, Ty) + a_2 d(Ty, T^2 y) \\ &\quad\; +b_1 d(y, Tx) + b_2 d(Tx, T^2 x) + r, \end{aligned}$$

for all $x, y \in X$, where $a_1, a_2, b_1, b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ and $r \geq 1$. Assume that T is α- admissable and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has approximate fixed point property. In addition, if T is α- continuous and (X, d) is an α- complete metric space, then T has a fixed point. T has a unique fixed point if it has property (H).

Conclusion : This research introduces a new class of partial Chatterjea convex contraction mapping of type 2 in metric spaces. The approximate fixed point theorem for these maps is established. Examples are given to validate our results. The result has potential for future work as common fixed points of these maps can be prove in different abstract spaces.

Conflicts of Interest

No conflict of interest was declared by the author.

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