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Marshall-Olkin Generalized Inverse Log-Logistic Distribution: Its properties and applications

O. L. Aako[1*], J. A. Adewara[2], E. B. Nkemnole[3]

1 Department of Mathematics and Statistics, Federal Polytechnic, Ilaro, Nigeria.
2 Distance Learning Institute, University of Lagos, Akoka, Nigeria.
3 Department of Mathematics, University of Lagos, Akoka, Nigeria.
* Corresponding author: olubisi.aako@federalpolyilaro.edu.ng

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Abstract

Marshall-Olkin G method of generalization is used to develop a new distribution named Marshall-Olkin Inverse log-logistic distribution (MOILLD), which has a more tractable form and can cope well with outliers in the upper tails. The statistical properties of the distribution, such as survival function, hazard function, moments, and order statistic, were investigated. The mean, variance, and mode of the distribution were also derived. The maximum likelihood estimation method was used to estimates the parameters of the distribution. The result of real-life data application showed that MOILLD has the least AIC, BIC, negative log-likelihood, and KS values compared with its competing distributions. Hence, an excellent alternative to Inverse log-logistic, Weibull, and log-normal distributions.

Keywords: Flexibility, Marshall-Olkin- G, Inverse log-logistic, Moments, Maximum likelihood estimation.
MSC2010: 76D05, 76D07, 76N10.

1 Introduction

Experts in probability distribution theory have made a frantic effort to ensure often, use of probability distributions to their mathematical simplicity or flexibility. Efforts made by the researchers over the past few years revealed that existing theoretical distributions are modified and extended [1]. The purpose of developing a probability distribution is to increase its flexibility and capability in modeling actual-life data. One of the various methods embraced in developing distribution is the generalized G families of distributions, which involves adding shape parameter(s) to the existing distribution to make the new distribution more flexible and fit than the original distribution. The objective of this paper is to generalize Inverse Log-logistic distribution using Marshall-Olkin G transformation to obtain MOILLD.
Marshall and Olkin's generalized family of distribution is a renowned method of adding a new parameter to an existing distribution [2]. The technique of applying Marshall-Olkin transformation

has helped in the extensions of a family of distributions for added flexibility and is considered by many researchers over the last few years. [3] presented Marshall-Olkin Half Logistic by extending half logistic distribution to increase flexibility. [4] proposed the Marshall-Olkin Inverse Lomax distribution by adding a new parameter to the inverse Lomax distribution. [6] introduced Marshall-Olkin Gumbel-Lomax distribution and derived some characterizations of the distribution. [6] introduced Marshall-Olkin Generalized Pareto Distribution, which can model non-monotonic failure rate functions. [7] introduced Marshall-Olkin Right Truncated Fréchet-Inverted Weibull Distribution using Marshall and Olkin transformation. This paper extends Inverse log-logistic distribution using the Marshall-Olkin generalization method. The results are compared with the existing distributions.

2 Materials and Methods

2.1 Marshall-Olkin Inverse Log-logistic distribution (MOILLD)

Marshall-Olkin Inverse Log-logistic Distribution (MOILLD) is a two-parameter distribution derived by generalizing Inverse Log-logistic Distribution using the Marshall-Olkin G family of distribution. The probability density function (pdf) of the Inverse Log-logistic (ILL) distribution as defined by [8] is giving by

$$f\left(x,\gamma\right)=\frac{\gamma}{x^{\gamma+1}{(1+x^{-\gamma})}^{2}};x>0,\gamma>0. \tag{2.1}$$

its corresponding cumulative distribution function (cdf) is given by

$$F\left(x,\gamma\right)=\frac{1}{1+x^{-\gamma}};x>0,\gamma>0. \tag{2.2}$$

and the survival function is given by

$$\bar{F}\left(x,\gamma\right)=\frac{x^{-\gamma}}{1+x^{-\gamma}};x>0,\gamma>0. \tag{2.3}$$

where γ is the shape parameter.

Let $\bar{F}(x)$ denote the survival or reliability function of a continuous random variable X, then the usual device of adding a new parameter results in another survival function $\bar{G}(x)$ defined by

$$\bar{G}(x)=\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)};x\in\Re, \tag{2.4}$$

where $\alpha>0$ is the tilt parameter, $\bar{\alpha}=1-\alpha$ and $-\infty<x<\infty$.
If $g(x)$ and $r(x)$ are the probability density function and hazard rate function corresponding to $\bar{G}(x)$, then

$$g(x)=\frac{\alpha f(x)}{(1-\bar{\alpha}\bar{F}(x))^{2}};x\in\Re, \tag{2.5}$$

where $\alpha>0$, $\bar{\alpha}=1-\alpha$ and $-\infty<x<\infty$ and

$$r(x)=\frac{h(x)}{1-\bar{\alpha}\bar{F}(x)}, \tag{2.6}$$

where $h(x)$ and $f(x)$ are the hazard rate and pdf respectively corresponding to $\bar{F}(x)$.
Marshall-Olkin Inverse Log-logistic Distribution (MOILLD) is derived by inserting (2.3) in (2.4) to get the survival function and, substituting (2.1) and (2.3) in (2.5) to get the corresponding density function.
The survival function of MOILLD is given by

$$\bar{G}\left(x\right)=\frac{x^{-\gamma}}{(1+x^{-\gamma})-(1-\alpha)x^{-\gamma}}.$$

Therefore,

$$\bar{G}(x) = \frac{\alpha x^{-\gamma}}{1+\alpha x^{-\gamma}}; x>0, \gamma>0, \alpha>0. \tag{2.7}$$

The cumulative distribution function (cdf) is given by

$$G(x) = 1-\bar{G}(x) = 1-\frac{\alpha x^{-\gamma}}{1+\ \alpha x^{-\gamma}}$$

$$= \frac{1}{1+\ \alpha x^{-\gamma}}; x>0, \gamma>0, \alpha>0. \tag{2.8}$$

and the corresponding probability density function (pdf) of MOILLD is given by

$$g(x) = \frac{\alpha\gamma}{x^{\gamma+1}\ (1+\ \alpha x^{-\gamma})^2}; x>0, \gamma>0, \alpha>0. \tag{2.9}$$

Figure 1 illustrates some of the possible shapes of the pdf plot of MOILLD for selected values of the parameters α and γ. Figure 2.1 gives the pdf plot of MOILLD when α is constant and γ is varied, figure 2.1 gives the pdf plot of MOILLD when α is varied and γ is constant and figure 2.1 gives the pdf plot of MOILLD when α and γ are varied. The figures show that the distribution of the Marshall-Olkin inverse Log-logistic random variable X is positively skewed and it is unimodal. The skewness tends to zero as the parameters increase simultaneously. It is a more suitable model to obtain accurate probability values at the tails.
By using (2.9), the MOILLD can be expressed to have proper probability density function as follows:

$$\int_0^\infty \frac{\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2}dx$$

$$= \int_0^\infty \frac{\alpha\gamma}{x^{\gamma+1}(1-cot^2\theta)^2}.\frac{-2cot\theta cosec^2\theta x^{\gamma+1}}{\alpha\gamma}d\theta.$$

(we let $cot^2\theta = -\alpha x^{-\gamma}$)

$$= \int_0^{\frac{\pi}{2}} sec2\theta tan2\theta d\theta.$$

$$= 1.$$

Hence, the MOILLD is a proper distribution.

3 Statistical Properties of MOILLD

3.1 Moments of the MOILLD

Theorem 1: Let X be a random variable that has the MOILLD, then, the r^{th} non-central moments is given by

$$E(X^r) = \alpha^{\frac{r}{\gamma}} \frac{\pi r}{\gamma} \csc\left(\frac{\pi(\gamma-r)}{\gamma}\right); r>0, \gamma>0, \alpha>0, \tag{3.1}$$

where csc(.) is the cosecant function.

Proof:

$$E(X^r) = \int_0^\infty x^r g(x)dx$$

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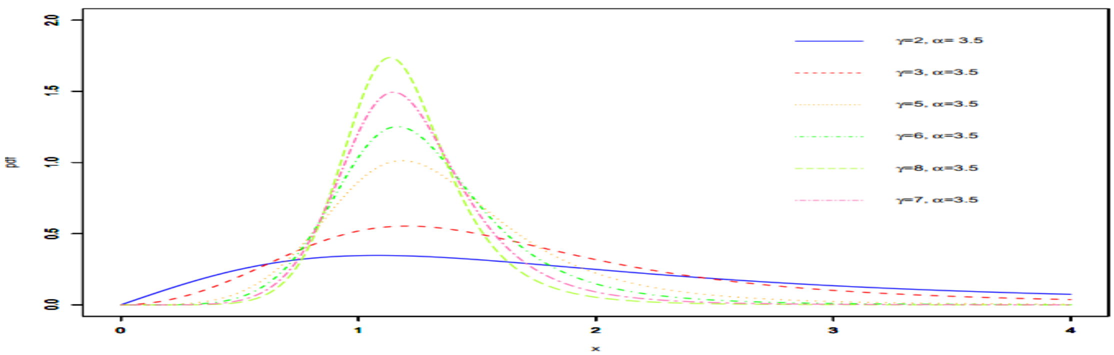
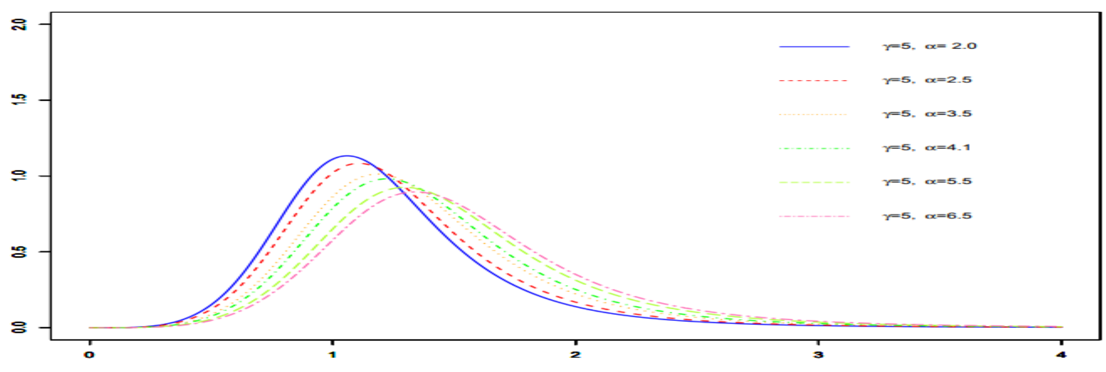
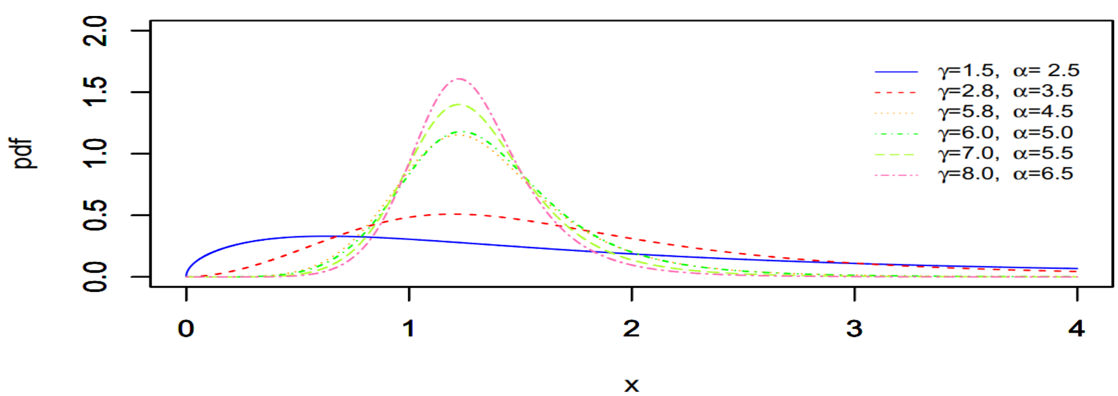


Alpha is constant



Gamma is constant



Parameters are varied

Figure 1: Probability Density Plots of MOILLD.

$$= \int_0^\infty x^r \frac{\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2} dx$$

$$= \int_0^\infty \frac{\alpha\gamma}{x^{\gamma-r+1}(1+\alpha x^{-\gamma})^2} dx$$

$$= \alpha\gamma \int_{0}^{\infty} x^{r-\gamma-1}(1+\alpha x^{-\gamma})^{-2}dx.$$

Let $y = 1 + \alpha x^{-\gamma}$, then, $\frac{dy}{dx} = -\alpha\gamma x^{-\gamma-1}$ and $x = \left(\frac{y-1}{\alpha}\right)^{-\frac{1}{\gamma}}$.
When $x = 0$, $y = \infty$ and when $x = \infty$, $y = 1$.
On substitution,

$$E(X^r) = \alpha\gamma \int_{\infty}^{1} \left(\frac{y-1}{\alpha}\right)^{-\frac{1}{\gamma}\, r-\gamma-1} .y^{-2}.\frac{dy}{-\alpha\gamma x^{-\gamma-1}}$$

$$= -\int_{\infty}^{1} \left(\frac{y-1}{\alpha}\right)^{-\frac{r}{\gamma}} y^{-2}dy. \tag{3.2}$$

Using a property of definite integral, (3.2) becomes

$$E(X^r) = \alpha^{\frac{r}{\gamma}} \int_{1}^{\infty} (y-1)^{-\frac{r}{\gamma}} y^{-2}dy$$

$$= \alpha^{\frac{r}{\gamma}} \int_{1}^{\infty} \frac{(y-1)^{-\frac{r}{\gamma}}}{y^2}dy$$

$$= \alpha^{\frac{r}{\gamma}} \int_{1}^{\infty} \frac{(y-1)^{(1-\frac{r}{\gamma})-1}}{y^2}dy.$$

Using the Beta function integral over a half-line ([11])

$$\int_{1}^{\infty} \frac{(t-1)^{p-1}}{t^2}dt = \frac{\pi(1-p)}{\sin(p\pi)}.$$

Then,

$$E(X^r) = \alpha^{\frac{r}{\gamma}}.\frac{\pi\left(1-\left(1-\frac{r}{\gamma}\right)\right)}{\sin\left(\pi\left(1-\frac{r}{\gamma}\right)\right)}$$

$$= \alpha^{\frac{r}{\gamma}} \frac{\frac{\pi r}{\gamma}}{\sin\left(\frac{\pi(\gamma-r)}{\gamma}\right)}$$

$$= \alpha^{\frac{r}{\gamma}} \frac{\pi r}{\gamma} \csc\left(\frac{\pi(\gamma-r)}{\gamma}\right)$$

where csc(.) is the cosecant function.
This complete the proof.

The first two moments ($r = 1$ and $r = 2$) about the origin for MOILLD are given by

$$E(X) = \alpha^{\frac{1}{\gamma}} \frac{\pi}{\gamma} \csc\left(\frac{\pi(\gamma-1)}{\gamma}\right) \tag{3.3}$$

and

$$E(X^2) = 2\alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \csc\left(\frac{\pi(\gamma-2)}{\gamma}\right). \tag{3.4}$$

Using (3.3) and (3.4), the variance of MOILLD is given by

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 2\alpha^{\frac{2}{\gamma}}\frac{\pi}{\gamma}\csc\left(\frac{\pi(\gamma-2)}{\gamma}\right) - \left(\alpha^{\frac{1}{\gamma}}\frac{\pi}{\gamma}\csc\left(\frac{\pi(\gamma-1)}{\gamma}\right)\right)^2$$

$$= \alpha^{\frac{2}{\gamma}}\frac{\pi}{\gamma}\left(2\csc\left(\frac{\pi(\gamma-2)}{\gamma}\right) - \frac{\pi}{\gamma}\csc^2\left(\frac{\pi(\gamma-1)}{\gamma}\right)\right). \tag{3.5}$$

3.2 Moment Generating Function of MOILLD

Theorem 2: Let X be a random variable that has the MOILLD, then the moment generating function X at t is

$$M_x(t) = \sum_0^\infty \frac{t^m}{m!}\alpha^{m/\gamma}\frac{\pi m}{\gamma}\csc\left(\frac{\pi(\gamma-m)}{\gamma}\right); m>0, \gamma>0, \alpha>0. \tag{3.6}$$

where csc(.) is the cosecant function.

Proof: We have that the moment generating function is given by

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}g(x)dx$$

$$= \int_0^\infty e^{tx}\frac{\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2}dx.$$

Using exponential series, we have

$$M_x(t) = \left(\int_0^\infty \sum_0^\infty \frac{t^m x^m}{m!}\right)\frac{\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2}dx$$

$$= \sum_0^\infty \frac{t^m}{m!}E(X^m)$$

$$= \sum_0^\infty \frac{t^m}{m!}\alpha^{\frac{m}{\gamma}}\frac{\pi m}{\gamma}\csc\left(\frac{\pi(\gamma-m)}{\gamma}\right).$$

To get the first moment, we differentiate $M_x(t)$ with respect to t

$$M_x'(t) = \frac{d}{dt}(M_x(t)) = \sum_0^\infty \frac{t^{m-1}}{m!}\alpha^{\frac{m}{\gamma}}\frac{\pi m^2}{\gamma}\csc\left(\frac{\pi(\gamma-m)}{\gamma}\right)$$

$$= \alpha^{\frac{1}{\gamma}}\frac{\pi}{\gamma}\csc\left(\frac{\pi(\gamma-1)}{\gamma}\right) + 2\alpha^{\frac{2}{\gamma}}\frac{\pi t}{\gamma}\csc\left(\frac{\pi(\gamma-2)}{\gamma}\right) + ...$$

$$E(x) = M_x'(t)|_{t=0}$$

thus,

$$E(X) = M'(0) = \alpha^{\frac{1}{\gamma}}\frac{\pi}{\gamma}\csc\left(\frac{\pi(\gamma-1)}{\gamma}\right). \tag{3.7}$$

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To find the variance, we need to determine the second moment which is the second derivative of $M_x(t)$ with respect to t

$$M''_x(t) = \frac{d^2}{dt^2}(M_x(t)) = \sum_{0}^{\infty} \frac{(m-1)t^{m-2}}{m!} \alpha^{\frac{m}{\gamma}} \frac{\pi m^2}{\gamma} \csc\left(\frac{\pi(\gamma-m)}{\gamma}\right)$$

$$= 0 + 2\alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \csc\left(\frac{\pi(\gamma-2)}{\gamma}\right) + 6\alpha^{\frac{3}{\gamma}} \frac{\pi t}{\gamma} \csc\left(\frac{\pi(\gamma-3)}{\gamma}\right) + \ldots.$$

But

$$E(X^2) = M''_x(0),$$

thus,

$$E(X^2) = M''_x(0) = 2\alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \csc\left(\frac{\pi(\gamma-2)}{\gamma}\right).$$

The variance

$$Var(X) = M''_x(0) - [M'_x(0)]^2$$

$$= 2\alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \csc\left(\frac{\pi(\gamma-2)}{\gamma}\right) - \left(\alpha^{\frac{1}{\gamma}} \frac{\pi}{\gamma} \csc\left(\frac{\pi(\gamma-1)}{\gamma}\right)\right)^2.$$

$$= \alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \left(2\csc\left(\frac{\pi(\gamma-2)}{\gamma}\right) - \frac{\pi}{\gamma}\csc^2\left(\frac{\pi(\gamma-1)}{\gamma}\right)\right) \tag{3.8}$$

3.3 Order statistics

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the order statistics from the sample $X_1, X_2, X_3, \ldots, X_n$ of size n from a continuous population with cdf $F_x(X)$ and pdf $f_x(X)$, then the pdf of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_x(x) \left[F_x(x)\right]^{r-1} \left[1 - F_x(x)\right]^{n-r}; r = 1, 2, \ldots, n. \tag{3.9}$$

The pdf of the r^{th} order statistic for a MOILLD is given by

$$f_{x_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2} \left[\frac{1}{1+\alpha x^{-\gamma}}\right]^{r-1} \left[\frac{\alpha x^{-\gamma}}{1+\alpha x^{-\gamma}}\right]^{n-r}, \tag{3.10}$$

for

$$r = 1, 2, \ldots, n, \gamma > 0, \alpha > 0.$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ for a MOILLD is given by

$$f_{x_{(n)}}(x) = \frac{n\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2} \left[\frac{1}{1+\alpha x^{-\gamma}}\right]^{n-1}, \tag{3.11}$$

and the pdf of the smallest order statistic $X_{(1)}$ for a MOILLD is given by

$$f_{x_{(1)}}(x) = \frac{n\alpha\gamma}{x^{\gamma+1}(1+\alpha x^{-\gamma})^2} \left[\frac{\alpha x^{-\gamma}}{1+\alpha x^{-\gamma}}\right]^{n-1}, \tag{3.12}$$

for

$$r = 1, 2, ..., n, \gamma > 0, \alpha > 0.$$

3.4 Quantile and Random Number Generation from MOILLD

The random numbers from a particular distribution are generated by solving the equation obtained on equating the cdf of a distribution to a number q. Let q be a uniform variate on the interval $(0, 1)$, the procedure for the generation of random numbers from the MOILLD proceeds as:
Let

$$G(x) = q$$

from Equation (2.8);

$$G(x) = \frac{1}{1 + \alpha x^{-\gamma}} = q$$

i.e.

$$\frac{1}{q} = 1 + \alpha x^{-\gamma}$$

$$\alpha x^{-\gamma} = \frac{1}{q} - 1$$

$$x^{-\gamma} = \frac{1}{\alpha}\left(\frac{1}{q} - 1\right)$$

$$x_q = \left[\frac{1}{\alpha}\left(\frac{1}{q} - 1\right)\right]^{-\frac{1}{\gamma}} \tag{3.13}$$

If $q = 0.25, q = 0.5$ and $q = 0.75$, the resulting solutions will be the first quartile (Q_1), Median (Q_2) and third quartile (Q_3) respectively. Hence,

$$Q_1 = x_{0.25} = \left(\frac{3}{\alpha}\right)^{-\frac{1}{\gamma}}. \tag{3.14}$$

$$Q_2 = x_{0.5} = \left(\frac{1}{\alpha}\right)^{-\frac{1}{\gamma}}. \tag{3.15}$$

$$Q_3 = x_{0.75} = \left(\frac{1}{3\alpha}\right)^{-\frac{1}{\gamma}}. \tag{3.16}$$

3.5 Mode of MOILLD

The mode of a distribution is given by solving the first derivative of its probability density function for x. Consider the density of MOILLD given in (2.9) and solve $\frac{\partial \ln g(x)}{\partial x} = 0$ for x, to obtain the mode of Marshall-Olkin Inverse Log-logistic distribution as follows.
That is,

$$\frac{\partial \ln g(x)}{\partial x} = \frac{-\alpha(\gamma+3)x^{2\gamma} - (\gamma+1)x^{3\gamma} + \left((\gamma-3)x^{\gamma} + \alpha(\gamma-1)\alpha^2\right)}{x(x^{\gamma}+\alpha)^3}$$

Setting $\frac{\partial \ln g(x)}{\partial x} = 0$, implies

$$\frac{-\alpha(\gamma+3)x^{2\gamma} - (\gamma+1)x^{3\gamma} + \left((\gamma-3)x^{\gamma} + \alpha(\gamma-1)\alpha^2\right)}{x(x^{\gamma}+\alpha)^3} = 0 \tag{3.17}$$

The maxima can be obtained by solving (3.17) to have

$$(mode = x_{mo}) = \left(\frac{\gamma+1}{\alpha(\gamma-1)}\right)^{\frac{-1}{\gamma}}. \tag{3.18}$$

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3.6 Maximum Likelihood Estimation of MOILLD

The parameters of the MOILLD using the method of Maximum Likelihood Estimation (MLE) are estimated as follows;
Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from MOILLD, then the likelihood function is given by

$$L(x|\alpha,\gamma) = \prod_{i=1}^{n} \frac{\alpha\gamma}{x_i^{\gamma+1}(1+\alpha x_i^{-\gamma})^2}; x>0, \gamma>0, \alpha>0. \tag{3.19}$$

By taking logarithm of the likelihood function, the log-likelihood function is given by

$$\ell = logL(x|\alpha,\gamma) = nlog(\alpha\gamma) - (\gamma+1)\sum_{i=1}^{n}\log(x_i) - 2\sum_{i=1}^{n}\log(1+\alpha x_i{}^{-\gamma}). \tag{3.20}$$

$$= nlog(\alpha\gamma) - (\gamma+1)\left(logx_1 + logx_2 + \ldots + logx_n\right)$$
$$-2\left(log(1+\alpha x_1{}^{-\gamma}) + log(1+\alpha x_2{}^{-\gamma}) + \ldots + log(1+\alpha x_n{}^{-\gamma})\right).$$

To obtain the MLE's of $\hat{\alpha}$ and $\hat{\gamma}$, differentiate the log-likelihood function with respect to α and γ. Thus, we have

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - 2\left(\frac{x_1{}^{-\gamma}}{1+\alpha x_1{}^{-\gamma}} + \frac{x_2{}^{-\gamma}}{1+\alpha x_2{}^{-\gamma}} + \ldots + \frac{x_n{}^{-\gamma}}{1+\alpha x_n{}^{-\gamma}}\right).$$

Therefore,

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\frac{x_i{}^{-\gamma}}{1+\alpha x_i{}^{-\gamma}}. \tag{3.21}$$

and

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} logx_i + 2\alpha\left(\frac{x_1{}^{-\gamma}logx_1}{1+\alpha x_1{}^{-\gamma}} + \frac{x_2{}^{-\gamma}logx_2}{1+\alpha x_2{}^{-\gamma}} + \ldots + \frac{x_n{}^{-\gamma}logx_n}{1+\alpha x_n{}^{-\gamma}}\right).$$

Hence,

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} logx_i + 2\alpha\sum_{i=1}^{n}\frac{x_i{}^{-\gamma}logx_i}{1+\alpha\ x_i{}^{-\gamma}}. \tag{3.22}$$

To find the estimate of α and γ, we set (3.21) and (3.22) to zero. The two estimates were obtained numerically by maximizing the log-likelihood function using the Newton's method.
The Fisher Information is a way of measuring the amount of information that an observable random variable X carries about an unknown parameter θ of a distribution that model X. The Fisher information matrix is used to calculate the covariance matrices associated with maximum likelihood estimates. The second partial derivatives which are useful to obtain the Fisher's information matrix can be computed as follows.

$$\frac{\partial^2\ell}{\partial\alpha\partial\gamma} = \frac{\partial^2\ell}{\partial\gamma\partial\alpha} = -2\left(\sum_{i=1}^{n}\left(-\frac{\alpha x_i^{-\gamma}\ln(x_i)}{1+\alpha x_i^{-\gamma}} + \frac{\alpha(x_i^{-\gamma})^2\ln(x_i)}{(1+\alpha x_i^{-\gamma})^2}\right)\right). \tag{3.23}$$

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{n}{\alpha^2} - 2\left(\sum_{i=1}^{n}\left(-\frac{(x_i^{-\gamma})^2}{(1+\alpha x_i^{-\gamma})^2}\right)\right). \tag{3.24}$$

$$\frac{\partial^2\ell}{\partial\gamma^2} = -\frac{n}{\gamma^2} - 2\left(\sum_{i=1}^{n}\left(\frac{\alpha x_i^{-\gamma}\ln(x_i)^2}{1+\alpha x_i^{-\gamma}} + \frac{\alpha^2(x_i^{-\gamma})^2\ln(x_i)}{(1+\alpha x_i^{-\gamma})^2}\right)\right). \tag{3.25}$$

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The Fisher's information matrix can be computed using the approximation

$$I_x(\hat{\alpha},\hat{\gamma}) = -E\begin{bmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\gamma} \\ \frac{\partial^2\ell}{\partial\gamma\partial\alpha} & \frac{\partial^2\ell}{\partial\gamma^2} \end{bmatrix}, \tag{3.26}$$

where $\hat{\alpha}$ and $\hat{\gamma}$ are the MLEs of α and γ respectively. Using this approximation, we may construct confidence intervals for the parameters of Marshall-Olkin inverse Log-logistic model. The approximate $100(1-\lambda)\%$ confidence intervals for α and γ are given by

$$\hat{\alpha} \pm Z_{\frac{\lambda}{2}}\sqrt{I_{11}^{-1}(\hat{\alpha},\hat{\gamma})} \tag{3.27}$$

and

$$\hat{\gamma} \pm Z_{\frac{\lambda}{2}}\sqrt{I_{22}^{-1}(\hat{\alpha},\hat{\gamma})} \tag{3.28}$$

where $Z_{\frac{\lambda}{2}}$ is the upper $(\frac{\lambda}{2})^{th}$ percentile of the standard normal distribution. The Hessian matrix and its inverse can easily be computed using R statistical software, and hence, find the values of the standard error and asymptotic confidence intervals.

4 Results and Discussion

4.1 Simulation Study of MOILLD

This section compares the parameters for different sample sizes at different combination of parameters on the basis of bias and MSE of MOILLD. All the algorithms are coded in *R* language to generate 10,000 samples by using Monte Carlos simulation. ML estimates for $\alpha = 0.5, \gamma = 2.0$ and $\alpha = 0.5, \gamma = 3.0$ are calculated based on generated samples. Mean of these estimates with bias and MSE for sample sizes 50, 100 and 200 are presented in the table below.

Procedure for Simulation

The following describes the steps to simulate from MOILLD.

i. Generate a random sample of size $r = 50$ from MOILLD for parameters $\alpha = 0.5, \gamma = 2.0$ and $\alpha = 0.5, \gamma = 3.0$ respectively.

ii. Use MLE method to find the estimate of α, $\hat{\alpha}$ and estimate of γ, $\hat{\gamma}$ from the generated data.

iii. Repeat steps (i) and (ii) 1000 times.

iv. Find the mean of $\hat{\alpha_i}$ and $\hat{\gamma_i}$ for $i = 1, 2, 3, ..., 1000$.

v. Calculate the bias of α and γ which is the average of deviation of $\hat{\alpha}$ and $\hat{\gamma}$ from α and γ respectively.

vi. Calculate the MSE of mean estimates which is the mean of square deviation of $\hat{\alpha}$ and $\hat{\gamma}$ from α and γ respectively.

vii. Repeat steps (i) to (vi) for $r = 100$ and $r = 200$ respectively.

The values in Table 1 indicate that the MSE of ML estimators of α and γ decreases and their biases reduce towards zero as sample size increases. While the increase in shape parameters, increases bias and MSE of estimated parameters. These are as usually expected under standard regularity conditions. As the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the estimates.

Table 1: Estimates, bias and MSE of Estimated parameters of MOILLD.

α	γ	Sample size	Parameter	Mean	Bias	MSE
		50	α	0.5071	0.0071	0.0181
			γ	2.0530	0.0530	0.0642
		100	α	0.5032	0.00316	0.00863
0.5	2.0		γ	2.0253	0.0253	0.0303
		200	α	0.5012	0.00128	0.00427
			γ	2.0130	0.0130	0.0147
		50	α	0.5071	0.0071	0.0181
			γ	3.0795	0.0795	0.1445
		100	α	0.5032	0.0032	0.0086
0.5	3.0		γ	3.0380	0.0380	0.0682
		200	α	0.5013	0.0013	0.0043
			γ	3.0196	0.0196	0.0331

4.2 Application of MOILLD to Real Life Data Sets

In this section, the performance of the Marshall-Olkin Inverse Log-logistic distribution is compared with Transmuted Inverse Log-logistic, Inverse Log-logistic, Weibull and log-normal distributions on some lifetime data sets already in literature.

Data set I: Data set generated from MOILLD$(4, 8.5)$. The data are:

1.3044, 1.9670, 3.7129, 1.8000, 4.7132, 1.7272, 3.2961, 1.5888, 1.6830, 1.5134, 2.8873, 1.1359, 1.6638, 2.6407, 2.6741, 2.2169, 1.4676, 2.0460,2.6426, 1.8805, 1.4641, 1.5745, 1.9251, 2.0862, 2.2469, 1.9276, 1.5185, 1.7478, 2.0789, 0.9225, 2.4588, 1.0785, 4.0290, 2.0026, 3.5017, 1.0499 1.8024, 1.6561, 2.6978, 1.9220, 0.9929, 3.8558, 1.5772, 1.6292, 1.8048, 1.2305, 1.5268, 1.2375, 2.2803, 2.0803, 1.2473, 1.1060, 3.2061, 0.5223 2.4182, 0.9931, 1.4543, 2.3761, 0.9166, 1.7527, 0.9193, 1.9618, 1.4970, 1.6260, 1.9136, 1.0575, 2.2904, 2.6740, 2.0319, 1.5696, 1.4826, 2.2582 1.9237, 1.8990, 1.5379, 2.1529, 2.9970, 1.9550, 1.7183, 1.9826, 1.4355, 1.0794, 1.9909, 1.6111, 1.7357, 2.8366, 1.6676, 1.1622, 0.9787, 1.3664 0.6126, 1.0664, 1.1801, 2.4157, 2.1894, 3.1645, 2.1623, 0.5218, 1.7035, 2.1128.

Data set II: The data set represents the survival times (in days) of seventy-two (72) guinea pigs infected with virulent tubercle bacilli. It has been previously used by Shankeretal15. The data are:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70,70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263,297, 341, 341, 376.

Data set III: Data set which represents the relief times of twenty patients receiving an analgesic. This data set was taken from Rodriguesetal15. The data are:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1,1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

Table 2 gives the summary statistics such as minimum and maximum values, first, second and third quartiles, skewness and kurtosis of the three set of data. The skewness coefficients show that the three data sets were positively skewed.
The goodness-of-fit statistics, Akaike information criterion (AIC), Bayesian information criterion (BIC), Log-likelihood (LL) and Kolmogrov-Smirnov (KS) statistic are computed to compare the fitted models. The KS statistic compares the empirical cumulative distribution of the data to any specified continuous distribution when its parameters are estimated by maximum likelihood. This

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Table 2: Summary of Real Life data sets.

Data set	Min.	Q_1	Median	Mean	Q_3	Max.	Skewness	Kurtosis
I	0.5218	1.4617	1.7763	1.8890	2.1963	4.7132	1.0407	1.5966
II	12.00	54.75	70.00	99.82	112.75	376.00	1.7590	2.4596
III	1.10	1.475	1.70	1.90	2.05	4.10	1.5924	2.3465

comparison of the two cdfs looks only at the point of maximum discrepancy of this statistic. The lower the K-S values, the more evidence we have that the two cdfs are from the same distribution.

$$KS = Sup|F_n(x) - F_0(x)| \tag{4.1}$$

where $F_n(x)$ is the empirical distribution function. Generic function calculating AIC and BIC for the model having number of parameters p are respectively given by

$$AIC = 2p - 2LL \tag{4.2}$$

and

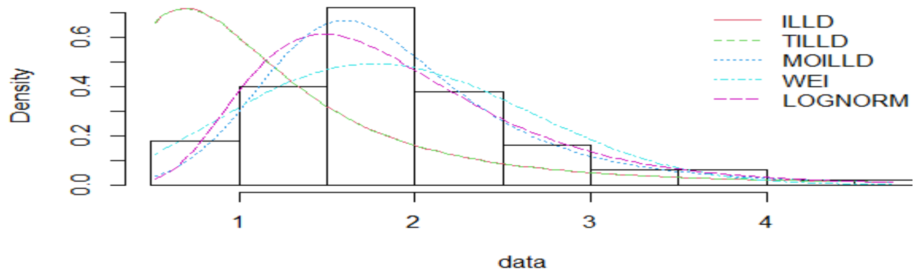
$$BIC = p\log(n) - 2LL \tag{4.3}$$

The smaller the AIC, BIC and $-$LL the better the distribution. All computations are carried out using the R-software (stat and fitdistrplus packages)
Table 3 presents the AIC, BIC, KS statistics and Negative log-likelihood values for fitted data set 1, II and III. The AIC, BIC, Negative Log-likelihood and KS Statistic values favour MOILLD in comparison with two-parameter transmuted inverse log-logistic distribution, one parameter ILLD, Weibull distribution and log-normal distribution. The result shows that the proposed model fit better than the competing distributions.

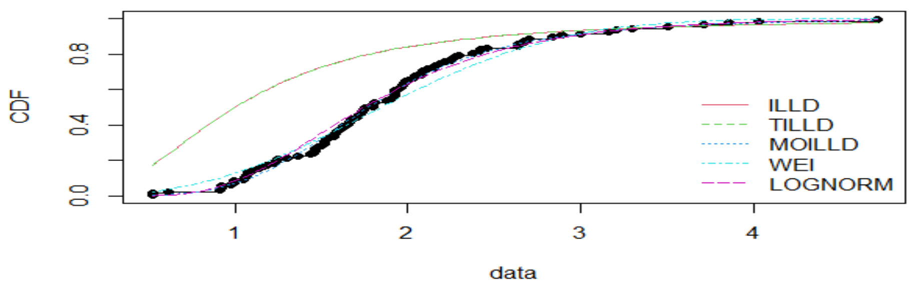
Table 3: AIC,BIC, KS Statistic and Negative Log-likelihood values for fitted data set 1, II and III.

Data	Fitness Measure	ILLD	TILLD	MOILLD	WEIBULL	LOGNORM
I	AIC	331.13	333.13	**215.65**	225.94	225.94
	BIC	333.74	338.35	**220.86**	231.15	223.55
	-LL	164.57	164.57	**105.82**	110.97	107.17
	KS	0.4741	0.4741	**0.0535**	0.0965	0.0854
II	AIC	1055.94	1057.94	**783.93**	798.30	784.67
	BIC	1058.22	1062.50	**788.48**	802.85	789.22
	-LL	526.97	526.97	**389.96**	397.15	390.34
	KS	0.7210	0.7211	**0.0866**	0.1463	0.0956
III	AIC	67.60	69.60	**36.95**	45.17	37.54
	BIC	68.60	71.60	**38.94**	47.16	39.53
	-LL	32.80	32.80	**16.48**	20.59	16.78
	KS	0.5617	0.5616	**0.1108**	0.1849	0.1519

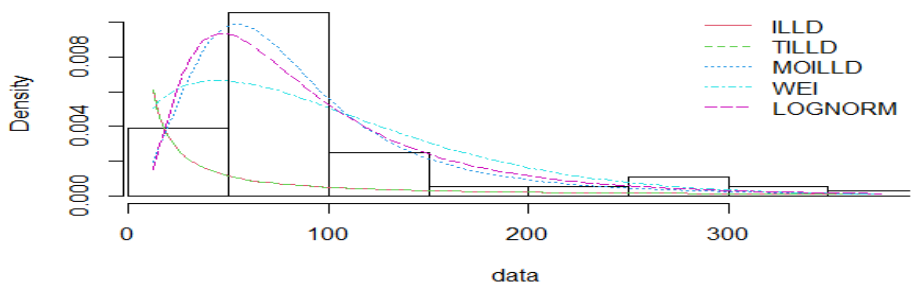
Figures 2 displays the histogram and probability density function and empirical cumulative density function (ecdf) plot of the competing distributions for the three data set to complement the results of AIC, BIC, Negative Log-likelihood and KS Statistic values. Figure 4.2 presents histogram and probability density function and Figure 4.2 presents ecdf plot for data I, Figure 4.2 presents histogram and probability density function and Figure 4.2 presents ecdf plot for data II while Figure 4.2 presents histogram and probability density function and Figure 4.2 presents ecdf plot for data III. The probability distribution function of MOILLD fits the histogram of the data better than the other distributions and the cumulative density function of the distribution fits the empirical distribution of the data better than that of other distributions. The plots show that MOILLD fits the three data set best.



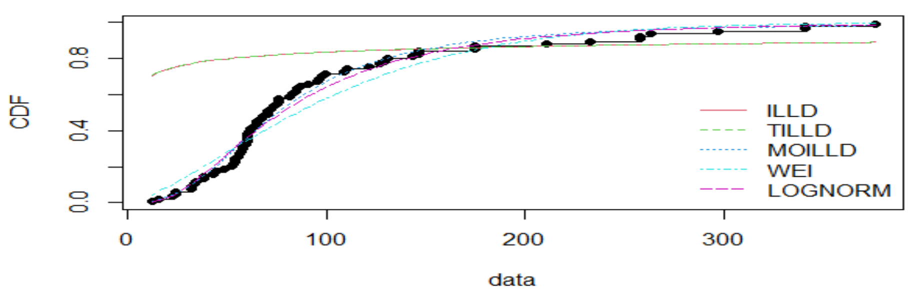
Histogram and probability density function for data I



ecdf plot for data I



Histogram and probability density function for data II



ecdf plot for data II

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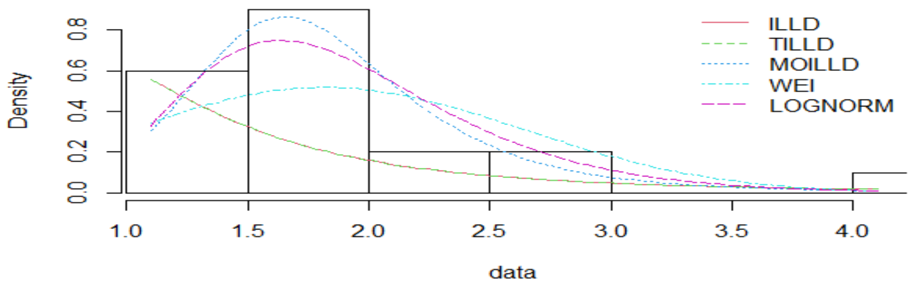
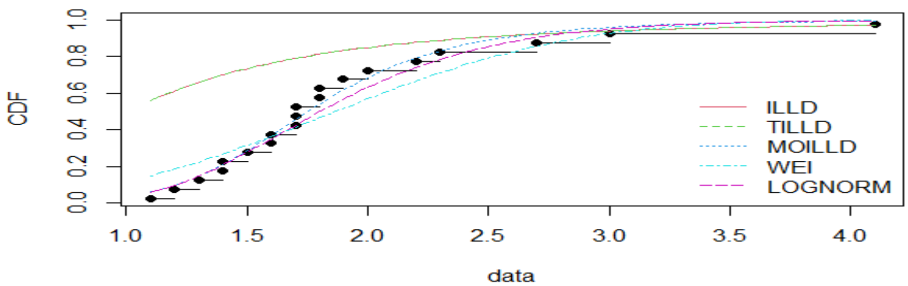


Histogram and probability density function for data III



ecdf plot for data III

Figure 2: The Histogram and fitted pdf and ecdf plot of the competing distributions for data set I, II and III.

5 Conclusion

The study proposed a new distribution named Marshal-Olkin Inverse Log-logistic distribution. The pdf showed that the distribution is positively skewed and unimodal. Some of the statistical properties of the distribution were derived. The expressions for mean, mode, and variance were also derived. The simulation result indicated that the MSE and estimates biases reduce towards zero as the sample size increases. The result also revealed that an increase in shape parameters would increase the bias and MSE of the estimated parameter. The AIC, BIC, KS statistics, and -LL values of the fitted data sets showed that the new model consistently fit better than the competing distributions. Hence, MOILLD is a better substitute for Inverse log-logistic, transmuted inverse log-logistic, Weibull, and log-normal distributions in modeling skewed distributions.

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