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Approximate Identities in Algebras of Compact Operators on Frechet Spaces

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Abstract

Bounded approximate identity(*bai*) is a key concept in the theory of amenability of algebras. In this paper, we show that algebra of compact operators on Frechet space X has both the right and left locally bounded approximate identities. Sufficient conditions for the existence of these identities are established based on the geometry properties of the Frechet space X and its dual space X' respectively.

Keywords: Algebra of compact operators; bounded compact approximation property; Frechet spaces; locally bounded approximate identity.
MSC2010: 46H99.

1 Introduction

Johnson [1], in his memoir connects bounded approximate identity to amenability of Banach algebras, where he showed the importance of *bai* to the notion of amenable Banach algebras. In other to extend the notion of amenability of Banach algebras to Frechet algebras, Pirkovskii [2] introduced the notion of locally bounded approximate identity(*lbai*) for Frechet algebras, where he emphasized that every Frechet algebra admits *lbai*. To further the work of Pirkovskii, Fatemah et al. [3] considered the notion of weak amenable Frechet algebras and the work of Ranjbaril and Rejali [4] looked at ideal amenability of Frechet spaces. Based on the importance of (locally) *bai* to amenability of algebras (see [5]), Dixon [6], in his paper proved that the algebra of compact operator $K(X)$ where X is a Banach space possesses a left *bai* if and only if the Banach space X possesses bounded compact approximation property ($BCAP$). In line with this, Samuel [7] later established that $K(X)$ admits a right *bai* if and only if X' the dual of Banach space X has $BCAP$. Moving in the direction of the notion of *lbai* as introduced by Pirkovskii [2], we extend the works of Dixon [6] and Samuel [7] to algebra of compact operators on Frechet space X. We show that the algebra of compact operators on a Frechet space X possesses left *lbai* based on the geometric property of X and possesses right *lbai* based on the geometric property of X'. Section 2 gives definitions, notations, notions and preliminaries on operator algebras relevant for subsequent work. In section 3 we have the main results which comprises two subsections. Subsection 3.1 looks at some results on the algebras of operators that have direct bearing to the work. In subsection 3.2, we use the geometry properties of the Frechet space X and its dual space X' to show that $K_I(X)$, the algebra of compact operators on a Frechet space X possesses right and left locally bounded approximate identities

2 Preliminaries

Here, we consider some definitions, notions, notations and results that are of interest to this work. We also give concepts and results on linear operators and algebra of operators on locally convex spaces(lcs). For detail see [8–16] .

A topological linear space X is referred to as a lcs if it has a local neighbourhood base comprises convex sets. A lcs X is referred to as reflexive if it coincides with the continuous dual of its continuous dual space i.e., $X = X''$.
A lcs is called *a metrizable lcs* if it possesses countable local neighbourhood base. A *Frechet space* X is a complete, *metrizable lcs*.

A is a *topological algebra* (ta) if and only if it is an algebra equipped with a structure of lcs respect to which the product is separately continuous.

A *Frechet algebra* is a complete topological algebra of which an increasing countable collection $\{p_i : i \in \mathbb{N}\}$ of sub-multiplicative continuous seminorms determines its topology.

For a continuous seminorm p on the lcs X and

$$V = \{s \in X | p(s) \leq 1\}$$

a 0-neighbourhood in X, we define by

$$V^0 = \{t \in X' | \sup_{s \in V} |t(s)| \leq 1\}$$

the polar of V in X' the dual of X.
Set

$$p'(t) = \sup_{s \in V} |t(s)|$$

and

$$V^0 = \{t | \; p'(t) \leq 1\}.$$

Let $X_1 \supset X_2 \supset X_3 \supset \cdots$ be a sequence of Banach spaces. Then we can have a projective limit $\lim_{\longleftarrow} X_k = X \quad (k = 1, 2, \dots)$ which is a Frechet space and also a space of linear form X'_k for X_k. We note that $X = \bigcap X_k$ while $X' = \bigcup X'_k$ its dual is a lcs and takes a strong topology.

A Frechet space X is called quasinormable if for each $n \in \mathbb{N}$ with $m \geq n$ where for each $r > 0$ there is a bounded subset $B_0 \subset X$, neighbourhoods U_m, U_n with $U_m \subset B_0 + rU_n$. Equivalently, if for each equicontinuous set $B_0 \subset X'$ there is a 0-neighbourhood U in X with $B_0 \subset U^0$ where topology of Banach space X'_{U^0} and strong topology on X' are equal on B_0

A net $(e_\alpha)_{\alpha \in I}$ in a ta A is a *right* (*left resp.*) *approximate identity* (ai) if $a = lim_\alpha a e_\alpha$ ($a = lim_\alpha e_\alpha a$ resp.). If a net $(e_\alpha)_{\alpha \in I}$ is a left and a right ai, then it is called *an approximate identity* (ai). A net $(e_\alpha)_{\alpha \in I}$ is called *a bounded ai* (bai) if it is bounded. An algebra A is said to possess a *left* (*right resp.*) *locally bai* ($llbai$) (($rlbai$) resp.) if for each 0-neighborhood $B \subset A$ and for each finite subset $R \subset A$ there is some $c > 0$ where $b \in cB$ with $r - br \in B \quad (r - rb \in B)$ for all $r \in R$. A is said to possess a $lbai$ if it possesses right $lbai$ and left $lbai$.
Given a topological algebra A, let Y be an A-bimodule. $d : A \longrightarrow Y$ is referred to as a *derivation* if $d(rq) = rd(q) + d(r)q, \quad r, q \in A$ and a continuous derivation d is called *inner* if $d(r) = rq - qr$

for each $q \in Y$ and $r \in A$.
A Frechet algebra A is called *amenable* if given an A-bimodule Y, every continuous derivation from A to the dual bimodule Y' is inner.

Given *lcs* X and Y. Let $T : X \longrightarrow Y$ be a linear operator.Then

(i) $T : X \longrightarrow Y$ is called *continuous* if there is U some neighbourhood in X and V a neighbourhood in Y with $T(U) \subset V$. We represent the space of continuous linear operators from X to Y as $L(X, Y)$.

(ii) $T : X \longrightarrow Y$ is called *compact* if there is U some neighbourhood in X where $T(U)$ is a relatively compact set in Y. $K(X, Y)$ represents the space of compact operators

(iii) $T : X \longrightarrow Y$ is called $Montel$ if for some bounded set B in X, $T(B)$ is relatively compact in Y. $M(X, Y)$ represents the space of Montel operators

(iv) $T : X \longrightarrow Y$ is called *bounded* if for some neighbourhood U in X, $T(U)$ is bounded in Y. Space of bounded operators is represented by $LB(X, Y)$

(v) A family, $\mathcal{U}(X, Y)$ of operators associated to each pair of X and Y is called an *operator ideal* if (a) $\mathcal{U}(X, Y)$ is a non-zero subspace of $L(X, Y)$; (b) $RTS \in \mathcal{U}(X_0, Y_0)$ whenever $R \in L(Y, Y_0)$, $T \in \mathcal{U}(X, Y)$ and $S \in L(X_0, X)$ for spaces X_0, X, Y_0 and Y.

(vii) The operator ideal $\mathcal{U}(\mathcal{X}, \mathcal{Y})$ is *closed* if $\mathcal{U} = \bar{\mathcal{U}}$.

A *lcs* X is said to possess the BAP if there exists an equicontinuous net $\{T_j\}_j \subset L(X)$ with $dimT_j(x)$ finite for every $j \in J$ and $limT_j(s) = s$ for every $s \in X$.
A Frechet space X is said to have an *unconditional partition of the identity* (UPI) if for a sequence $\{T_n\}_n$ of continuous linear operators $T_n : X \longrightarrow X$ we have $dimT_n(X)$ finite and $\sum_i (T_i s) = s$ where convergence is unconditional, $s \in X$
A Frechet space X is said to possess the BCAP if and only if there is a sequence $\{T_n\}_n$ of compact operators where $dim(T_n(s)) < \infty$ for every $n \in \mathbb{N}$ and $lim_{n\to\infty} T_n(s) = s$ for each $s \in X$.
Let a Frechet space X have a UPI. We have that for continuous seminorm p on X and a sequence $\{T_i\}_i$ of continuous linear operators, $\sum_i p(T_i s) < \infty \ \forall \ s \in X$.
For a Frechet (barreled) space X, $U = \{s | \sum_i p(T_i s) \leq 1\}$ is a 0-neighbourhood in X. Hence, if $\{p_i\}_{i\in\mathbb{N}}$ is a fundamental system of 0-neighbourhoods for X, then for all k, we have j and $c > 0$ for which

$$\sum_i p_k(T_i s) \leq c p_j(s), \qquad (i)$$

$s \in X$ and so

$$\sum_i |\lambda(T_i s)| \leq c p'_k(\lambda) p_j(s) \qquad (ii)$$

$\lambda \in X'$, $s \in X$ and $p'_k(\lambda) = \sup\{|\lambda(s)| : p_k(s) \leq 1\}$.

Remark 2.1. The following implications hold: $UPI \Longrightarrow BAP \Longrightarrow BCAP$. See ([17], p.181) and ([18], p.826).

We consider next the operator algebra $L(X)$ where X is a *lcs*.

Remark 2.2. The algebra $L(X)$ where X is a *lcs* has no locally convex topology with jointly continuous multiplication, (see [16]). The importance of operator algebra $L(X)$ to the operator theory led to the search for subalgebra of $L(X)$ that can enjoy the locally convex topology with jointly continuous multiplication. In view of this, the definitions of an operator algebra and of a topologizable operator with some other notable results are presented here. For details see [8,9,16]. The following definitions and theorems can be found in [16].

Definition 2.3. Let X be a *lcs*. A unital subalgebra $A \subset L(X)$ which is a *lcs* is an operator algebra on X such that

$$(T, s) \longmapsto Ts \tag{iii}$$

is jointly continuous from $A \times X \to A$. We present this in form of defining family of continuous seminorms. Suppose that A is an algebra of operators on X. If $(p_\alpha)_{\alpha \in I}$ is a family of continuous seminorm on X and $(p_\beta)_{\beta \in I}$ is a family of continuous seminorm on A, then (iii) has the same meaning as: For each

$$\alpha \in I, \exists \alpha' = \alpha'(\alpha), \beta = \beta(\alpha) > 0$$

where

$$p_\alpha(Ts) \leq p_\beta(T) p_{\alpha'}(s) \; \forall T \in A$$

and $\forall s \in X$.

Definition 2.4.. Let X be a *lcs*. An operator T in $L(X)$ can be topologized if it is a member of some operator algebra on X.

Remark 2.5. Subalgebra of $L(X)$ can be treated as an operator algebra with different topologies. The following result established that a unital *lc* algebra is topologically isomorphic with some operator algebra.

Theorem 2.6. Let A be a unital *lc* algebra. Then A is topologically isomorphic with some operator algebra.

Let $L_I(X)$ denote the operator algebra endowed with *lc* topology such that it is a locally multiplicatively convex algebra where I represents any family of continuous seminorm for the topology of a *lcs* X. Then the following theorem is identical with Theorem 2.6.

Theorem 2.7. Let A ba a unital m-convex algebra. Then A is topologically isomorphic with some operator algebra of the form $L_I(X)$.

Remark 2.8. We shall give $L_I(X)$ a topology and consider its algebra. For details see [8] and [9]. Let $\{p_\alpha : \alpha \in I\}$ be a collection of continuous seminorms on a *lcs* X, we define family $\{\epsilon V_\alpha : \alpha \in I, \; \epsilon > 0\}$ where $(V_\alpha = \{s \in X | p_\alpha(s) \leq 1\})$ is the base of 0-neighbourhood for the topology on X.
Hence, the collection of linear operators $T : X \longrightarrow X$ denoted by $L_I(X)$ is an algebra of operators given that for a real number $c_\alpha > 0$ there exists $TV_\alpha \subset c_{\alpha,T} V_\alpha$ for all $\alpha \in I$.
For each $p_\alpha \in \{p_\alpha : \alpha \in I\}$ the real valued function p_α on $L_I(X)$ defined by $p_\alpha(T) = \inf\{\lambda_{\alpha,T} : p_\alpha(Ts) \leq \lambda_{\alpha,T} \, p_\alpha(s) \; \forall \; s \in X\} = \sup_{s \in X}\{p_\alpha(Ts) : p_\alpha(s) \leq 1\}$ is a seminorm.
Addition on $L_I(X)$ is defined pointwise with multiplication being defined by composition. Since we are familiar with the fact that $L_I(X)$ is a *lcs*, we shall only verify that the operator seminorm is submultiplicative. For $T_1, T_2 \in L_I(X)$

$$\begin{aligned}
p_\alpha(T_1T_2) &= \sup_{s \in X}\{p_\alpha(T_1T_2s) : p_\alpha(s) \leq 1\} \\
&\leq \sup_{s \in X}\{p_\alpha(T_1)p_\alpha(T_2s) : p_\alpha(s) \leq 1\} \\
&\leq \sup_{s \in X}\{p_\alpha(T_1)p_\alpha(T_2)p_\alpha(s) : p_\alpha(s) \leq 1\} \\
&= p_\alpha(T_1)p_\alpha(T_2) \\
\therefore \quad & \quad p_\alpha(T_1T_2) \leq p_\alpha(T_1)p_\alpha(T_2).
\end{aligned}$$

This shows that we can have the following relation

$$p_\alpha(Ts) \leq p_\alpha(T)p_\alpha(s) \;\forall\, T \in L_I(X) \;\; \alpha \in I \;\text{ and }\; s \in X.$$

Set $\beta_\alpha = \{T \in L_I(X) : TV_\alpha \subset V_\alpha\}$.
The following topology is given to $L_I(X)$ as base of 0-neighbourhood.

$$\left\{ \epsilon \bigcup_{j=1}^{k} \beta_{\alpha_j} | \epsilon > 0 \; \alpha_j \in I, \;\; k \text{ is finite} \right\}$$

We note that $\beta_\alpha = \{T \in L_I(X) : p_\alpha(T) \leq 1\}$.
Hence, the family $\{p_\alpha : \alpha \in I\}$ of continuous seminorm is defined on $L_I(X)$. Under this topology, $L_I(X)$ become a Hausdorff locally convex topological algebras. Suppose X is Frechet, then $L_I(X)$ becomes a complete locally multiplicatively convex algebra

3 Main Results

In subsection 3.1, we first establish some results on the algebra of compact operators on locally convex spaces. Subsection 3.2 considers the existence of *llbai* and *rlbai* for algebras of compact operators on Frechet space.

3.1 Algebras of operators

We establish here some results on algebras of operators on locally convex spaces that are of interest to our work in this paper.

The following result identifies $L_I(X')$ with the direct limit of inductive system of operators.

Theorem 3.1.1. Suppose X' is the dual of a Frechet space X, there exists a bijection between operators on X' and the strict inductive limit of the inductive system of continuous linear operators of Banach spaces.

Proof. Let $i, j \in I$ such that $j \geq i$, we define a map $f_{ij} : X_i \longrightarrow X_j$ such that $U_i \subset U_j$ where U_i and U_j are 0-neighbourhoods in X_i and X_j respectively and f_{ij} is continuous with $\{X_i\}_i$ being family of Banach spaces. Hence, we identify X' as the strict direct limit of sequence of Banach spaces $\{X_i\}$. That is $X' = \lim_{\longrightarrow} X_i = \bigcup X_i = X' \quad (i = 1, 2, \dots)$ with $f_{ij} \circ f_{jk} = f_{ik}$ satisfied for $j \geq i$, $k \geq j$. X' is endowed with strict inductive limit topology where $f_i : X_i \longrightarrow X'$ is continuous such that $f_i(s_i) = s'$ and $f_{ij}(s_i) = s_j$. Hence, X' is a complete *lcs*. We identify X' as the dual of a Frechet space X.
Moreover, given $i \in I$. Let $T_i : D(T_i) \subset X_i \longrightarrow X_i$. $\{T_i : i \in I\}$ can be seen as an inductive system of operators in such a way that for $s_i \in D(T_i) \subset X_i$ and $i > j$.

$$T_i(f_{ji}(s_j)) = f_{ji}(T_j(s_j)).$$

We then define T' as the inductive limit of the inductive system $\{D(T_i) : i \in I\}$ using $T'(s') = f_i(T_i(s_i))$ or $f_i^{-1}(T'(s')) = T_i(f_i^{-1}(s'))$ where $s' \in D(X')$ with $i \in I$. Therefore, we refer to T' as the direct limit of $\{T_i : i \in I\}$. We have that T' is a linear operator. Hence, for each i, $T_i \in L(X_i)$, there exists $T' \in L_I(X')$.
In the sequel, we finally have the following relation. $T \in L_I(\lim_{\longleftarrow} X_i) = L_I(X)$ and $T' \in L_I(\lim_{\longrightarrow} X_i) = L_I(X')$.

Remark 3.1.2.

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(i) Note that the following inclusion suffices. $K_I(X,Y) \subseteq M_I(X,Y) \subseteq L_I(X,Y)$ and $K_I(X,Y) \subseteq LB_I(X,Y) \subseteq L_I(X,Y)$

(ii) Given that X, Y and Y_0 are Frechet spaces and define $R \in LB_I(X,Y_0)$ and $S \in M_I(Y_0,Y)$ then, the following map commutes.

$$S \circ R : X \longrightarrow Y_0 \longrightarrow Y$$

. Hence, $S \circ R = T \in K_I(X,Y)$.

Proposition 3.1.3. Suppose X and Y are Frechet spaces where X_0 and Y_0 are subspaces of X and Y respectively. Let X be quasi normable and Y be reflexive. If $R \in LB_I(X_0,X) \subseteq L(X_0,X)$ and $S \in M_I(Y,Y_0) \subseteq L(Y,Y_0)$, then the algebra of compact operators $K_I(X,Y)$ is an ideal in $L_I(X,Y)$.
Proof. Suppose $R \in LB_I(X_0,X)$, $S \in M_I(Y,Y_0)$ and $T \in K_I(X,Y)$. We need to show that $K_I(X,Y)$ is an ideal.
Since $K_I(X,Y) \subseteq L_I(X,Y)$, it is not empty.
By definition, there exists some neighbourhood $U_0 \subset X_0$ and a bounded subset $B \subset X$ such that

$$RU_0 \subset B, \qquad (i)$$

Since X is quasi normable, there are 0-neighbourhoods U and V with $V \subset U$ such that for every $\epsilon > 0$ we have $V \subset B + \epsilon U$. Hence, by definition there exists a compact set $W \subset Y$ where TV is relatively compact in Y.That is

$$TV \subset W, \qquad (ii)$$

Lastly, since Y is reflexive, the relatively compact set TV is a bounded set in Y. Hence, there exists by definition a compact set $G \subset Y_0$ where $S(TV)$ is relatively compact in Y_0. That is

$$S(TV) \subset G. \qquad (iii)$$

From relations (i) and (ii), $RU_0 \subset B + \epsilon U$. Hence,

$$T(RU_0) \subset W, \qquad (iv)$$

From relations (iii) and (iv), since $T(RU_0)$ is relatively compact, which implies that it is bounded in a reflexive Frechet space Y. Hence,

$$ST(RU_0) \subset G. \qquad (v)$$

This implies that $STR \in K_I(X_0,Y_0)$. There fore, $K_I(X,Y)$ is an ideal. □

Theorem 3.1.4. Let X and Y be Frechet spaces, where X is quasi normable and Y is reflexive, then $K_I(X,Y)$ is a closed subspace of $L_I(X,Y)$.

Proof. There is a uniform convergence topology τ on set given to $L_I(X,Y)$ as a result of continuity of linear operators in $L_I(X,Y)$. Let $\lambda, \gamma \in$ with $T_1, T_2 \in K_I(X,Y)$. Let $(s_i)_i^\infty$ be a sequence in X. Since T_1 and T_2 are compact operators, then, we let $\{s_{ij}\}_j^\infty$ be a convergent subsequence generated by T_1 and $\{s_{jk}\}_k^\infty$ be a convergent subsequence of $\{s_{ij}\}_j^\infty$ generated by T_2. Then, we have $\{(\lambda T_1 + \gamma T_2)\}_k^\infty$ to be a convergent subsequence in Y. $\lambda T_1 + \gamma T_2 \in K_I(X,Y)$ since $K_I(X,Y)$ is a Frechet space. With this, we next show that $K_I(X,Y)$ is closed.
Let $T_n \in K_I(X,Y)$ for $n \geq 1$ and $\lim_{n\to\infty} T_n = T \in L_I(X,Y)$. If we can show that T is a compact operator, we are done. Suppose set B_0 is bounded in X. Since X is quasi normable, there are 0-neighbourhoods V and U with $V \subset U$ and $r > 0$ such that $V \subset B_0 + rU$. We need to establish relative compactness of $T(V)$. Similarly, for every $r > 0$ we have $T(V) \subset T(B_0) + rT(U)$. $T(B_0)$ is bounded in Y. Let $T(U) = U_0$ in Y. Let V_0 and W_0 be 0-neighborhoods in Y such that $V_0 + V_0 + V_0 \subset U$ and $W_0 \subset V_0$. Hence, there is a set $F \subset T(B_0)$ which is finite in Y where $T_N s - Ts \in F + rV_0$ and $T_1, \ldots, T_N \in F$, since $T_N V$ is bounded and relatively compact by

definition. Let $\{t_j\}_{j=1}^{\infty} = \{T_N s_j\}_{j=1}^{\infty}$ such that $\{t_j\}_{j=1}^{m}$ is a finite cover for $T_N(V)$, for some $N > 0$. Now for all $s \in V$, $(T_N s_j - T_N s) \in F + rW_0$. Then,

$(Ts_j - Ts) = (Ts_j - T_N s_j + T_N s_j - T_N s + T_N s - Ts)$
$= ((T - T_N)s_j) + (T_N s_j - T_N s) + (T_N - T)s)$
$\in (F + rV_0) + (F + rW_0) + (F + rV_0) \subseteq F + r(V + W_0 + V)$
$\subset F + rU$

i.e. $(Ts_j - Ts) \in (F + rU)$. Since $T(B_0)$ is relatively compact, there exists a compact set K_r such that $(Ts_j - Ts) \in (F + rU) \subset (T(B_0) + rU_0) \subset K_r + rU_0$. Hence, $T(V)$ is relatively compact so $T \in K_I(X, Y)$. □

Corollary 3.1.5. Suppose X is a Frechet space, then a closed two sided ideal of $L_I(X)$ is $K_I(X)$.

Definition 3.1.6. Let X and Y be Frechet spaces. For $T \in L_I(X, Y)$, an adjoint of T is an operator $T' \in L_I(Y', X')$. We shall use the notation $\langle s, s' \rangle = s'(s)$ for $s' \in X'$ and $\lambda \in Y'$ define T' so that $\langle s, T'(\lambda) \rangle = \langle Ts, \lambda \rangle$ whereby $(T'\lambda)(s) = \lambda(Ts)$ for all $s \in X$, $\lambda \in Y'$.

3.2 Existence of Locally Bounded Approximate Identities

We shall look at some results that establish bounded compact approximation properties for Frechet space X and its dual space X' which consequently give rise to the existence of locally bounded approximate identity for $K_I(X)$.

The Frechet space X considered here is reflexive and quasi normable.

Proposition 3.2.1. Suppose X is a Frechet space. An UPI for X implies an UPI for X'.

Proof. Let a Frechet space X has UPI. That is, for a continuous linear sequence of operators $\{T\}_i \subset L_I(X)$ with $\dim(T_i(s)) < \infty$ and $i \in \mathbb{N}$, we have $\sum_i^n T_i(s) = s$. Let s_i converge to s weakly in X. Then, we have

$$\sum_i T_i(s_i - s) = \sum_i (T_i s_i - T_i s) \longrightarrow 0.$$

Hence, from section 2 inequality (i), we have for all k there exists j and $c > 0$ such that

$$\sum_i p_k(T_i(s_i - s)) \leq c p_j(s_i - s).$$

Therefore,

$$\begin{aligned} |cp_j(s_i) - cp_j(s)| &\geq |\sum_i p_k(T_i(s_i)) - \sum_i p_k(T_i(s))| \\ &\leq \sum_i p_k(T_i(s_i - s)) \leq cp_j(s_i - s), \end{aligned}$$

Hence,

$$|\sum_i p_k(T_i(s_i)) - \sum_i p_k(T_i(s))| \leq cp_j(s_i - s).$$

Then for $\lambda \in X', s \in X$ we have

$$\begin{aligned} |cp'_k(\lambda)p_j(s_i)| - |cp'_k(\lambda)p_j(s)| &\geq \sum_i |\lambda(T_i(s_i)| - \sum_i |\lambda(T_i(s)| \leq \sum_i |\lambda(T_i(s_i - s)) \\ &\leq cp'_k(\lambda)p_j(s_i - s) \end{aligned}$$

i.e $\sum_i |\lambda(T_i(s_i))| - |\lambda \sum_i T_i(s)| \leq cp'_k(\lambda)p_j(s_i - s)$.
Since the summation is over i and also since X has UPI

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i.e.$\sum_i T_i s = s$, we then have

$$\sum_i |\lambda(T_i(s_i)| - |\lambda \sum_i T_i(s)| \leq cp'_k(\lambda)p_j(s_i - s)$$

then,

$$\sum_i |\lambda(T_i(s_i))| - |\lambda(s)| \leq cp'_k(\lambda)p_j(s_i - s) \qquad (*)$$

Let define an operator $T'_i : X' \to X'$ from Theorem 3.1.1 such that $T'_i(\lambda s_i) = \lambda(T_i(s_i))$ where $\lambda \in X'$.
$(*)$ now becomes

$$\begin{aligned} \sum_i |T'_i(\lambda s_i)| - |\lambda(s)| &\leq cp'_k(\lambda)p_j(s_i - s) \\ |cp'_k(\lambda)p_j(s_i)| - |cp'_k(\lambda)p_j(s)| &\geq \sum_i |T'_i(\lambda s_i)| - |\lambda(s)| \\ &\leq |\sum_i T'_i(\lambda s_i) - \lambda(s)| \leq cp'_k(\lambda)p_j(s_i - s). \end{aligned}$$

This implies that

$$|\sum_i T'_i(\lambda s_i) - \lambda(s)| \leq cp'_k(\lambda)p_j(s_i)$$

Let $\epsilon = \max(m, cp'_k(\lambda)p_j(s_i))$ for $m > 0$

$$\therefore \quad |\sum_i T'_i(\lambda s_i) - \lambda(s)| \leq \epsilon$$

Hence, it implies that

$$\sum_i T'_i(\lambda s_i) - \lambda s \longrightarrow 0.$$

For $s \in X$, we have

$$\sum_i T'_i(\lambda s) - \lambda s = 0$$

that is,

$$\sum_i T'_i(\lambda s) = \lambda s \text{ for } \lambda(s) \in X'.$$

Therefore X' has an UPI. $\square$.

Proposition 3.2.2. Suppose X is a Frechet space, the BCAP for X' implies the BCAP for X.

Proof. Let X' has the BCAP, then we have a compact operator

$$T' : X' \longrightarrow X'$$

. Define an operator

$$T : X \longrightarrow X$$

. We shall show first that T is compact. Since X is quasi normable, there exists 0 neighbourhoods U, V and a bounded set B_0 and $r > 0$ such that

$$V \subseteq B_0 + rU$$

. Hence

$$T(V) \subseteq T(B_0) + rT(U)$$

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. Define

$$T'' : X'' \longrightarrow X''$$

. Since X' is a strong dual of X, every bounded set B_0 is equicontinuous in the strong bidual topology on X''. Therefore, the set $T''(B_0)$ which is a subset of X is relatively compact in X''. Furthermore, in the topology of uniform convergence on equicontinuous subsets of X', $T''(B_0)$ is relatively compact. Hence, as a result of the induced topology on X, $T(B_0)$ is relatively compact in X. To this end, there is a compact set E in X so that

$$T(V) \subseteq E + rT(U)$$

. This implies that $T(V)$ is relatively compact in X. This implies that T is a compact operator. Moreover, since X' has a BCAP, then by definition there is an equicontinuous bounded compact operator net $\{T'_n\}_n \subset K_I(X')$ with $\dim(T'_n) < \infty$ and $\lim_n T'_n(s') = s'$ for every $s' \in X'$. Let $\{s'_1, \ldots, s'_n\} \subset X'$. Let $E = \{s_1, \ldots, s_n\}$ be in the collection of compact convex and balanced sets in X with $T'_i = \sum_i^n s'_i \otimes s_i$. Thus $T_i = \sum_i^n s_i \otimes s'_i$.

Since E is compact in X, then we have a sequence of compact sets E_i in X where $\bigcup E_i$ covers E and span $E \subset X$. Hence for $\{T_i\}_i \subset K_I(\bigcup E_i)$, there exists $s \in X$ such that $\lim_i T_i(s) = s$ weakly on X. Then we have a compact operator
$T_n : X \longrightarrow X$ such that

$$T_n(s) = \sum_i^n r_i T_i(s),\ r_i \in, r_i \longrightarrow 0$$

and

$$\lim_n T_n(s) = \lim \sum_i^n r_i T_i(s) = s.$$

Hence, $\lim_n T_n(s) = s$. $\square$.

Theorem 3.2.3. Suppose X is a Frechet space. $K_I(X)$, the algebra of compact operator on X, possesses a *llbai* if and only if X has the BCAP.

Proof. Suppose X is a Frechet space that has the BCAP. Then, there is a sequence of equicontinuous bounded compact operators $\{T_n\}_{n\in\mathbb{N}} \subset K_I(X)$ with $\dim\{T_n\} < \infty$ where $\lim_n T_n(s) = s$. Given a family $\P(X)$ of all compact absolutely convex sets in X, we can find $C_0 \subset X$ with $E \in \P(X)$ where $C_0 \subset E$. Hence, the linear $SpanE \subset X$. We define algebra of bounded operators $T \in LB_I(SpanE, X)$. By defintion we have a 0-neighbourhood $V \subset SpanE \subset X$ where $T(V)$ is bounded in X. Hence, we define a finite subset $F \subset X$ where $T \in F$.

Suppose U is a 0-neighbourhood in X. Choose the balance 0-neighbourhood W where $V \subseteq W$, $V^2 \subseteq W$ and $W + W \subseteq U$. Since T_n is a compact operator there exists $\lambda > 0$ such that $T_n(s) \in \lambda V_n \subset \lambda V$. We define a projection $B : X \longrightarrow Span\ E$ which is a Montel operator where $B \in M_I(X, SpanE)$. Hence, we have the following composition from proposition 3.1.3.
$SpanE \xrightarrow{T} X \xrightarrow{T_n} X \xrightarrow{B} SpanE$
$T - BT_n \in \lambda^{-1}W$, $T_n - T_nT \in \lambda^{-1}V$ and $BT_n - T_n \in \lambda^{-1}V$ therefore

$$\begin{aligned} T - T_nT &= T - BT_n + BT_n - BT_nT + BT_nT - T_nT \\ &= (T - BT_n) + B(T_n - T_nT) + (BT_n - T_n)T \\ &\in \lambda^{-1}W + \lambda V \cdot \lambda^{-1}V + \lambda^{-1}V \cdot \lambda V \\ &\subset \lambda^{-1}W + V^2 + V^2 \subset \lambda^{-1}W + W + W \\ &\subseteq U + U \subset U \end{aligned}$$

$\therefore\ T - T_nT \in U$.
Therefore, T_n is a *llbai* for $K_I(X)$. Conversely, suppose $K_I(X)$ has a *llbai*. That is $T - T_nT \subset U$ for U a 0-neighbourhood in (X). We then consider a compact subset E of X where the $SpanE \subset X$

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and a projection $T : SpanE \longrightarrow X$ is defined by $T(s) = s$ for every $s \in SpanE$. Therefore, $\lim_n(T_nT(s) - T(s)) = 0$. This shows that

$$\lim_n T_nT(s) - T(s) = \lim_n T_n(s) - s = 0$$

i.e. $\lim_n T_n(s) = s$. Hence X possesses a BCAP. □.

Theorem 3.2.4. Suppose X is a Frechet space. Then the algebra $K_I(X)$ of compact operator on X has a *rlbai* if and only if X' has a BCAP.

Proof. Suppose X is a Frechet space and let its dual X' has an UPI. By Proposition 3.2.1 and Remark 2.1, X' has BCAP and this implies that for a sequence of compact operators $\{T_i\}_{i\in\mathbb{N}} \subset K_I(X)$ with $\dim\{T_i\} < \infty$ we have a net of compact bounded equicontinuous operators $(T_i')_i \subset K_I(X')$ with $\dim\{T_i'\}_i < \infty$ where $\lim_i T_i'(s') = s'$ for every $s' \in X'$. From $T \in LB_I(SpanE, X)$ in (Theorem 3.2.3) we define a projection $B' : X' \longrightarrow \text{Span } E'$ such that $T' \in LB_I(X', SpanE')$ for every $s' \in X'$, so also from $B \in M_I(X, SpanE)$ in (Theorem 3.2.3) there is an inclusion operator $B' : \text{Span } E' \to X'$ such that $B' \in M_I(SpanE', X')$ (see [19] Corollary 2.3).
For a bounded set D in $SpanE'$, $B'(D)$ is relatively compact in X'. Since the strong dual of a reflexive Frechet space is reflexive, $B'(D)$ is bounded. Hence, there exist a finite set $P \subset X'$ where $B' \in P$. So also, since the Frechet space X is quasinormable, given a 0-neighbourhood V in X, we have $B'(D) \subset V'$ where V' is the polar of V. This implies that $T_i(V')$ is relatively compact in X'. Let U' be a 0-neighbourhood. We choose the balanced 0-neighbourhood V' and W' such that $V' \subseteq W'$, $V'^2 \subseteq W'$ and $W' + W' \subseteq U'$. Since we have $T_i'(V')$ being relatively compact (bounded) implies that we can have $\lambda > 0$ where $T_i' \in \lambda V_i' \subset \lambda V'$. So also for $V' \subset X'$, we have $B' \in \lambda V'$. For the bounded operator T', and for 0-neighbourhood $V' \subset X'$ such that $T' \in \lambda V$.
We use the following composition

$$Span\ E' \xrightarrow{B'} X' \xrightarrow{T_i'} X' \xrightarrow{T'} SpanE'.$$

We note that, $B' - T'T_i' \in \lambda^{-1}W'$, $T_i' - T_i'B' \in \lambda^{-1}V'$ and $(T'T_i' - T_i') \in \lambda^{-1}V'$. Hence,

$$\begin{aligned} B' - T_i'B' &= (B' - T'T_i') + (T'T_i' - T'T_i'B') + (T'T_i'B' - T_i'B') \\ &= (B' - T'T_i') + T'(T_i' - T_i'B') + (T'T_i' - T_i')B' \\ &\in \lambda^{-1}V' + \lambda V' \cdot \lambda^{-1}V' + \lambda^{-1}V' \cdot \lambda V' = \lambda^{-1}V' + V'^2 + V'^2 \subset \lambda^{-1}V' + W' + W' \\ &\subset U' + U' \subset U' \end{aligned}$$

$\therefore\ B' - T_i'B' \in U'$.
Therefore, $B' - T_i'B' \in U'$. This means that $B - BT_i \in U$. Hence, we can find a bounded subset $M \subset X$ for which $B(M)$ is totally bounded in $SpanE$ and also a finite subset $A \subset B(M)$ where $B \in A$. Hence, $B - BT_i \in U$ implies that T_i is a *rlbai* for $K_I(X)$.
Conversely, suppose $K_I(X)$ has a *rlbai*. This implies that $B - BT_i \subset U$ for U a 0-neighbourhood in $K_I(X)$. Suppose $E' \subset X'$ is a finite compact subset of X' and $\text{Span } E' \subset X'$. We define a projection $B' : Span\ E' \longrightarrow X'$ by $B'(s') = s'$ for every $s' \in\ Span\ E'$. Hence,

$$\lim_i (T_i'B'(s') - B'(s')) = 0$$

implies

$$\lim T_i'B'(s') - B'(s') = \lim_i T_i'(s') - s' = 0.$$

Hence, $\lim_i T_i'(s') = s'$. Therefore, X' has a bounded compact approximation property. □

Conflicts of Interest

No conflict of interest was declared by the authors.

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