Minimization of material volume of three layer compound cylinder having same materials subjected to internal pressure

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Abstract

A thick cylinder storing fluid with large internal pressure has second order non-linear variation in the hoop stress across the wall. Utilization of material of the cylinder is not up to its full capacity. For more uniform hoop stress distribution, compound cylinders are formed by shrinkage process where outer cylinder is heated until it will slide freely over inner cylinder thus exerting the required shrinkage pressure on cooling. Using the calculated shrinkage pressures between two contacting cylinders, it is possible to reduce the hoop stress and make it more or less uniform over the thickness. This paper introduces the methodology for minimization of volume of shrink-fitted three layer compound cylinder and to get equal maximum hoop stresses in all the cylinders. The analytical results are validated in comparison with FEM in ANSYS Workbench. Both the results agree with each other. Thus methodology can be applied to multi-layer compound cylinders used in real-world applications.

Keywords: multi-layer cylinders, hoop stress, residual stress, shrink fit, shrinkage (contact) pressure, optimum design.

1. Introduction

Compound cylinders have wide applications in hydraulic presses, forging presses, power plants, gas storages, chemical and nuclear plants, military applications etc. To increase the pressure capacity of thick-walled cylinders, two or more cylinders (multi-layer) are shrunk into each other with different diametric differences to form compound cylinder. Manufacturing and assembly process of such compound cylinders of real-world applications introduces some residual stresses. The level of such residual stresses can reach quite significant value and cannot be neglected. These residual stresses can be summed up with hoop stress developed due to internal pressure to find maximum hoop stress in all the cylinders.

The optimization of bi-metal compound cylinders has been proposed and minimized the weight of compound cylinder for a specific pressure by Majzoobi *et al.* (2006). The variables were shrinkage radius and shrinkage tolerance. They have used SQP technique for optimization, the finite element code, ANSYS for numerical simulation and concluded that high pressure cylinders can partially be replaced by a lower, lighter and cheaper material to reduce the cost and weight of cylinders. Sequential quadratic programming (SQP) is an iterative method for nonlinear optimization. SQP methods solve a sequence of optimization subproblems, each which optimizes a quadratic model of the objective subject to a linearization of the constraints. Two layer compound cylinder has been introduced and optimized intermediate, outer diameter and shrinkage tolerance to get minimum volume of two layer compound cylinders by Patil (2005).

Figure 1 shows that radial (σ_r) and hoop (σ_θ) stress distribution across the wall of thick cylinder is non-linear in nature from inner to outer radius of the cylinder. Dimensions and material of the cylinder are usually designed to tolerate the stresses which occur in the inner radius of the cylinder while the stresses reduce sharply towards the outer radius. This necessitates that two or more cylinders are shrunk into each other forming a compound cylinder (multi-layer). However, as a result of shrinking, stress redistribution may occur across the wall of the compound cylinder.
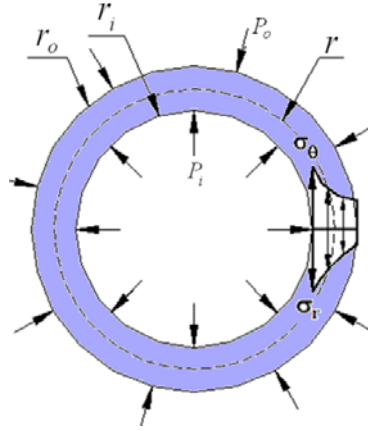
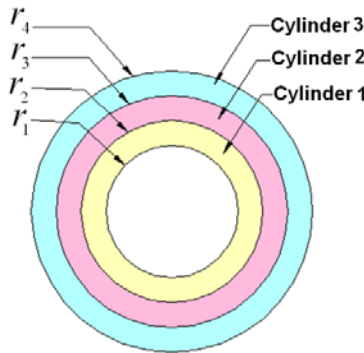**Figure** 1. Stress distribution across a single thick-walled cylinder

Figure. 2. Three Layer Compound Cylinder

In this paper three cylinders are considered (Figure 2). Cylinder 2 is shrink-fitted on cylinders 1 and cylinder 3 is shrink-fitted on cylinders 2. The stress redistribution is strongly influenced by shrinking radii (the outer radii of the inner cylinders and the inner radii of the outer cylinders) and shrinking allowances (shrink fit). The main objective of this work is to optimize the values of these parameters so that the minimum material volume for the three layer compound cylinder is achieved and maximum hoop stresses in all three cylinders become equal. This ensures best advantage of the material of the cylinder. Shrink fit is a very efficient way to extend compressive residual stress.

2. Application of Lame's theory to compound cylinder

Referring to Figure 1, let

r_i	Inner radius of cylinder
r_o	Outer radius of cylinder
r	Radius in between inner and outer radius of cylinder
P_i	Internal pressure in the cylinder
P_o	External pressure in the cylinder
σ_θ	Hoop stress in the cylinder
σ_r	Radial stress in the cylinder
σ_z	Axial stress in the cylinder

Radial stress in the cylinder made of homogeneous material is given by

$$\sigma_r = \frac{P_i r_1^2 - P_{s12} r_2^2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2 (P_{s12} - P_i)}{r_2^2 - r_1^2} \frac{1}{r^2} \tag{A}$$

while hoop stress in the cylinder is given by

$$\sigma_\theta = \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2} - \frac{r_i^2 r_o^2 (P_o - P_i)}{r_o^2 - r_i^2} \frac{1}{r^2} \tag{B}$$

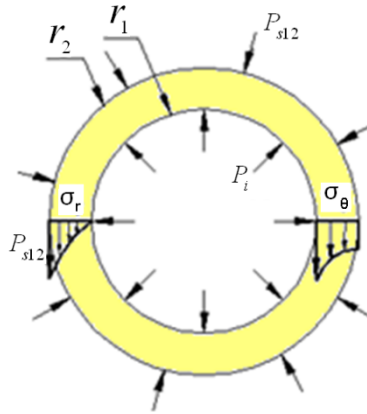*2.1 Radial and hoop stresses in cylinder 1*

Figure 3. Radial and Hoop Stress distribution in Cylinder 1

Referring to Figures 2 and 3, let

$r_i = r_1$ Inner radius of cylinder 1

$r_o = r_2$ Outer radius of cylinder 1

$P_i = P_i$ Internal pressure acting in the cylinder 1

$P_o = P_{s12}$ Contact pressure between cylinder 1 and 2 acting as external pressure on the cylinder 1

Using equation (A), radial stress in the cylinder 1 is given by

$$\sigma_r = \frac{P_i r_1^2 - P_{s12} r_2^2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2 (P_{s12} - P_i)}{r_2^2 - r_1^2} \frac{1}{r^2} \tag{1 a}$$

while hoop Stress in the cylinder 1 is given by

$$\sigma_\theta = \frac{P_i r_1^2 - P_{s12} r_2^2}{r_2^2 - r_1^2} - \frac{r_1^2 r_2^2 (P_{s12} - P_i)}{r_2^2 - r_1^2} \frac{1}{r^2} \tag{1 b}$$

If $P_i = \mathbf{0}$ i. e. there is no internal pressure, radial stress σ_r and hoop stress σ_θ reduce to

$$\sigma_r = -P_{s12} \frac{r_2^2}{r_2^2 - r_1^2} \left(1 - \frac{r_1^2}{r^2}\right) \tag{2 a}$$

$$\sigma_\theta = -P_{s12} \frac{r_2^2}{r_2^2 - r_1^2} \left(1 + \frac{r_1^2}{r^2}\right) \tag{2 b}$$

σ_r is maximum at outer radius $r_o = r_2$. Using equation (2 a)
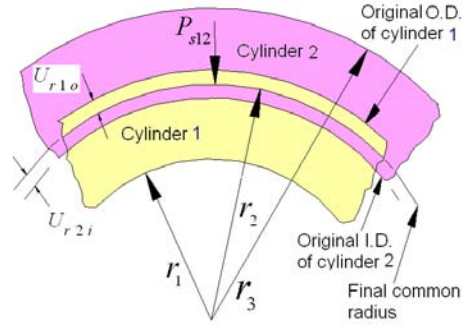
$$\sigma_{r\max(at\ r_2)} = -P_{s12} \tag{3}$$

σ_r is zero at inner radius $r_i = r_1$.

Using equation (2 b), σ_θ at outer radius $r_o = r_2$ is written as follows

$$\sigma_{\theta(at\ r_2)} = -P_{s12} \left[\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right] \tag{4}$$

σ_θ is maximum at inner radius $r_i = r_1$ which can be written **by** using equation (2 b)

$$\sigma_{\theta\max(at\ r_1)} = -\left[\frac{2P_{s12} r_2^2}{r_2^2 - r_1^2}\right] \tag{5}$$

Figure 4.   Shrinkage allowance between cylinders 1 and 2

Referring to Figure 4, let

$\varepsilon_{\theta lo}$ Hoop strain in the outer wall of cylinder 1

U_{rlo} Radial displacement at outer wall of cylinder 1

ν Poisson's ratio

In the presented study open type compound cylinder is considered, wherein axial stress σ_z produced due to internal pressure is zero. Values of hoop strain by using plane stress hypothesis and plane strain hypothesis at the same interference comes out to be very close to each other. Hence plane stress hypothesis can be assumed.

The radial displacement U_{rlo} at outer wall (at $r_o = r_2$) of cylinder 1 is given by using equations (3) and (4)

$$\varepsilon_{\theta lo} = \frac{U_{rlo}}{r_2} = \frac{1}{E}[\sigma_\theta - \nu\sigma_r] = \frac{1}{E}\left[-P_{s12}\left(\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right) - \nu\left(-P_{s12}\right)\right]$$

$$U_{rlo} = \frac{-P_{s12}r_2}{E}\left[\left(\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right) - \nu\right] \tag{6}$$

2.2 Radial and hoop stresses in cylinder 2
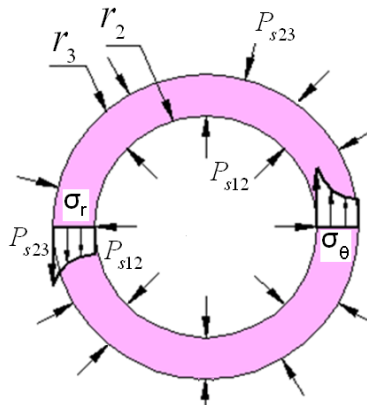
Figure 5. Radial and hoop stress distribution in Cylinder 2 due to contact pressure P_{s12}
(acting as internal pressure) and contact pressure P_{s23} (acting as external pressure)

Referring to Figure 5, let

$r_i = r_2$ Inner radius of cylinder 2

$r_o = r_3$ Outer radius of cylinder 2

$P_i = P_{s12}$ Contact pressure between cylinder 1 and 2 acting as internal pressure on the cylinder 2

$P_o = P_{s23}$ Contact pressure between cylinder 2 and 3 acting as external pressure on the cylinder 2

Using equation (A), radial stress in the cylinder 2 is given by

$$\sigma_r = \frac{P_{s12}r_2^2 - P_{s23}r_3^2}{r_3^2 - r_2^2} + \frac{r_2^2 r_3^2 (P_{s23} - P_{s12})}{r_3^2 - r_2^2}\frac{1}{r^2}$$

Radial stress in the cylinder 2 at inner radius $r_i = r_2$ is given by

$$\sigma_{r(at\ r_2)} = -P_{s12} \tag{7}$$

Radial stress in the cylinder 2 at outer radius $r_o = r_3$ is given by

$$\sigma_{r(at\ r_3)} = -P_{s23} \tag{8}$$

Using equation (B), hoop stress in the cylinder 2 is given by

$$\sigma_\theta = \frac{P_{s12}r_2^2 - P_{s23}r_3^2}{r_3^2 - r_2^2} - \frac{r_2^2 r_3^2 (P_{s23} - P_{s12})}{r_3^2 - r_2^2}\frac{1}{r^2}$$

Hoop stress in the cylinder 2 at inner radius $r_i = r_2$ is given by

$$\sigma_{\theta\ max\ (at\ r_2)} = \frac{P_{s12}(r_3^2 + r_2^2)}{r_3^2 - r_2^2} - \frac{2P_{s23}(r_3^2)}{r_3^2 - r_2^2} \tag{9}$$

Hoop stress in the cylinder 2 is maximum at inner radius r_2 (See Figure 5). Hoop stress in the cylinder 2 at outer radius $r_o = r_3$ is given by

$$\sigma_{\theta(at\ r_3)} = \frac{2P_{s12}(r_2^2)}{r_3^2 - r_2^2} - \frac{P_{s23}(r_3^2 + r_2^2)}{r_3^2 - r_2^2} \tag{10}$$

Referring to Figure 4, let

$\qquad \varepsilon_{\theta2i}$ Hoop strain in the inner wall of cylinder 2

$\qquad U_{r2i}$ Radial displacement at inner wall of cylinder 2

Radial displacement U_{r2i} at inner wall (at $r = r_2$) of cylinder 2 is given by using equations (7) and (9).

$$\varepsilon_{\theta2i} = \frac{U_{r2i}}{r_2} = \frac{1}{E}[\sigma_\theta - \nu\sigma_r] = \frac{1}{E}\left[\frac{P_{s12}(r_3^2 + r_2^2)}{r_3^2 - r_2^2} - \frac{2P_{s23}(r_3^2)}{r_3^2 - r_2^2} - \nu(-P_{s12})\right]$$

$$U_{r2i} = \frac{r_2}{E}\left[P_{s12}\left(\frac{r_3^2 + r_2^2}{r_3^2 - r_2^2} + \nu\right) - \frac{2P_{s23}(r_3^2)}{r_3^2 - r_2^2}\right] \tag{11}$$

Total interference δ_{12} at the contact between cylinder 1 and 2 is given by algebraic sum of radial displacement U_{r2i} at inner wall of cylinder 2 and radial displacement U_{r1o} at outer wall of cylinder 1.

Using equations (6), (11) and referring to Figure 4, we can write δ_{12} as follows

$$\delta_{12} = U_{r2i} - U_{r1o} = \frac{P_{s12}r_2}{E}\left[\left(\frac{r_3^2 + r_2^2}{r_3^2 - r_2^2}\right) + \left(\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right)\right] - \frac{2P_{s23}r_2}{E}\left[\frac{(r_3^2)}{r_3^2 - r_2^2}\right] \tag{12}$$
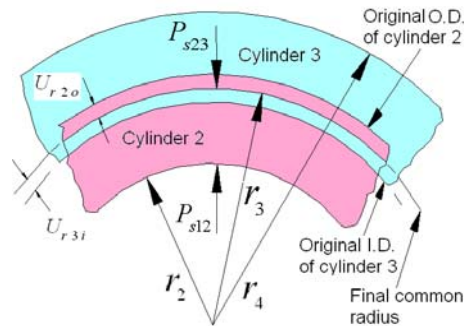

**Figure** 6. Shrinkage allowance between cylinders 2 and 3

Referring to Figure 6, let

$\qquad \varepsilon_{\theta2o}$ Hoop strain in the outer wall of cylinder 2

$\qquad U_{r2o}$ Radial displacement at outer wall of cylinder 2

Radial displacement U_{r2o} at outer wall (at $r = r_3$) of cylinder 2 is given by using equations (8) and (10).

$$\varepsilon_{\theta 2o} = \frac{U_{r2o}}{r_3} = \frac{1}{E}[\sigma_\theta - \nu\sigma_r] = \frac{1}{E}\left[\frac{2P_{s12}(r_2^2)}{r_3^2 - r_2^2} - \frac{P_{s23}(r_3^2 + r_2^2)}{r_3^2 - r_2^2} - \nu(-P_{s23})\right]$$

$$U_{r2o} = \frac{r_3}{E}\left[\frac{2P_{s12}\,r_2^2}{r_3^2 - r_2^2} - P_{s23}\left(\frac{r_3^2 + r_2^2}{r_3^2 - r_2^2} - \nu\right)\right] \tag{13}$$

2.3 Radial and hoop stresses in cylinder 3
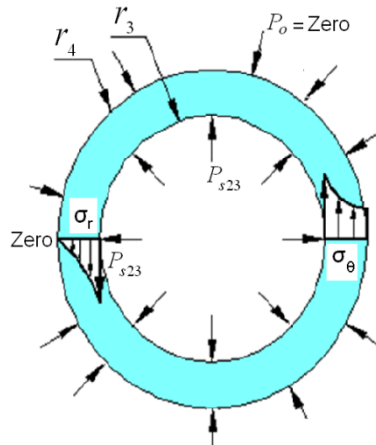
Figure 7.   Radial and Hoop Stress distribution in Cylinder 3
due to contact pressure P_{s23} only (acting as internal pressure)

Referring to Figure 7, let

$r_i = r_3$ 	Inner radius of cylinder 3

$r_o = r_4$ 	Outer radius of cylinder 3

$P_i = P_{s23}$ 	Contact pressure between cylinder 2 and 3 acting as internal pressure on the cylinder 3

$P_o = $ Zero 	External pressure on the cylinder 3

Using equation (A), radial stress in the cylinder 3 at inner radius $r_i = r_3$ is given by

$$\sigma_{r(at\ r_3)} = -P_{s23} \tag{14}$$

and radial stress in the cylinder 3 at outer radius $r_o = r_4$ is given by

$$\sigma_{r(at\ r_4)} = 0 \tag{15}$$

Using equation (B), hoop stress in the cylinder 3 at inner radius $r_i = r_3$ is given by

$$\sigma_{\theta\,max(\ at\ r_3)} = \frac{P_{s23}(r_4^2 + r_3^2)}{r_4^2 - r_3^2} \tag{16}$$

Hoop stress in the cylinder 3 at outer radius $r_o = r_4$ is given by

$$\sigma_{\theta(at\ r_4)} = \frac{2P_{s23}(r_3^2)}{r_4^2 - r_3^2} \tag{17}$$

σ_θ is maximum at inner radius r_3 as compared to at outer radius r_4.

Contact pressure P_{s23} is acting as internal pressure on the cylinder 3 and external pressure on the cylinder 3 is assumed as zero. Residual hoop stress σ_θ due to P_{s23} on the cylinder 3 at $r_i = r_3$ and $r_o = r_4$ is given by equations (16) and (17).
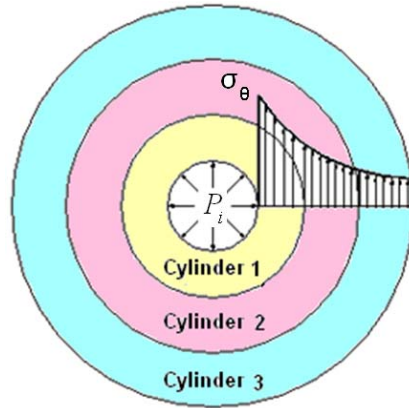
Referring to Figures 6 and 7, let

$\varepsilon_{\theta 3i}$ 	Hoop strain in the inner wall of cylinder 3

U_{r3i} 	Radial displacement at inner wall of cylinder 3

Radial displacement U_{r3i} at inner wall (at $r_i = r_3$) of cylinder 3 is given by using equations (14) and (16).

$$\varepsilon_{\theta 3i} = \frac{U_{r3i}}{r_3} = \frac{1}{E}\left[\sigma_\theta - \nu\sigma_r\right] = \frac{1}{E}\left[\frac{P_{s23}(r_4^2 + r_3^2)}{r_4^2 - r_3^2} - \nu(-P_{s23})\right]$$

$$U_{r3i} = \frac{P_{s23}r_3}{E}\left[\frac{(r_4^2 + r_3^2)}{(r_4^2 - r_3^2)} + \nu\right] \tag{18}$$

Total interference δ_{23} at the contact between cylinder 2 and 3 is given by algebraic sum of radial displacement U_{r3i} at inner wall of cylinder 3 and radial displacement U_{r2o} at outer wall of cylinder 2. Using equations (13) and (18),

$$\delta_{23} = U_{r3i} - U_{r2o} = \frac{P_{s23}r_3}{E}\left[\left(\frac{r_4^2 + r_3^2}{r_4^2 - r_3^2}\right) + \left(\frac{r_3^2 + r_2^2}{r_3^2 - r_2^2}\right)\right] - \frac{2P_{s12}\,r_3}{E}\left[\frac{r_2^2}{r_3^2 - r_2^2}\right] \tag{19}$$

**Figure** 8. Tensile hoop stress σ_θ in all three cylinders due to internal pressure P_i only

Referring to Figure 8, if the cylinder is subjected to internal pressure P_i (and no external pressure $P_o = 0$), tensile hoop stress is caused in all three cylinders which is given by equation (B), where $P_i = P_i$ at $r_i = r_1$ and $P_o = 0$ at $r_o = r_4$.

$$\sigma_\theta = \frac{P_i r_1^2}{r_4^2 - r_1^2}\left[\frac{r_4^2}{r^2} + 1\right] \tag{20}$$

Tensile hoop stress due to internal pressure P_i and residual hoop stress due to shrink fit have been superimposed to get the resultant hoop stresses. The resultant maximum hoop stress occurred at the inner surfaces of all three cylinders.

Maximum hoop stress at the inner surfaces of cylinder 1 (at r_1) is given by adding equations (20) and (5). (Refer to Figure 9).

$$\sigma_{\theta 1} = P_i\left[\frac{r_4^2 + r_1^2}{r_4^2 - r_1^2}\right] - 2P_{s12}\left[\frac{r_2^2}{r_2^2 - r_1^2}\right] \tag{21}$$

(a) Tensile hoop stress   (b) Residual hoop stress due to   (c) Resultant hoop stress
   due to internal pressure P_i contact pressure P_{s12} ($\sigma_{\theta 1}$)
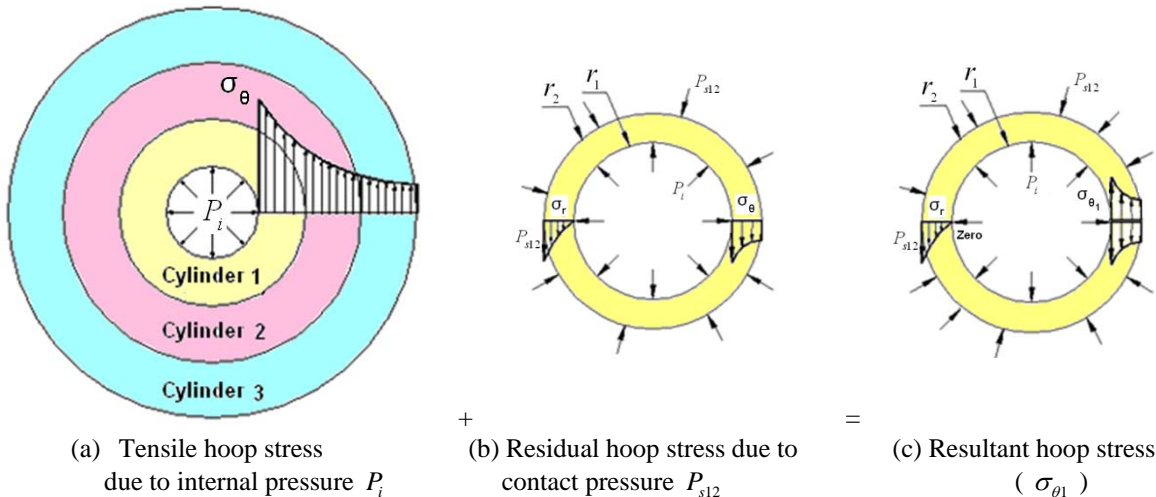
Figure 9. Superposition of tensile hoop stress due to internal pressure P_i &
residual hoop stress due to contact pressure P_{s12} in cylinder 1

Maximum hoop stress at the inner surfaces of cylinder 2 (at r_2) is given by adding σ_θ given by equations (20) and (9). (Refer to Figure 10).

$$\sigma_{\theta 2} = \frac{P_i r_1^2}{r_2^2}\left[\frac{r_4^2 + r_2^2}{r_4^2 - r_1^2}\right] + \frac{P_{s12}(r_3^2 + r_2^2) - 2P_{s23}r_3^2}{r_3^2 - r_2^2} \tag{22}$$

a) Tensile hoop stress   (b) Residual hoop stress due to   (c) Resultant hoop stress
    due to internal pressure P_i contact pressure P_{s12} and P_{s23} ($\sigma_{\theta 2}$)
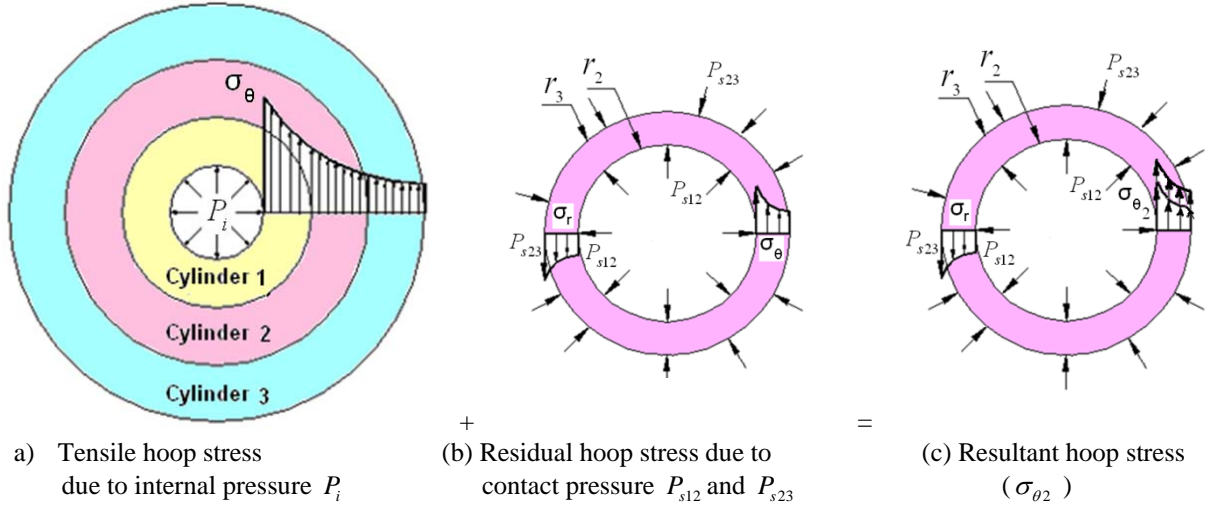
Figure 10. Superposition of tensile hoop stress due to internal pressure P_i & residual hoop stress
due to contact pressure P_{s12} and P_{s23} in cylinder 2

Maximum hoop stress at the inner surfaces of cylinder 3 (at r_3) is given by adding σ_θ given by equations (20) and (16). (Refer to Figure 11).

$$\sigma_{\theta 3} = \frac{P_i r_1^2}{r_3^2}\left[\frac{r_4^2 + r_3^2}{r_4^2 - r_1^2}\right] + P_{s23}\left[\frac{r_4^2 + r_3^2}{r_4^2 - r_3^2}\right] \tag{23}$$

(a) Tensile hoop stress   (b) Residual hoop stress due to   (c) Resultant hoop stress
   due to internal pressure P_i contact pressure P_{s23} ($\sigma_{\theta 3}$)
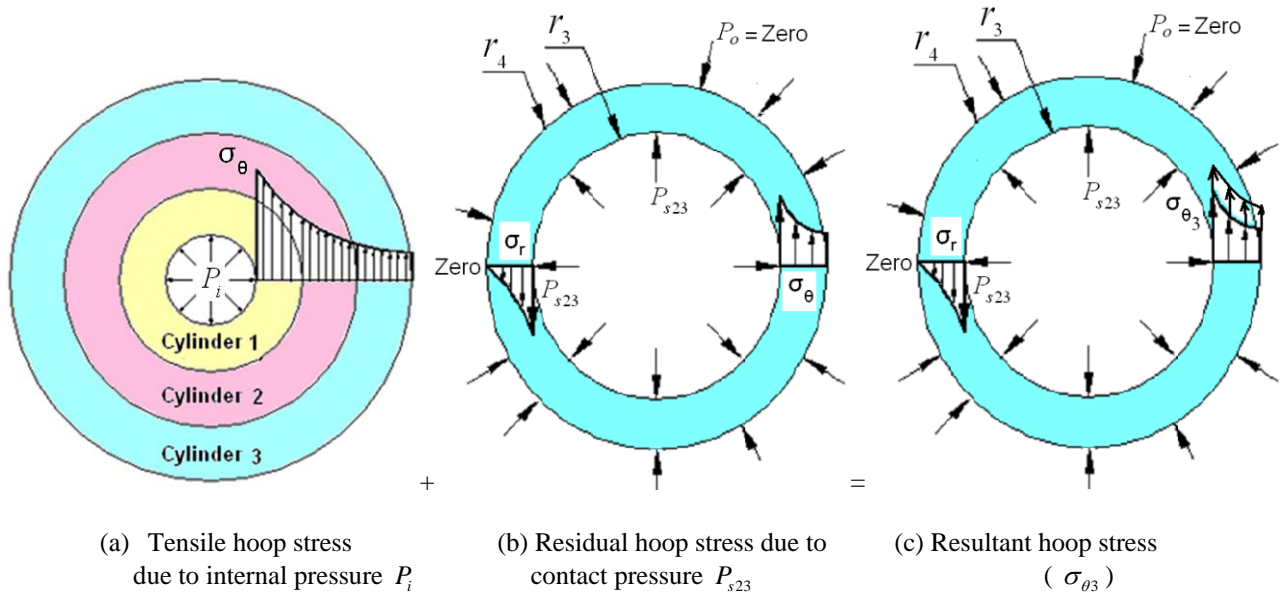
Figure 11. Superposition of tensile hoop stress due to internal pressure P_i and
residual hoop stress due to contact pressure P_{s23} in cylinder 3

3. Optimum Design Methodology for Three Layer Compound Cylinder

After finding hoop stresses at all the radii, the principle of superposition is applied, i.e. the various stresses are then combined algebraically to produce the resultant hoop stresses in the compound cylinder subjected to both shrinkage pressures and internal pressure P_i.
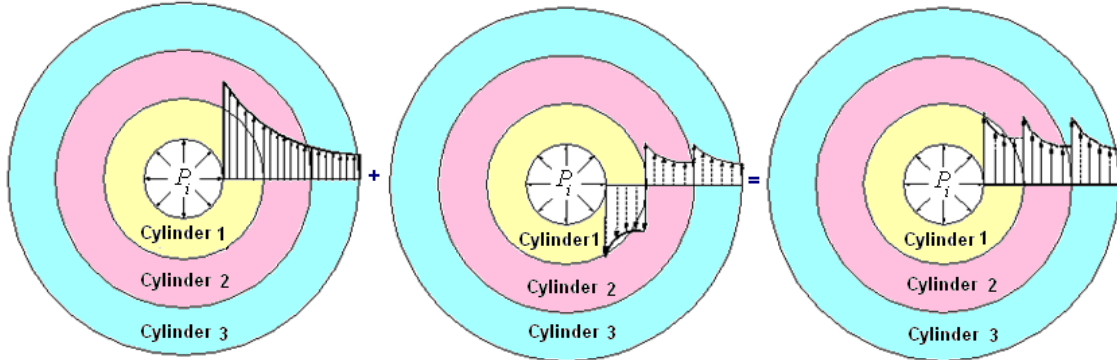
a) Hoop stress due to P_i b) Residual hoop stress due to P_{s12} & P_{s23} (c) Resultant hoop stress

Figure 12: Superposition of hoop stress due to P_i & residual hoop stresses due to P_{s12} & P_{s23} in all cylinders

To obtain optimum values of the contact pressures P_{s12} and P_{s23} which will produce equal hoop stresses in all the three cylinders, maximum hoop stresses given by the equations (21), (22) and (23) must be equated.

By equating equations (21) and (22) i. e. $\sigma_{\theta 1} = \sigma_{\theta 2}$ and rearranging,

$$P_{s12}\left[\frac{2r_2^2}{r_2^2 - r_1^2} + \frac{r_3^2 + r_2^2}{r_3^2 - r_2^2}\right] = P_i\left[\frac{r_4^2 + r_1^2}{r_4^2 - r_1^2} - \frac{r_1^2}{r_2^2}\left[\frac{r_4^2 + r_2^2}{r_4^2 - r_1^2}\right]\right] + P_{s23}\frac{2r_3^2}{r_3^2 - r_2^2}$$

Let $C_1 = \dfrac{r_2}{r_1} = \dfrac{d_2}{d_1}$, $C_2 = \dfrac{r_3}{r_2} = \dfrac{d_3}{d_2}$, $C_3 = \dfrac{r_4}{r_3} = \dfrac{d_4}{d_3}$ (24)

where d_1, d_2, d_3, d_4 are diameters corresponding to radii r_1, r_2, r_3, r_4.

Hence $C_1 C_2 = \dfrac{r_2}{r_1}.\dfrac{r_3}{r_2} = \dfrac{r_3}{r_1}$, $C_2 C_3 = \dfrac{r_3}{r_2}\dfrac{r_4}{r_3} = \dfrac{r_4}{r_2}$, $C_1 C_2 C_3 = \dfrac{r_2}{r_1}.\dfrac{r_3}{r_2}\dfrac{r_4}{r_3} = \dfrac{r_4}{r_1}$ (25)

Let $j = \dfrac{2r_2^2}{r_2^2 - r_1^2} + \dfrac{r_3^2 + r_2^2}{r_3^2 - r_2^2} = \dfrac{2C_1^2}{C_1^2 - 1} + \dfrac{C_2^2 + 1}{C_2^2 - 1}$ (26)

$$k = \frac{r_4^2 + r_1^2}{r_4^2 - r_1^2} - \frac{r_1^2}{r_2^2}\left[\frac{r_4^2 + r_2^2}{r_4^2 - r_1^2}\right] = \frac{C_1^2 C_2^2 C_3^2 + 1}{C_1^2 C_2^2 C_3^2 - 1} - \frac{C_2^2 C_3^2 + 1}{C_1^2 C_2^2 C_3^2 - 1}$$ (27)

$$l = \frac{2r_3^2}{r_3^2 - r_2^2} = \frac{2C_2^2}{C_2^2 - 1}$$ (28)

Hence $P_{s12} = P_i\left[k / j\right] + P_{s23}\left[l / j\right]$ (29)

By equating equations (22) and (23) i. e. $\sigma_{\theta 2} = \sigma_{\theta 3}$ and rearranging,

$$P_{s12}\frac{r_3^2 + r_2^2}{r_3^2 - r_2^2} = P_i\left[\frac{r_1^2(r_4^2 + r_3^2)}{r_3^2(r_4^2 - r_1^2)} - \frac{r_1^2(r_4^2 + r_2^2)}{r_2^2(r_4^2 - r_1^2)}\right] + P_{s23}\left[\frac{r_4^2 + r_3^2}{r_4^2 - r_3^2} + \frac{2r_3^2}{r_3^2 - r_2^2}\right]$$

Let $m = \dfrac{r_3^2 + r_2^2}{r_3^2 - r_2^2} = \dfrac{C_2^2 + 1}{C_2^2 - 1}$ (30)

$$n = \frac{r_1^2(r_4^2 + r_3^2)}{r_3^2(r_4^2 - r_1^2)} - \frac{r_1^2(r_4^2 + r_2^2)}{r_2^2(r_4^2 - r_1^2)} = \frac{C_3^2 + 1}{C_1^2 C_2^2 C_3^2 - 1} - \frac{C_2^2 C_3^2 + 1}{C_1^2 C_2^2 C_3^2 - 1} \tag{31}$$

$$o = \frac{r_4^2 + r_3^2}{r_4^2 - r_3^2} + \frac{2r_3^2}{r_3^2 - r_2^2} = \frac{C_3^2 + 1}{C_3^2 - 1} + \frac{2C_2^2}{C_2^2 - 1} \tag{32}$$

Hence $P_{s12} = P_i[n/m] + P_{s23}[o/m]$ (33)

Solving equations (29) and (33) to get P_{s12} and P_{s23} in terms of P_i as follows,

$$P_{s12} = P_i \left[\frac{(n/o) - (k/l)}{(m/o) - (j/l)} \right] \tag{34}$$

$$P_{s23} = P_i \left[\frac{(n/m) - (k/j)}{(l/j) - (o/m)} \right] \tag{35}$$

Putting the values of C_1, C_2 and C_3, the equations (12) and (19) can be written as

$$\delta_{12} = \frac{P_{s12} r_2}{E} \left[\left(\frac{C_2^2 + 1}{C_2^2 - 1} \right) + \left(\frac{C_1^2 + 1}{C_1^2 - 1} \right) \right] - \frac{2 P_{s23} r_2}{E} \left[\frac{C_2^2}{C_2^2 - 1} \right] \tag{36}$$

$$\delta_{23} = \frac{P_{s23} r_3}{E} \left[\left(\frac{C_3^2 + 1}{C_3^2 - 1} \right) + \left(\frac{C_2^2 + 1}{C_2^2 - 1} \right) \right] - \frac{2 P_{s12} r_3}{E} \left[\frac{1}{C_2^2 - 1} \right] \tag{37}$$

3.1 Analytical Calculations for Optimum Design

Material for all the three cylinders is assumed to be the same i. e. steel. For the given volume of fluid to be stored, the internal diameter of cylinder 1 (d_1) is known. Here it is assumed as 100 mm. Yield strength of the steel material is σ_y = 250 MPa. In case of pressure vessels it is observed that failure occurs across the thickness of cylinder where the hoop stress is acting. Although the von Mises stress is more than the hoop stress, it is acting on larger area than hoop stress. So chances of failure due to hoop stress is more in pressure vessel. Hence maximum hoop stress criteria is used for material suffering. Maximum principal stresses in all the cylinders (here it is maximum hoop stress) should not exceed the yield stress of the material to avoid the failure of the compound cylinder. Optimum material volume (length is assumed as unity) can be calculated using the following steps.

Steps in Optimum Design

1. Assume internal diameter of cylinder 1 (d_1) say 100 mm.
2. Select the ratios $C_1 = \dfrac{r_2}{r_1} = \dfrac{d_2}{d_1}$, $C_2 = \dfrac{r_3}{r_2} = \dfrac{d_3}{d_2}$, $C_3 = \dfrac{r_4}{r_3} = \dfrac{d_4}{d_3}$
3. For the given internal pressure P_i, one can find contact (shrinkage) pressures P_{s12} and P_{s23} in terms of ratios C_1, C_2 and C_3 using equations (34) and (35).
4. Find the volume of the compound cylinder using $Fx = \pi.(d_4^2 - d_1^2)/4$
5. Minimize the volume subjected to the constraints,
   i) $\sigma_{\theta 1} <= \sigma_y$ ii) $\sigma_{\theta 2} <= \sigma_y$ iii) $\sigma_{\theta 3} <= \sigma_y$ iv) $\delta_{12} > 0$ v) $\delta_{23} > 0$
6. Optimized parameters C_1, C_2, C_3 and δ_{12}, δ_{23} are used for the design.

By selecting the various values if C_1, C_2, C_3 and using iterative numerical method (with the help of computer programming), contact (shrinkage) pressures P_{s12} and P_{s23} for given internal pressure P_i are calculated using equations (34) and (35) resp. These contact pressures are then used to find interferences δ_{12}, δ_{23} with the help of equations (36) and (37) resp. Volume is minimized subjected to the above said constraints.

A lot of combinations for C_1, C_2, C_3 starting from 1.1 to 1.5 with increment of 0.10, 0.05, 0.02 have been tried. Actually the condition that maximum hoop stresses in all three cylinders being equal is met by number of combinations of C_1, C_2, C_3. Due to this, the output of the computer program is very long. Out of these some selected combinations have been included in this paper where volume is relatively less as compared to rest of the combinations. But for only one set, the volume is minimum. To show that, six sets of combination of C_1, C_2, C_3 have been listed. Using the 6 selected combinations of C_1, C_2, C_3 diameters d_2, d_3, d_4 and δ_{12}, δ_{23} are calculated through computer program and results are listed in the table1.

Table 1: Analytical Results of numerical method using computer program (for d_1 = 100 mm and σ_y = 250 MPa)

Combination of C_1, C_2, C_3 → Parameters ↓	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
C_1	1.200	1.200	1.250	1.450	1.160	1.200
C_2	1.300	1.450	1.250	1.300	1.360	1.300
C_3	1.450	1.300	1.450	1.200	1.430	1.440
d_2 (mm)	120.000	120.000	125.000	145.000	116.000	120.000
d_3 (mm)	156.000	174.000	156.250	188.500	157.760	156.000
d_4 (mm)	226.200	226.200	226.562	226.200	225.597	224.640
Fx (Volume) (mm^3)	32331.400	32331.330	32460.230	32331.330	32117.320	**31778.980**
$\sigma_{\theta 1}$ (MPa)	247.496	247.496	246.135	247.496	249.868	249.719
$\sigma_{\theta 2}$ (MPa)	247.496	247.496	246.135	247.496	249.868	249.719
$\sigma_{\theta 3}$ (MPa)	247.496	247.496	246.135	247.496	249.868	249.719
δ_{12} (mm)	0.022	0.022	0.027	0.045	0.018	0.022
δ_{23} (mm)	0.032	0.046	0.027	0.034	0.038	0.032
P_{s12} (MPa)	18.939	18.939	22.471	32.501	15.718	18.922
P_{s23} (MPa)	20.996	21.581	20.778	17.910	21.867	20.894

From the table it is observed that minimum volume is 31778.980 mm^3 corresponding to values in set number **6** where C_1 = **1.200**, C_2 = **1.300**, C_3 = **1.440**.

4. Validation Procedure by FEM

Now-a-days in the industry FEM is widely used due to its reliability. In the present analysis, certain assumptions are made and to validate these assumptions, FEM solutions is used.

By taking the corresponding values of C_1, C_2, C_3 from the set number 6 of the table 1 and taking d_1 = 100 mm remaining diameters d_2, d_3, d_4 are calculated. Using the values of δ_{12}, δ_{23} shrink fit is applied between cylinders 1 & 2 and between cylinders 2 & 3 respectively in ANSYS Workbench. Contact between cylinders 1 & 2 as well as between cylinders 2 & 3 is applied using contact tool in ANSYS Workbench. The values of δ_{12}, δ_{23} are radius based. These values are doubled to take diametric effect in the Finite Element Model.

Table 2: Data for modeling in ANSYS (for set number 6) to validate analytical results

C_1	C_2	C_3	d_2	d_{2i}	d_3	d_{3i}	d_4	δ_{12}	δ_{23}
1.200	1.300	1.440	120.000	119.956	156.000	155.936	224.640	0.022	0.032

where d_1, d_2 = inner & outer diameters of cylinder 1 respectively.

d_{2i} , d_3 = inner & outer diameters of cylinder 2 respectively for shrink fit.

d_{3i} , d_4 = inner & outer diameters of cylinder 3 respectively for shrink fit.

FEM model of three layered compound cylinder is prepared in ANSYS Workbench using values of the diameters from table 2. Results of FEM by ANSYS Workbench are listed in the figures 13, 14, 15, 16, 17.
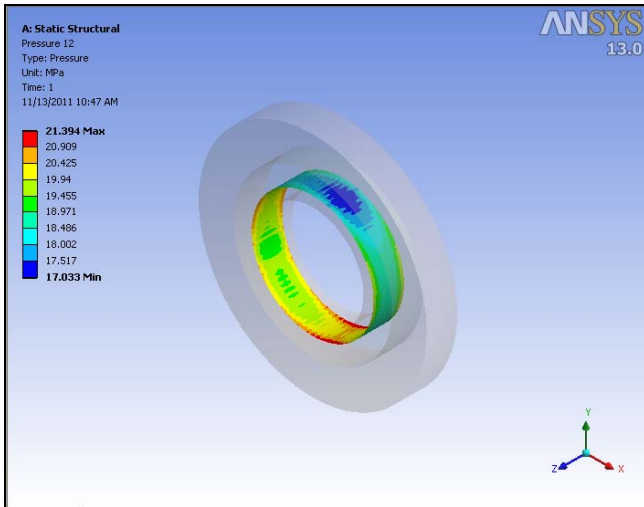
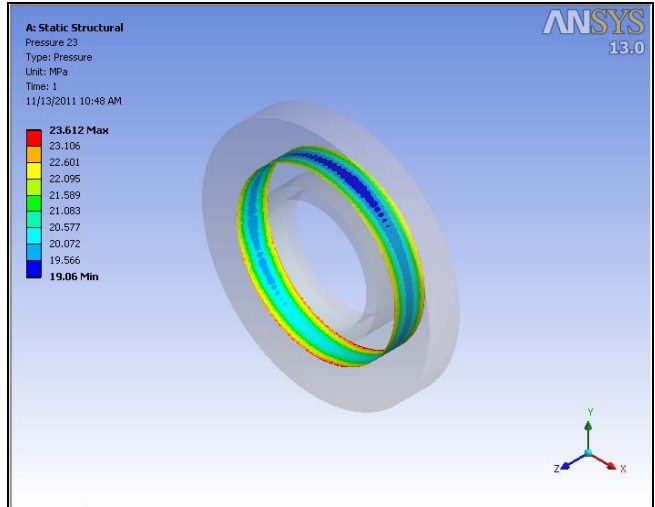**Figure** 13. P_{s12} (without P_i) = 19.21 MPa (Avg)

Figure 14. P_{s23} (without P_i) = 21.33 MPa (Avg)
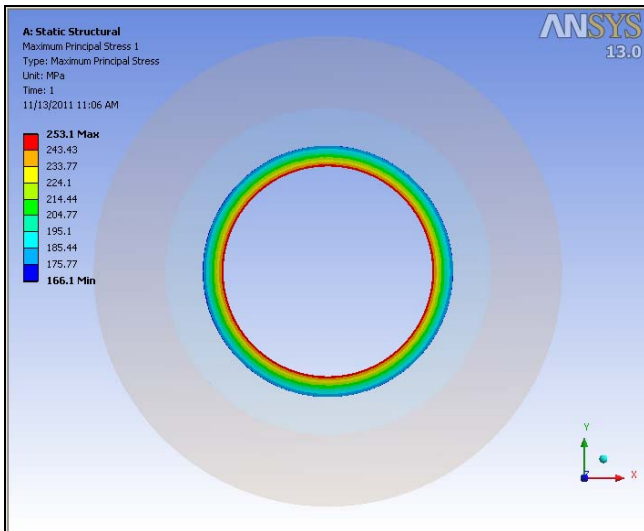

Figure 15. Maximum Principal Stress in cylinder 1
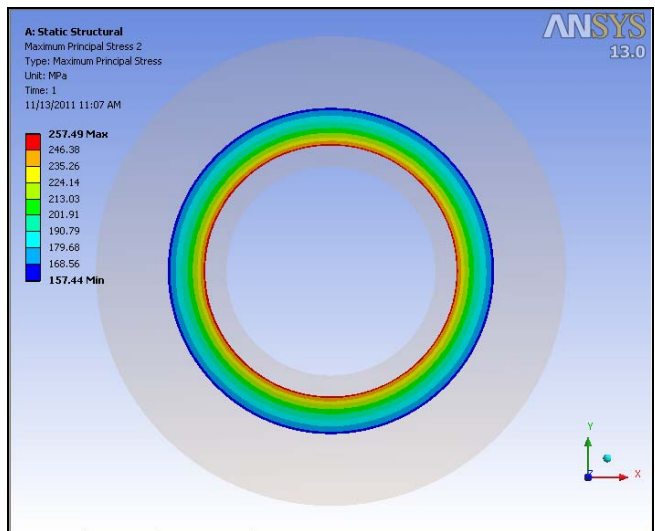

Figure 16. Maximum Principal Stress in cylinder 2
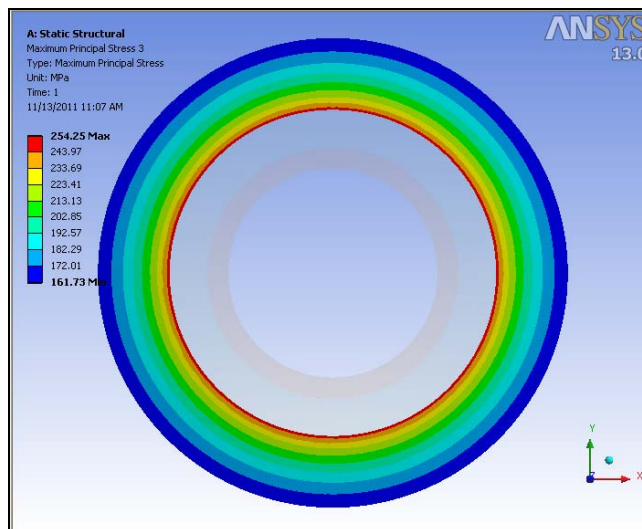

**Figure** 17. Maximum Principal Stress in cylinder 3

5. Results

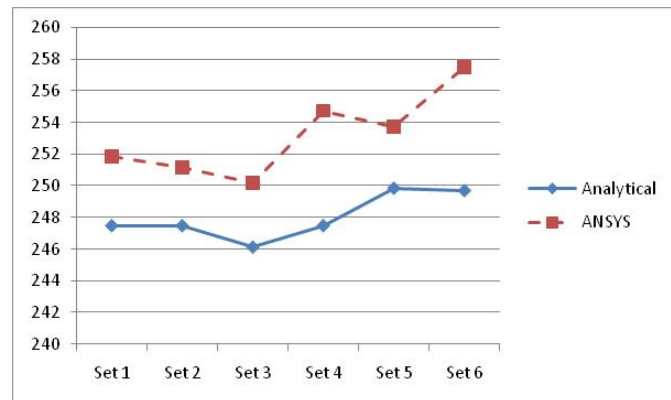Analytical results and FEM (ANSYS) results are summarized in Table 3.

Table 3: Comparison of Analytical and ANSYS results for 6 combination sets of C_1, C_2, C_3

Set No	Results	P_{s12} (MPa)	P_{s23} (MPa)	$\sigma_{\theta1}$ (MPa)	$\sigma_{\theta2}$ (MPa)	$\sigma_{\theta3}$ (MPa)
Set 1	Analytical	18.939	20.996	247.496	247.496	247.496
	ANSYS	19.620 (Avg)	21.705 (Avg)	250.270	251.850	251.730
	Difference	**3.59 %**	**3.37 %**	**1.12 %**	**1.77 %**	**1.71 %**
Set 2	Analytical	18.939	21.581	247.496	247.496	247.496
	ANSYS	19.530 (Avg)	22.085 (Avg)	250.990	251.190	253.270
	Difference	**3.12 %**	**2.33 %**	**1.41 %**	**1.49 %**	**2.33 %**
Set 3	Analytical	22.471	20.778	246.135	246.135	246.135
	ANSYS	22.630 (Avg)	21.510 (Avg)	249.230	250.200	250.520
	Difference	**0.70 %**	**3.52 %**	**1.25 %**	**1.65 %**	**1.78 %**
Set 4	Analytical	32.501	17.910	247.496	247.496	247.496
	ANSYS	32.593 (Avg)	18.073 (Avg)	248.960	254.740	250.750
	Difference	**0.28 %**	**0.91 %**	**0.59 %**	**2.92 %**	**1.31 %**
Set 5	Analytical	15.718	21.867	249.868	249.868	249.868
	ANSYS	16.203 (Avg)	22.165 (Avg)	254.490	253.730	254.350
	Difference	**3.08 %**	**1.36 %**	**1.84 %**	**1.54 %**	**1.79 %**
Set 6	Analytical	18.922	20.894	249.719	249.719	249.719
	ANSYS	19.210 (Avg)	21.330 (Avg)	253.100	257.490	254.250
	Difference	**1.52 %**	**2.08 %**	**1.35 %**	**3.11 %**	**1.81 %**

From the table 3, it is observed that there is very small difference in analytical and FEM results, i. e. FEM gave virtually similar results as the analytical results (Refer to Figure 18).
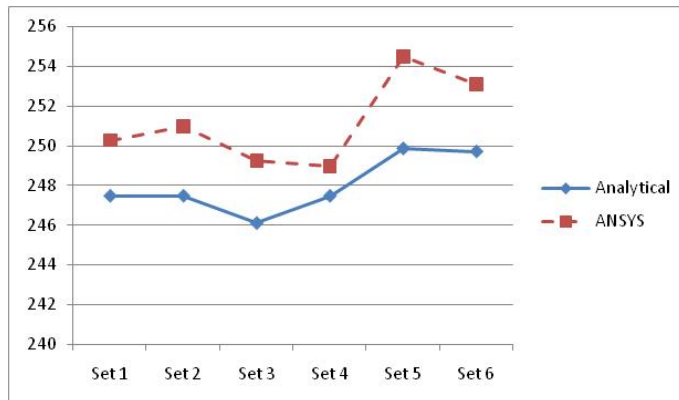
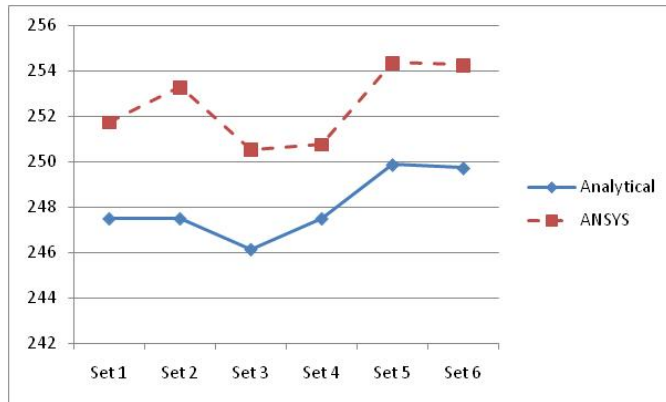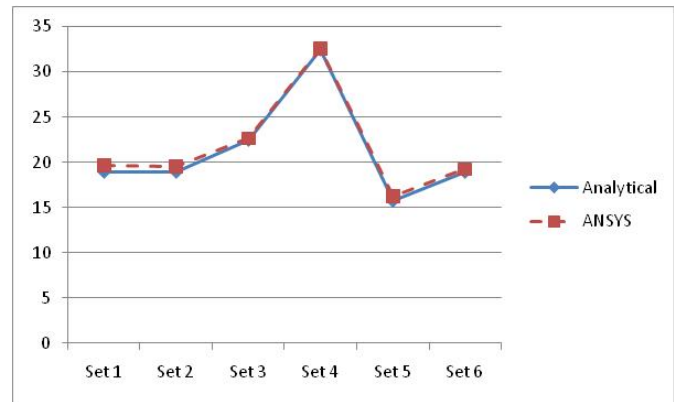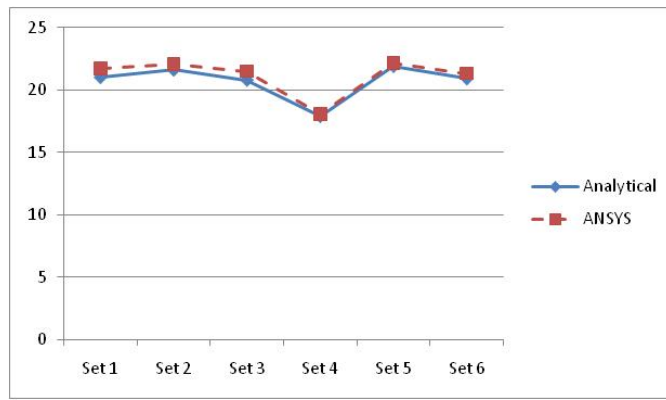(a) $\sigma_{\theta1}$ in MPa   (b) $\sigma_{\theta2}$ in MPa

(c) $\sigma_{\theta 3}$ in MPa

(d) P_{s12} in MPa (without internal pressure)

(e) P_{s23} in MPa (without internal pressure)

Figure 18: Difference in analytical and ANSYS Software results

6. Conclusion

The difference between analytical results and ANSYS results is below 4 % with respect to analytical results. This difference is due to numerical techniques of Finite Element Method in ANSYS. Since analytical results are validated by FEM calculations, the design methodology proposed in this paper can be successfully applied into the real-world mechanical applications for minimizing the material volume of multi-layered compound cylinders to assure best utilization of material. Patil (2005) proposed the optimization of intermediate, outer diameter and shrinkage tolerance to get minimum volume of two layer compound cylinders as 37974.94 mm^3. In comparison with this, proposed methodology for three layer compound cylinders gave volume as 31778.980 mm^3 (for same internal diameter = 100 mm and same material of σ_y = 250 MPa) which is quite significant to save the material and serve the purpose.

Acknowledgement

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