MultiCraft

International Journal of Engineering, Science and Technology
Vol. 3, No. 6, 2011, pp. 220-225

INTERNATIONAL JOURNAL OF ENGINEERING, SCIENCE AND TECHNOLOGY
www.ijest-ng.com

Some multiple generating functions involving Mittag-Leffler's functions

M. Kamarujjama, N. U. Khan and Waseem A. Khan*

Department of Applied Mathematics, Z.H. College of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, INDIA
[*]Corresponding Author: e-mail: waseem08_khan@rediffmail.com (Waseem A. Khan)

Abstract

In the paper it will be shown that generating functions of hyper Bessel functions due to Humbert and Delerue can be extended to a new class of generating relations for generalized Mittag-Leffler's functions. A number of new and known double and multiple generating functions involving the product of classical polynomials and functions are considered as special cases.
.

Key words: Mittag-Leffler's function and related functions, Generalized hypergeometric function and hyper Bessel function.
2000 AMS Subject Classification**:** Primary 33E12 Secondary 33C99

DOI: http://dx.doi.org/10.4314/ijest.v3i6.17

1. Introduction and Definition

Recently, Shukla and Prajapati (2007) introduced the function $E_{\alpha,\beta}^{\gamma,\delta}(Z)$ and discussed their properties and their explicit representations. These functions are generalizations of Mittag-Leffler's functions (Mittag-Leffler's, 1905) and Prabhakar functions (Prabhakar, 1971) and are connected with Wright functions (Srivastava and Manocha, 1984) and hyper Bessel functions (Delerue, 1953). Motivated and inspired by the above mentioned work on the generalization of the Mittag- Leffler's functions, we derive some partly unilateral and partly bilateral generating functions Shukla and Prajapati functions and hyper Bessel functions.

In the usual notation let ${}_pF_q$ denote a generalization hypergeometric function of one variable ${}_pF_q$ with p and q (positive integer or zero), defined by (Srivastava and Manocha, 1984; p. 42(1)).

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,...,\alpha_p; \\ & z \\ \beta_1,\beta_2,....,\beta_q;\end{matrix}\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_p)_n}\frac{z^n}{n!}={}_pF_q(\alpha_1,...,\alpha_p;\beta_1,....,\beta_q;z), \tag{1.1}$$

$$(\beta_j \neq 0,-1,-2,...,\ \ j=1,\ 2,...,q)$$

where $(a)_n$ is the Pochhammer symbol, defined by

$$ (\mathrm{a})_n=\begin{cases}1, & n=0 \\ a(a+1)...(a+n-1),\ if & n=1,2,3,...\end{cases} \tag{1.2}$$

(i) converges for all $|z|<\infty\, if\ \ p\leq q$

(ii) converges for $|z|<1 if\ \ p=q+1$

(iii) absolutely convergent $|z| = 1 \, if \;\; p = q + 1$ and

$$\mathrm{Re}\,(\sum_{j=1}^{q} \beta_j - \sum_{j=1}^{n} \alpha_j) > 0$$

The function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha > 0 \tag{1.3}$$

was introduced by Mittag-Leffler (1905).

The function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; \alpha, \; \beta > 0 \tag{1.4}$$

has properties very similar to those of Mittag-Leffler's function (z) (See Agarwal, 1953).

In 1971, Prabhakar (Prabhakar, 1971) introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \; \alpha, \; \beta, \; \gamma > 0 \tag{1.5}$$

In continuation of his work, Shukla and Prajapati (2007) investigated the function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ which is defined for $\alpha, \beta, \gamma \in C;$ Re(α)>0,Re(β)>0,Re(γ)>0 and $\delta \in (0,1) \cup N$ as

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{\delta k}}{\Gamma(\alpha k + \beta)}, \; \alpha, \; \beta, \; \gamma, \; \delta > 0 \tag{1.6}$$

Where $(\gamma)_{\delta k} = \dfrac{\Gamma(\gamma + \delta k)}{\Gamma(\gamma)}$ denote the generalized Pochammer symbol which in particular reduces to $\delta^{\delta k} \prod_{r=1}^{\delta} \left(\dfrac{\gamma + r - 1}{\delta} \right) k$ if $\delta \in N$. The function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ converges absolutely for all z if $\delta <$ Re $\alpha + 1$ and for $|z| < 1$ if $\delta =$ Re $\alpha + 1$. It is an entire function of order $(\mathrm{Re}\,\alpha)^{-1}$. The function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ is most natural generalization of the exponential function exp(z), Mittag-Leffler function $E_{\alpha}(z)$ and Wiman's function $E_{\alpha,\beta}(z)$.

$$E^{\gamma,1}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z), E^{1,1}_{\alpha,\beta}(z) = E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), E_{\alpha,1}(z) = E_{\alpha}(z),$$
$$E^{1,1}_{1,1}(z) = E_{1,1}(z) = E_1(z) = e^z, E_2(z^2) = \cosh z \tag{1.7}$$

An interesting generating function due to Humbert (1936), is recalled here in the following form:

$$\exp\left[\frac{z}{3}(x + y \pm \frac{1}{xy})\right] = \sum_{m,n=-\infty}^{\infty} \frac{x^m \, y^m \, (z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} \; {}_0F_2\,[-; m+1, n+1; \pm(\frac{z}{3})^3], \tag{1.8}$$

where the hyper-Bessel function $J_{m,n}(z)$ and modified hyper-Bessel function $I_{m,n}(z)$ of order 2 are defined by

$$J_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} \; {}_0F_2\,[-; m+1, n+1; -(\frac{z}{3})^3] \tag{1.9}$$

And

$$I_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} \; {}_0F_2\,[-; m+1, n+1; (\frac{z}{3})^3] \tag{1.10}$$

respectively.

Further generalization of equation (1.8) is given by Pathan (Pathan, 2007; p. 41.eq.(2.2))

$$\exp[(\frac{z}{n+1})(x_1+...+x_n \pm \frac{1}{x_1...x_n})]=\sum_{m_1...m_n=-\infty}^{\infty} \frac{x_1^{m_1}....x_n^{m_n}(\frac{z}{n+1})^{\sum_{i=0}^{n} m_i}}{m_1!...m_n!}$$

$${}_0F_n[-;m_1+1,....,m_n+1;\pm(\frac{z}{n+1})^{(n+1)}] \qquad (1.11)$$

where the hyper-Bessel function $J_{m_1...m_n}(z)$ and modified hyper-Bessel function $I_{m_1...m_n}(z)$ of order n are defined by

$$J_{m_1...m_n}(z)=\frac{(\frac{z}{n+1})^{\sum_{i=0}^{n} m_i}}{m_1!...m_n!}{}_0F_n[-;m_1+1,....,m_n+1;-(\frac{z}{n+1})^{(n+1)}] \qquad (1.12)$$

and

$$I_{m_1...m_n}(z)=\frac{(\frac{z}{n+1})^{\sum_{i=0}^{n} m_i}}{m_1!...m_n!}{}_0F_n[-;m_1+1,....,m_n+1;(\frac{z}{n+1})^{(n+1)}] \qquad (1.13)$$

respectively.

2. Generating functions involving generalized Mittag-Leffler's functions

Result- 1.

If $E_{\alpha\beta}^{\gamma\delta}(z)$ is defined by (1.6), then

$$E_{\alpha_1,\beta_1}^{\gamma_1,\delta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}^{\gamma_2,\delta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}^{\gamma_3,\delta_3}\left(\frac{\pm z/3}{xy}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m*}^{\infty}\frac{x^m y^m (z/3)^{m+n}}{m!n!\Gamma(\beta_1)\Gamma(\beta_1)\Gamma(\beta_3)}$$

$$\sum_{k=0}^{n}\frac{(\pm1)^k(\gamma_1)\,\delta_{1(m+k)}(\gamma_2)\,\delta_{2(n-k)}(\gamma_3)\,\delta_{3k}}{(m+1)_k\,(n+1)_k\,(\beta_1)_{\alpha_{1(m+k)}}\,(\beta_2)_{\alpha_{2(n-k)}}\,(\beta_3)_{\alpha_{3k}}}\frac{(z/3)^{3k}}{k!} \qquad (2.1)$$

Where

$$\begin{matrix}\gamma_i,\delta_i\\ \alpha_i,\beta_i\end{matrix} J_{m,n}(z)=\frac{(z/3)^{m+n}}{m!n!}\sum_{k=0}^{n}\frac{(-1)^k(\gamma_1)\,\delta_{1(m+k)}(\gamma_2)\,\delta_{2(n-k)}(\gamma_3)\,\delta_{3k}}{(m+1)_k\,(n+1)_k\,(\beta_1)_{\alpha_{1(m+k)}}\,(\beta_2)_{\alpha_{2(n-k)}}\,(\beta_3)_{\alpha_{3k}}}\frac{(z/3)^{3k}}{k!} \qquad (2.2)$$

$$\{i=1,2,3\}$$

And

$$\begin{matrix}\gamma_i,\delta_i\\ \alpha_i,\beta_i\end{matrix} I_{m,n}(z)=\frac{(z/3)^{m+n}}{m!n!}\sum_{k=0}^{n}\frac{(\gamma_1)\,\delta_{1(m+k)}(\gamma_2)\,\delta_{2(n-k)}(\gamma_3)\,\delta_{3k}}{(m+1)_k\,(n+1)_k\,(\beta_1)_{\alpha_{1(m+k)}}\,(\beta_2)_{\alpha_{2(n-k)}}\,(\beta_3)_{\alpha_{3k}}}\frac{(z/3)^{3k}}{k!} \qquad (2.3)$$

$$\{i=1,2,3\}$$

provided that both sides of (2.1) exist.

Proof of Result- 1.

If the function

$$V = E_{\alpha_1,\beta_1}^{\gamma_1,\delta_1}\left(\frac{z\,x}{3}\right)E_{\alpha_2,\beta_2}^{\gamma_2,\delta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}^{\gamma_3,\delta_3}\left(\frac{\pm z/3}{xy}\right)$$

is expanded by the definition (1.6), we have

$$V = \sum_{k=0}^{\infty}\frac{(\gamma_3)\delta_{3k}}{\Gamma(\alpha_3 k+\beta_3)}\frac{(\pm 1)^k}{k!}\sum_{i=0}^{\infty}\frac{(\gamma_1)i\delta_1}{\Gamma(\alpha_1 i+\beta_1)}\frac{(x)^{i-k}}{i!}\sum_{i=0}^{\infty}\frac{(\gamma_2)j\delta_2}{\Gamma(\alpha_2 j+\beta_2)}\frac{\left(\frac{z}{3}\right)^{i+j+k}(y)^{j-k}}{j!}$$

Replace i-k and j-k by m and n respectively, then after rearrangement justified by the absolute convergence of the above series it follows that

$$V = \sum_{m=-\infty}^{\infty}\sum_{n=m*}^{\infty}\frac{x^m y^m (z/3)^{m+n}}{m!n!\Gamma(\beta_1)\Gamma(\beta_1)\Gamma(\beta_3)}\sum_{k=0}^{n}\frac{(\pm 1)^k(\gamma_1)\,\delta_{1(m+k)}(\gamma_2)\,\delta_{2(n-k)}(\gamma_3)\,\delta_{3k}}{(m+1)_k(n+1)_k(\beta_1)_{\alpha_{1(m+k)}}(\beta_2)_{\alpha_{2(n-k)}}(\beta_3)_{\alpha_{3k}}}\frac{(z/3)^{3k}}{k!}$$

Thus the result (2.1) is proved.

Special Cases:

(1) On setting $\gamma_i = \delta_i = 1,$ for each i, equation (2.1) reduces to the following relation:

$$E_{\alpha_1,\beta_1}^{1,1}\left(\frac{z\,x}{3}\right)E_{\alpha_2,\beta_2}^{1,1}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}^{1,1}\left(\frac{\pm z/3}{xy}\right)=\frac{(z/3)^{m+n}}{m!n!}\sum_{k=0}^{n}\frac{(\pm 1)^k(m+k)!(n-k)!\left(\frac{z}{3}\right)^{3k}}{(m+1)_k(n+1)_k(\beta_1)_{\alpha_{1(m+k)}}(\beta_2)_{\alpha_{2(n-k)}}(\beta_3)_{\alpha_{3k}}} \qquad (2.4)$$

For $\alpha_{i=}\beta_{i=1,}$ for each I, equation (2.4) is equivalent to a know result (Kamarujjama and Khursheed, 2002).

(2) For $\alpha_1=\alpha_2=\alpha_3=1, \delta_1=\delta_2=\delta_3=2,$ equation (2.1) reduces to the following relation:

$$E_{1,\beta_1}^{\gamma_1,2}\left(\frac{z\,x}{3}\right)E_{1,\beta_2}^{\gamma_2,2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}^{\gamma_3,2}=\left(\frac{\pm z/3}{xy}\right)\sum_{m,n=-\infty}^{\infty}\frac{x^m y^n\left(\frac{\gamma_1}{2}\right)_m\left(\gamma_1\frac{1}{2}\right)_m\left(\frac{\gamma_1}{2}\right)_n\left(\gamma_2+\frac{1}{2}\right)_n}{m!n!\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\left(\frac{\beta_1}{2}\right)_m\left(\frac{\beta_{1+1}}{2}\right)_m\left(\frac{\beta_2}{2}\right)_n\left(\frac{\beta_{2+1}}{2}\right)_n}$$

$${}_6F_5\left[\begin{matrix}\frac{\gamma_1}{2}+m,\gamma_1+\frac{1}{2}+m,\frac{\gamma_2}{2}+n,\gamma_2+\frac{1}{2}+n,\frac{\gamma_3}{2},\gamma_3+\frac{1}{2};\pm\left(\frac{16z}{3}\right)^3\\ m+1,\ n+1,\beta_1+m,\beta_2+n,\ \beta_3;\end{matrix}\right] \qquad (2.5)$$

(3) For $\alpha_1=\alpha_2=1,\ \alpha_3=2, \delta_1=\delta_3=2, \delta_2=1, \beta_2=\beta_3=\gamma_2=\gamma_3=1,$ equation (2.1) reduces to the following relation:

$$E_{1,\beta_1}^{\gamma_1,2}\left(\frac{z\,x}{3}\right)E_{1,1}^{1,1}\left(\frac{zy}{3}\right)E_{2,1}^{1,2}=\left(\frac{\pm z/3}{xy}\right)\sum_{m,n=-\infty}^{\infty}\frac{x^m y^n 2^{2m}\left(\frac{\gamma_1}{2}\right)_m\left(\frac{\gamma_1+1}{2}\right)_m}{m!n!\Gamma(\beta_1)\Gamma(\beta_2)(\beta_1)_m}\ {}_2F_3\left[\begin{matrix}\frac{\gamma_1}{2}+m,\frac{\gamma_1+1}{2}+m;\\ \pm\left(\frac{4z}{3}\right)^3\\ m+1,n+1,\beta_1+m;\end{matrix}\right] (2.6)$$

(4) For $\alpha_1=\alpha_2=\beta_1=\beta_3=\gamma_1=\gamma_2=\gamma_3=\delta_1=1, \alpha_3=\delta_2=\delta_3=2,$ equations (2.1) reduces to the following relation:

$$E_{1,1}^{1,1}\left(\frac{zx}{3}\right)E_{1,\beta_2}^{1,2}\left(\frac{zy}{3}\right)E_{2,1}^{1,2}=\left(\frac{\pm z/3}{xy}\right)\sum_{m,n=-\infty}^{\infty}\frac{x^m y^n 2^{2n}}{m!n!\Gamma(\beta_2)(\beta_2)(\beta_2)_n}\ {}_2F_2\left[\begin{matrix}\frac{1}{2}+n,\frac{1}{2}; & \\ & \pm\left(\frac{16z}{3}\right)^3 \\ m+1,\ \beta_2+n; & \end{matrix}\right] \tag{2.7}$$

3. Further Generalization

$$E_{\alpha_1,\beta_1}^{\gamma_1,\delta_1}\left(\frac{zx_1}{n+1}\right)_{....}E_{\alpha_n,\beta_n}^{\gamma_n,\delta_n}\left(\frac{zx_n}{n+1}\right)E_{\alpha\beta}^{\gamma\delta}\left(\frac{\pm\frac{z}{n+1}}{x_1....x_n}\right)=\sum_{m_1,....,m_n=-\infty}^{\infty}x_1^{m_1}....x_n^{m_n}\left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n}m_i}$$

$$\sum_{k=0}^{n}\frac{(\pm1)^k(\gamma_1)_{\delta_1(m_1+k)}\cdots(\gamma_n)_{\delta_n(m_n+k)}(\gamma_{n+1})_{\delta_{n+1}k}\left(\frac{z}{n+1}\right)^{k(n+1)}}{\Gamma(\alpha_1(m_1+k)+\beta_1)_{....}\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha_{n+1}k+\beta_{n+1})} \tag{3.1}$$

where

$${}_{\alpha_i\,\beta_i}^{\gamma_i\,\delta_i}J_{m_1,....,m_n}(z)=\left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n}m_i}\sum_{k=0}^{n}\frac{(-1)^k(\gamma_1)_{\delta_1(m_1+k)}\cdots(\gamma_n)_{\delta_n(m_n+k)}(\gamma_{n+1})_{\delta_{n+1}k}\left(\frac{z}{n+1}\right)^{k(n+1)}}{\Gamma(\alpha_1(m_1+k)+\beta_1)_{....}\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha_{n+1}k+\beta_{n+1})} \tag{3.2}$$

and

$${}_{\alpha_i\,\beta_i}^{\gamma_i\,\delta_i}I_{m_1,....,m_n}(z)=\left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n}m_i}\sum_{k=0}^{n}\frac{(\gamma_1)_{\delta_1(m_1+k)}\cdots(\gamma_n)_{\delta_n(m_n+k)}(\gamma_{n+1})_{\delta_{n+1}k}\left(\frac{z}{n+1}\right)^{k(n+1)}}{\Gamma(\alpha_1(m_1+k)+\beta_1)_{....}\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha_{n+1}k+\beta_{n+1})} \tag{3.3}$$

For $\gamma_i = \delta_i = 1$, equation (3.1) is equivalent to a known result of (Kamarujjama and Khursheed, 2002) and further it reduces to equation (1.11) for $\alpha_i = \beta_i = 1$.

4. Conclusions

For further investigation based on this article we may use some other special functions in place of hyper Bessel function used in this paper. From several of examples discussed in this paper it will appear that our results (2.4) to (2.7) are extension of the generating functions involving the product of classical orthogonal polynomials. The resulting formula (3.1) allows the considerable unification of the results involving the product of various special functions which appear in the literature.

Acknowledgement

The authors are grateful to the referees for their valuable suggestions to make this paper in the present form.

References

Agarwal R.P., 1953. A product d'ume note de M.pierre Humbert, *C.R.Acad.Sci*.Paris.Vol.236, pp.2031-2032.

Delerue P., 1953. Ser calcul symbolique a n-variables et surles functions hyper Besselienness I.II, *Ann.Soc.Sci. Bruxelles*.Vol.67, pp. 83-104 and 229-274.

Erde'lyi e t al. A., 1953. Higher transcendental function, Vol.3, McGraw-Hill, New York.

Humbert P., 1936. Le calcul symbolique a variable, *Ann.Soc.Sci.Bruxelles, Ser.A*. Vol. 56, pp. 26-43.

Kamarujjama M and Khursheed Alam M., 2002. A Certain Class of Multiple generating functions involving Mittag-Leffler's functions, *SEA Jour Math and Math.Sc*. Vol. 1, pp. 29-38.

Mittag-Leffler G.M., 1905. Surla representation analyique d'unebranche uniforme d'une function monogene. *Acta Math*. Vol.29, pp. 101-182.

Pathan M.A., 2007. Multiple generating relations for hyper Bessel and hypergeometric functions, *Pacific Asian Jour.Mathematics*, Vol.1, No.1, pp. 38-44.

Prabhakar T.R., 1971. A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohana Math.J*. Vol. 19, pp. 7-15.

Shukla A.K and Prajapati J.C., 2007. On a generalization of Mittag-Leffler function and its properties, *J.Math.Anal and Appl*. Vol. 336, pp. 797-811.

Srivastava H.M and Manocha H.L., 1984. A treatise on generating functions, Halsted Press (Ellis Horwood Ltd., Chichester), John Wiley and Sons, New York.

Biographical notes

M.Kamarujjama has received Ph.D degree in 1993 form Aligarh Muslim University, Aligarh, India. He is an Associate Professor in the Department of Applied Mathematics, Z.H.College of Engineering and Technology, Aligarh Muslim University, Aligarh. He has more than 28 research papers. He has supervised one M.Phil and three Ph.D students in the field of special functions. He is a life member of Indian Mathematical Society and Society for Special functions and their Applications.

N.U.Khan has received M.Sc, M.Phil and Ph.D from Department of Mathematics, Aligarh Muslim University, Aligarh, India. He is an Assistant Professor in the Department of Applied Mathematics, Z.H.College of Engineering and Technology, Aligarh Muslim University, Aligarh. He has published more than 13 research papers in referred National and International journals. He has also attended and delivered talks in many National and International Conferences, Symposiums. He is a life member of Indian Mathematical Society, Society for Special functions and their Applications and Indian Society of Mathematics and Mathematical Sciences etc.

Waseem A. Khan has received M.Phil and Ph.D in 2008 and 2011, respectively from Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India. He has published more than 4 research papers in the field of special functions in National and International journals. He has participated in many National and International Conferences, Symposiums. He is a life member of Society for Special functions and their Applications (SSFA).

Received January 2011
Accepted February 2011
Final acceptance in revised form December 2011