Discrete-time nonlinear sliding mode controller

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Abstract

The objective of this paper is to design nonlinear sliding surface with investigation of the state delays. A particular linear transformation is defined to transform the time-delay and stability is proven for nonlinear sliding surface, which leads to show the asymptotic stability of the system.

Keywords: Discrete-time delay system, Sliding mode control, nonlinear sliding surface, optimal control.

1. Introduction

The time delay is a quite common in all types of engineering systems such as chemical process control, delay in the actuator operation of mechanical systems and economic systems etc. Compared to the systems without delay, the presence of the delay makes the system more difficult to achieve its satisfactory performance of the systems. In recent years there is a great interest of researchers for processes having the state delays.

The concept of the sliding mode control in recent years has drawn the attention of researchers (Bartoszewicz, 1998; Utkin, 1977; Gao. 1977; Lee et al., 2001). Sliding mode control is a methodology in which an appropriate input is provided so that the system states are confined to the desired sub-manifold of the state space. Furuta (1990) proposed discrete time sliding mode control laws where the control input includes state feedback terms in three different feedback coefficients. The discrete time sliding mode control has been mainly applied to stabilization problems. However in stabilization problems only bounded stability is guaranteed if the uncertainties in the system do not vanish at the equilibrium points. Won and Hedrick (1995) deal with the mismatch uncertainties for discrete adaptive sliding mode control for the nonlinear system for the output to track a sufficiently smooth desired trajectory. In Bandhyopadhyay and Deepak (2009), nonlinear sliding surface is created for varying the damping factor for improving the performance of the multi-input and multi-output linear systems with matched conditions. Some of the concepts and theoretical advances of continuous time sliding mode control are covered in literature (Utkin, 1976; Hung et al., 1993; El.Ghezawi et al., 1983; Young et al., 1999; Edward and Spurgeon, 1998) and the references therein. Due to flexibility of implementation, large classes of continuous systems are controlled by digital signal processors and microprocessors. To analyze the sampling time effect, discrete sliding mode control is well studied in the literature (Gao et al., 1995a,b; Spurgeon, 1992; Golo and Milosavljevic, 2000; Hu et al., 2009; Furukawa and Shimemura, 1983; Janardhanan and Bandhyopadhyay, 2006; Fiagbedzi and Pearson, 1986; Jafarov, 2000; Sarpturk et al., 1987). This paper has been organized as follows: Section II briefly introduces state about the description of the discrete –time plant with state delay and its transformation, section III illustrates proposed design of nonlinear sliding surface, section IV illustrates result analysis and section V describes the conclusion.

2. Problem Statement:

Consider the following discrete time-delay systems:

$$x(k+1) = \bar{A}x(k) + \bar{A}_d x(k-d) + \bar{B}u(k)$$

$$y(k) = \bar{C}x(k)$$

(1)

Where $x(k) \in R^n$ is the state vector, $u(k) \in R^m$ is the control input, $y(k) \in R^p$ is the output respectively. The term $\bar{A}$, $\bar{A}_d$, $\bar{B}$ and $\bar{C}$ are some constant matrices of appropriate dimensions and parameter $d>0$ is a constant.

We assume the following assumptions.

Assumption I: the pair ($\bar{A}$, $\bar{B}$) is controllable, the pair ($\bar{A}$, $\bar{C}$) is observable, matrices $\bar{B}$ and $\bar{C}$ have full rank.

Assumption II: p>m and rank ($\bar{C}\bar{B}$) = m.

For analyzing the system (1), we will transform it in to its regular form. Let suppose $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}^T$

And satisfies the det (B_2) $\neq 0$, there exists a linear non-singular transformation

$$z = Tx = \begin{bmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & B_2^{-1} \end{bmatrix} x \tag{2}$$

It transforms the system dynamics (1) to its regular form

$$z(k+1) = Az(k) + A_d z(k-d) + Bu(k)$$
$$y(k) = Cz(k)$$

Where

$$z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \tag{3}$$

The transform of the system can be shown with out loss of generality.

$$z_1(k+1) = A_{11} z_1(k) + A_{12} z_2(k) + A_{d11} z_1(k-d) + A_{d12} z_2(k-d)$$
$$z_2(k+1) = A_{11} z_1(k) + A_{22} z_2(k) + A_{d21} z_1(k-d) + A_{d22} z_2(k-d) + B_2 u(k) \tag{4}$$
$$y(k) = Cz(k)$$

Where $z_1(k) \in R^{(n-m)}$, $z_2(k) \in R^m$ are the state vectors and the matrices A, A_d, B_2 are the constant, with their appropriate dimensions. The following matrix can be written as follows after transformation. Regular form of the equation can be written as follows

$$A = T\bar{A}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{reg} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{5}$$

The regular form of the system's state delay is as follows

$$A_d = T\bar{A}_d T^{-1} = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}$$

$$A_{dreg} = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} \tag{6}$$

$$B = T\bar{B} \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

$$C = \bar{C}T^{-1}$$

$z_1 \in \Re^{n-m}$ and $z_2 \in \Re^m$ are the state vectors. $A_{11}, A_{12}, A_{21}, A_{22}$ are known constant matrices with appropriate dimensions. I_m is $m\times m$ identity matrix.

2.1 Calculating the optimal gain matrix : From the discrete-time delay system, the gain matrix can be calculated as follows:

$$x(k+1) = \bar{A}x(k) + \bar{A}_d x(k-d) + \bar{B}u(k)$$

$$y(k) = \bar{C}x(k)$$

To Minimized the cost function

$$J = \frac{1}{2}\sum_0^\infty (x^T(k)Qx(k) + u^T(k)Ru(k)) \tag{7}$$

To find out the gain with the help of LQR method and the resulting optimal gain matrix
R=1,
Q=diag(Q1,Q2,Q3)

G=lqr($\bar{A}$, $\bar{B}$,Q,R)

3. Designing of Nonlinear Sliding Surface

In this section the designing of the nonlinear sliding surfaces for the general MIMO systems. Without loss of generality, the system described by (1) can be transformed in to regular form as follows

$$z_1(k+1) = A_{11}z_1(k) + A_{12}z_2(k) + A_{d11}z_1(k-d) + A_{d12}z_2(k-d)$$
$$z_2(k+1) = A_{11}z_1(k) + A_{22}z_2(k) + A_{d21}z_1(k-d) + A_{d22}z_2(k-d) + B_2u(k) \qquad (8)$$
$$y(k) = Cz(k)$$

$$z = Tx = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \qquad (9)$$

The sliding surface for the system in regular form is proposed as follows

$$S(k) = C^T z(k) = 0 \qquad (10)$$

Without loss of generality of the system, the sliding mode control is as follows

$$S(k) = C^T z(k) = \begin{bmatrix} C_1 & I_m \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \qquad (11)$$

Where $C^T \in \Re^{m \times n}, C_1 \in \Re^{m \times (n-m)}$, $I_m \in \Re^{m \times m}$ dimension. For sliding motion the sliding condition is as follows

$$S(k) = C^T z(k) = 0$$

Then

$$S(k) = C^T z(k) = 0$$
$$S(k) = \begin{bmatrix} C_1 & I_m \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = 0 \qquad (12)$$
$$S(k) = C_1 z_1(k) + z_2(k) = 0$$
$$z_2(k) = -C_1 z_1(k)$$

$$S(k) = \begin{bmatrix} G - \psi(y(k))\{A_{12}^T P(A_{11} - A_{12}G))\} & I_m \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}$$
$$C_1 = \begin{bmatrix} G - \psi(y(k))\{A_{12}^T P(A_{11} - A_{12}G)\} \end{bmatrix}$$
$$z_1(k+1) = A_{11eq}z_1(k) + A_{12}\psi(y(k))A_{12}^T P A_{11eq}z_1(k)$$

Seeing $z_2(k)$ as dummy input of the subsystem (9). Substituting (12) into subsystem (9) yields

$$z_1(k+1) = A_{11}z_1(k) - A_{12}C_1 z_1(k) + A_{d11}z_1(k-d) - A_{d12}C_1 z_2(k-d)$$
$$z_1(k+1) = (A_{11} - A_{12}C_1)z_1(k) + (A_{d11} - A_{d12}C_1)z_1(k-d) \qquad (13)$$

The nonlinear sliding surface can be designed by incorporating the nonlinearity $\psi(y(k))$ in the system dynamics for proven for its stability. If the nonlinear surface is stable then the all the states trajectory will be on the stable surface for future all the time. The condition for the stability of the nonlinear surface as follows

$$2\psi(y(k)) + \psi(y(k))A_{12}^T P A_{12}\psi(y(k)) \le 0 \qquad (14)$$

3.1 Proof : The proof for the nonlinear sliding surface as stability follows.
Consider Lyapunov functional candidate for proving the nonlinear sliding surface as stable

$$V_1(k) = z_1^T(k)Pz_1(k)$$
$$\Delta V(k) = V(k+1) - V(k)$$
$$\Delta V_1(k) = z_1^T(k+1)Pz_1(k+1) - z_1^T(k)Pz_1(k)$$

$$z_1(k+1) = A_{11}z_1(k) + A_{12}(-C_1z_1(k))$$
$$z_1(k+1) = (A_{11} - A_{12}C_1)z_1(k)$$
$$= A_{11} - A_{12}(G - \psi(y(k))A_{12}^T P(A_{11} - A_{12}G))z_1(k)$$
$$= A_{11eq}z_1(k) + A_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k)$$

(15)

Where $A_{11eq} = (A_{11} - A_{12}G)$ and transform state dynamics becomes as follows

$$z_1(k+1) = A_{11eq}z_1(k) + A_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k)$$

$$S(k) = \left[G - \psi(y(k))\left\{A_{12}^T P(A_{11} - A_{12}G)\right\} \quad I_m \right] \times \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}$$

$$C_1 = \left[G - \psi(y(k))\left\{A_{12}^T P(A_{11} - A_{12}G)\right\} \right]$$

$$z_1(k+1) = A_{11eq}z_1(k) + A_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k)$$

For stability $(A_{11} - A_{12}G)$ must have all the eigenvalues inside the unit circle. Now, we have to prove that the nonlinear sliding surface is stable by the Lyapunov method for the present state.

3.2 Proof : Consider the Lyapunov functional candidate as follows for proving asymptotic stability of the transform system

$$V_1(k) = z_1^T(k)Pz_1(k)$$
$$V_1(k+1) = z_1^T(k+1)Pz_1(k+1)$$

$$\Delta V(k) = z_1^T(k+1)Pz_1(k+1) - z_1^T(k)Pz_1(k)$$

$$\Delta V(k) = \left\{A_{11eq}z_1(k) + A_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k)\right\}^T P * \left\{A_{11eq}z_1(k) + A_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k)\right\}$$
$$- z_1^T(k)Pz_1(k)$$
$$\Delta V(k) = z_1^T(k)A_{11eq}^T PA_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k) + z_1^T(k)A_{11eq}^T z_1(k) - z_1^T(k)Pz_1(k)$$
$$+ z_1^T(k)A_{11eq}^T PA_{12}\psi(y(k))A_{12}^T PA_{11eq}z_1(k) + e_1^T(k)A_{11eq}^T PA_{12}A_{12}^T \Psi(y(k))$$
$$* A_{12}^T PA_{11eq}e_1(k)$$
$$= -z_1^T Hz_1(k) + z_1^T(k)A_{11eq}^T PA_{12}\left\{2\psi(y(k)) + \psi(y(k))A_{12}^T PA_{12}\psi(y(k))\right\}$$
$$* A_{12}^T PA_{11eq}z_1(k)$$

(16)

Where H is some small positive definite matrix
Let $z_1^T(k)A_{11eq}^T PA_{12} = N^T$

$$\Delta V_1(k) = N^T \left\{2\psi(y(k)) + \psi(y(k+))A_{12}^T PA_{12}\psi(y(k+))\right\} N - z_1^T(k)Hz_1(k)$$

(17)

From (14)

$$2\psi(y(k)) + \psi(y(k))A_{12}^T PA_{12}\psi(y(k)) \le 0$$
$$\Delta V(k) = -z_1^T(k)Hz_1(k)$$

Here it is proven that the nonlinear sliding surface is asymptotically stable as the $\Delta V(k)$ is negative definite, therefore all the states will be tracking on stable nonlinear sliding surface for all the future time.

3.3 Lemma 1 [22] Given constant matrices D, E and a symmetric constant matrix Y of appropriate dimensions, the following inequality holds:

$$Y + DEF + E^T F^T D^T < 0$$

Where F satisfies $F^T F \le N$, if and only if for some $\zeta > 0$

$$Y + \left[\zeta^{-1}E^T \quad \zeta D \right] \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \zeta^{-1}E \\ \zeta D^T \end{bmatrix} < 0 \quad \text{holds.}$$

The main result of the asymptotic stability of the reduced order with time delay is summarized as per following theorem.

3.4 Theorem 1: If there exists a symmetric and positive definite matrix P, some matrix M and some scalar ζ such that reduced order discrete-time system is asymptotically stable via sliding mode surface (13):

$$\begin{bmatrix} -H & * & * & * \\ A_{11}H + A_{12}M & -H & * & * \\ E_1H + E_2M & 0 & -\zeta I & * \\ 0 & \zeta D^T & 0 & -\zeta I \end{bmatrix}$$

Where $-H = P^{-1}$, $M = C_1 P^{-1}$ and * denotes the transposed elements in the symmetric positions.

3.5 Proof: Consider the Lyapunov function candidate for delay system

$$V(k) = z_1^T(k) P z_1(k) + \sum_{\alpha=k-d}^{k-1} z_1^T(\alpha) Q z_1(\alpha) \tag{18}$$

Where P and Q are positive definite symmetrical matrices. The difference of V(k) is

$$\Delta V(k) = V(k+1) - V(k)$$

$$= z_1^T(k+1) P z_1(k+1) + \sum_{\alpha=k+1-d}^{k} z_1^T(\alpha) Q z_1(\alpha) \tag{19}$$

$$- z_1^T(k) P z_1(k) - \sum_{\alpha=k-d}^{k-1} z_1^T(\alpha) Q z_1(\alpha)$$

$$\Delta V(k) = \left[A_1^T z(k) + A_{d1}^T z(k-d) \right]^T \times P \left[A_1^T z(k) + A_{d1}^T z(k-d) \right]$$

$$+ z^T(k) Q z(k) - z^T(k-d) Q z(k-d)$$

$$= \Psi^T(k) \left[A_1^T \quad A_{d1}^T \right]^T P \left[A_1^T \quad A_{d1}^T \right] \Psi(k) - \Psi^T(k) \begin{bmatrix} P-Q & 0 \\ 0 & Q \end{bmatrix} \Psi(k)$$

Where

$$\Psi(k) = \left[z_1^T(k) \quad z_1^T(k-d) \right]^T \tag{20}$$

$$\Psi_1^T(k) \left(\left[A_1 \quad A_{d1} \right]^T P \left[A_1 \quad A_{d1} \right] + K \right) \Psi_1(k)$$

Where

$$\Psi(k) = \begin{bmatrix} z_1(k) \\ z_1(k-d) \end{bmatrix}$$

$$K = \begin{bmatrix} Q-P & 0 \\ 0 & -Q \end{bmatrix}$$

$$A_1 = A_{11} - A_{12}G$$

$$A_{d1} = A_{d11} - A_{d12}G$$

Therefore sliding mode dynamics (13) is asymptotically stable, if the following inequality holds

$$\left(\left[A_1 \quad A_{d1} \right]^T P \left[A_1 \quad A_{d1} \right] + K \right) < 0$$

Apply Schur complements in (14) is equal to

$$\begin{bmatrix} Q-P & * & * \\ 0 & -Q & * \\ A_1 & A_{d1} & -P^{-1} \end{bmatrix} < 0 \qquad \text{condition hold.}$$

*Denotes the transposed elements in the symmetric position. It will be negative definite and the change in Lyapunov function will be negative definite i.e.

$$\Delta V(k) < 0, \quad z_1(k) \neq 0, \quad k = 1, 2, \ldots.$$

Now the system will be asymptotically stable.

4. Designing of Control Law

The designing of the control law is based upon the stable nonlinear sliding mode control which already proven for its stability.

$$S(k) = C^T z(k)$$

$$S(k+1) = C^T z(k+1)$$

Sliding mode condition

$$S(k+1) = S(k) = 0$$
$$z(k+1) = Az(k) + A_d(k-d) + Bu(k)$$

$$C^T z(k+1) = C^T \left[Az(k) + A_d(k-d) + Bu(k) \right] = 0$$
$$= C^T Az(k) + C^T A_d(k-d) + C^T Bu(k)$$
$$C^T Az(k) + C^T A_d(k-d) + C^T Bu(k) = 0$$
$$u(k) = (C^T B)^{-1} \left[C^T Az(k) + C^T A_d(k-d) \right]$$

$$(21)$$

At the sampling instant of discrete-time system, the state trajectory will remain on the nonlinear sliding surface and system will be stable for ever in all the future time. It is proven that control law (21) drives the system trajectory on to the nonlinear sliding surface and maintains the trajectory on the sliding surface during subsequent time.

5. Conclusion

In this paper, nonlinear sliding mode controller has been designed for the time delay in the discrete-time system. The nonlinear systems are highly unstable and the proposed algorithm of nonlinear sliding mode controller will have more effectiveness for solving the nonlinear systems.

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