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Persistence and stability for the three species ratio-dependent food–chain model

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Abstract

In this paper, we have proposed and analyzed a tritrophic food chain model composed of a prey, a middle predator and a top predator. Ratio-dependent functional response is considered to model the interactions among the species of the system. Mathematical analysis of model equation with regard to the nature of equilibria, boundedness and persistence of the solution are carried out. To verify the analytical findings numerical simulation is performed. Furthermore, global stability of the system is shown graphically. It has been observed from numerical simulation that prey population decreases in the absence of top predators. However, presence of top predators in the system causes an increase in prey population with a depression in the population of middle predators.

Keywords: Food chain model, Ratio-Dependent, Boundedness, Stability, Persistence, Computer Simulation.

1. Introduction

Predator-prey functional responses have been facing a great challenge from biological and physiological researchers. At present it is clear that predator abundance also has ability to influence the functional response. Arditi and Ginzburg (1989) have suggested in situations characterized by strong space and time heterogeneities that the functional response can be approximated by a function of the prey to predator ratio. Several biologists have been able to establish the fact that functional responses over typical ecological time scales ought to be depending on the densities of both prey and predator, especially when predators have to search for food, and therefore have to share or compete for food. Actually prey dependent and ratio dependent models are extremes or limiting cases: prey dependent models are based on the daily energy balance of predators, on the other hand ratio-dependent models presuppose that prey are easy to find and the predators dynamics are governed by direct density dependence with the prey densities determining the size of defended territories. Within the natural environment both aspects have the ability to influence predator-prey dynamics, and the issue of which of the two extremes is closer to reality in which system is wide open. Arditi and Ginzburg (1989) first proposed following Michalies-Menten type ratio-dependent predator-prey model:

$$x'(t) = rx(1-\frac{x}{k}) - c\frac{mxy}{x+ay}, \qquad x(0) > 0,$$

$$y'(t) = y\left[-d + \frac{mx}{ay+x}\right], \qquad y(0) > 0.$$

The dynamics of ratio-dependent predator prey model has been studied in detail by Karev and Arditi (2001), Xiao and Ruan (2001), Hsu et al. (2001), and others. Hsu et al. (2001) have shown that the ratio-dependent models are capable of producing far richer and biologically more realistic dynamics. Ratio-dependent model requires high densities for both prey and predator while the most interesting dynamics is near the axes (Xiao and Ruan 2001) which allow the mutual extinction as possible outcome of a predator prey interaction (Kuang and Beretta, 1998; Jost et al. (1999)). Specifically, ratio dependent model do not produce the paradox of biological control and the paradox of enrichment. Ratio-dependent model always exhibits interesting dynamics in the vicinity of the origin due the fact that such models are undefined there. Hence, any type of solution initiating with positive populations is valid only if it can be shown that there does not exist a sequence of times tending to infinity such that either the prey values or predator values tend to zero. This can be expressed in terms of "Persistence". Biologically, persistence means the survival of all populations in future time. Mathematically, the persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non-negative cone.

For the mathematical model of multiple species interaction, Hsu et al. (2001a) studied a model of two predators competing for single prey with ratio-dependence functional response. Dubey and Upadhaya (2004) proposed a two predator one prey system with ratio-dependent predator growth rate. Criteria for local stability, instability and global stability of the non-negative equilibria are obtained. They also discussed about the permanent co-existence of the three species. Kar and Batabyal (2010) considered two prey one predator system in presence of time delay due to gestation. Criteria for local stability and global stability of the non-negative equlibria are obtained. They also obtain the criteria for the system to be persistent. Zhang et.al (2006) studied the stability of three species population model consisting of an endemic prey (bird), an alien prey (rabbit) and an alien predator (cat). Another important mathematical model of multiple species interactions is the so-called food chain model. Many simple two species food chain model have been thoroughly explored, while new discoveries continue to be examining with three or four trophic level (Moghadas and Gumel(2003)). In the paper of Freedman and Waltman (1977), studied the persistence of three species food chain model and provided necessary and sufficient condition for the persistence of Lotka-Volterra food chain model. Gard and Hallam (1979), obtained conditions for the persistence of food chain of arbitrary length. Later, Gard (1980-81) studied persistence of food chain with general interactions and in 1985 Freedman and So studied global stability and persistence of simple food chains. Hsu et al. (2002) studied the three trophic level food chain with ratio-dependent Michalies-Menten type functional response and its application to biological control. The model is shown to be rich in boundary dynamics and capable of generating extinction of both pest and control agent.

In this paper, we have considered a general three trophic level food chain model with ratio-dependent functional response in which prey is plant, middle predator is a pest and top predator is a natural enemy of pest. Here, we have obtained conditions which influence the boundedness, stability and persistence of the system. By a change of variables we transform our system in such a way that there is one to one correspondence between the positive values of prey, predator and top predator of the original system and the positive values of transformed system, so that the result which are true for transformed model also hold true for original model. This paper is concerned with questions of stability and persistence of populations. The stability theory of ordinary differential equations is used to analyze the model. Our results are illustrated by example.

The organization of this paper is as follows. In section 2, we describe our model and give conditions which guarantee the existence of an interior equilibrium. In section 3 we have determined the boundary equilibrium point and their stabilities; the boundedness of our model is also studied in this section. Persistence of system is studied in section 4. In section 5, we consider the numerical example to illustrate our results. Computer simulation of the system is presented in this section. In section 6, we have studied the system (1) in absence of top predator and obtained corresponding stability conditions of equilibrium points. At last general discussions of the paper and biological implications of our model are presented in section 7.

2. Mathematical Model

We consider a three trophic level food-chain model with ratio-dependence under the framework of the following set of nonlinear ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= xg(x) - yp\left(\frac{x}{y}\right), && x(0) = x_0 > 0, \\ \dot{y}(t) &= y\left[-d_1 + cp\left(\frac{x}{y}\right)\right] - zq\left(\frac{y}{z}\right), && y(0) = y_0 > 0, \\ \dot{z}(t) &= z\left[-d_2 + dq\left(\frac{y}{z}\right)\right], && z(0) = z_0 > 0. \qquad (1) \end{aligned}$$

where *x (t)* is the population density of lowest tropic level species or prey, *y (t)* is population density of middle tropic level species or middle predator and *z (t)* is population density of highest tropic level or (super) top predator in which *z* prey on *y* and only *y*, and *y* prey on *x* and nutrient recycling is not accounted for. Here the positive constant d_1, d_2, c and d respectively denote the death rate of the first predator, the death rate of the super predator, the conversion rate and maximal growth rate of top predator.

We take the following assumptions:

(H1): All functions have second order derivatives continuous in their augments on the interval $(0, \infty)$. This is sufficient to guarantee that solutions to positive initial value problems exist uniquely at least for some positive time.

(H2): *g(x)* is specific growth rate of prey and is always assumed to satisfy,

$g(0) > 0$, $g'(x) < 0$ for $x \geq 0$, and there exist $K > 0$ such that $g(K) = 0$.
For small values of the prey population, it will grow. However, there exists a carrying capacity of the environment beyond which the prey population cannot increase even in the absence of predators.

Before discussing our new assumptions we introduce new variables,

$$u = \frac{x}{y} \qquad \text{then} \qquad p\left(\frac{x}{y}\right) = p(u)\ . \tag{2}$$

And

$$v = \frac{y}{z} \qquad \text{then} \qquad q\left(\frac{y}{z}\right) = q(v). \tag{3}$$

(H3): $p(0) = 0$; $p'(u) > 0$ for $u \geq 0$.

u is large if *y* is small or *x* is large i.e. few predators or many prey. $p(u)$ is the functional response of the middle predators which increases with *u*, i.e. with the prey, and in the absence of the prey there can be no predation.

(H4): $q(0) = 0$; $q'(v) > 0$ for $v \geq 0$.

v is large if *z* is small or *y* is large i.e. few super predators or many first predators. *q (v)* is the functional response of the super predators which increases with *v*, i.e. with the first predators.

Now we derive the transformed system for the variables *x, u* and *v* by (1), (2) and (3) we have

$$\dot{u} = \frac{\dot{x}y - \dot{y}x}{y^2},$$

$$= ug(x) - p(u) + ud_1 - cup(u) + \frac{uq(v)}{v}.$$

If we take

$$R(u) = \begin{cases} \dfrac{p(u)}{u}, u > 0 \\ p'(0), u = 0. \end{cases}$$

and

$$Q(v) = \begin{cases} \dfrac{q(v)}{v}, v > 0 \\ q'(0), v = 0. \end{cases}$$

Then,

$$\dot{u} = u[g(x) - R(u) + d_1 - cp(u) + Q(v)],$$

Again

$$\dot{v} = \frac{\dot{y}z - \dot{z}y}{z^2},$$

$$\dot{v} = v(d_2 - d_1) + cvp(u) - q(v) - vdq(v).$$

Then our model (1) becomes

$$\begin{aligned} \dot{x}(t) &= x[g(x) - R(u)], & x(0) &= x_0 > 0, \\ \dot{u}(t) &= u[g(x) - R(u) + d_1 - cp(u) + Q(v)], & u(0) &= u_0 > 0, \\ \dot{v}(t) &= v[(d_2 - d_1) + cp(u) - Q(v) - dq(v)], & v(0) &= v_0 > 0. \end{aligned} \tag{4}$$

Where $\left(. = \dfrac{d}{dt} \right)$,

3. Boundedness, Boundary Equilibria and Stability

Computations of the boundary equilibria and their stabilities for system (4) provide the information needed to determine the persistence of the system (4). To do so, we compute the variational matrix of system (4). The signs of the real parts of the eigen values of the matrix evaluated at a given equilibrium point determine its stability. This matrix is given by

$$M = \begin{bmatrix} g(x) - R(u) + xg'(x) & -xR'(u) & 0 \\ ug'(x) & [g(x) - R(u) + d_1 - cp(u) + Q(v) - us'(u)] & uQ'(v) \\ 0 & vcp'(u) & d_2 - d_1 + cp(u) - Q(v) - dq(v) - vh'(v) \end{bmatrix},$$

Where, $s'(u) = R'(u) + cp'(u)$ and $h'(v) = Q'(v) + dq'(v)$.

System (4) has at most eight non-negative equilibrium points: $E_0(0,0,0)$, $E_1(x_1,0,0)$, $E_2(0,u_2,0)$, $E_3(0,0,v_3)$, $\overline{E}(\overline{x},\overline{u},0)$, $\widetilde{E}(\widetilde{x},0,\widetilde{v})$, $\hat{E}(0,\hat{u},\hat{v})$, and $E^*(x^*,u^*,v^*)$. Existence of E_0 is obvious. For E_1 to exist, $x_1 > 0$ must exist such that $g(x_1) = p'(0)$. This occurs if and only if $g(0) > p'(0)$. For E_2 to exist, a positive root of $g(0) + d_1 + q'(0) = cp(u_2) + R(u_2)$ must exist, i.e. $p(u_2) = \dfrac{[g(0) + d_1 + q'(0)]u_2}{1 + cu_2}$, has a positive root. For E_3 to exist, equation $Q(v_3) + dq(v_3) = d_2 - d_1$, must have a positive root.

Equilibrium $\overline{E}(\overline{x},\overline{u},0)$ exists if and only if the algebraic equations

$$g(\overline{x}) = R(\overline{u}),$$

$$g(\overline{x}) + d_1 + q'(0) = R(\overline{u}) + cp(\overline{u}),$$ has a positive solution.

Then $p(\overline{u}) = \dfrac{d_1 + q'(0)}{c}$ and $g(\overline{x}) = R(\overline{u}) = \dfrac{d_1 + q'(0)}{c\overline{u}}$. Hence $\overline{E}(\overline{x},\overline{u},0)$ exists if and only if $\dfrac{d_1 + q'(0)}{c}$ is in the range of $p(\overline{u})$ and $g(0) > \dfrac{d_1 + q'(0)}{c\overline{u}}$, where $p(\overline{u}) = \dfrac{d_1 + q'(0)}{c}$.

Equilibrium occurring in the $x - v$ plane is $\widetilde{E}(\widetilde{x},0,\widetilde{v})$. Here $\widetilde{x}$ and $\widetilde{v}$ are obtained by solving

$$g(\widetilde{x}) = p'(0),$$

$$d_2 - d_1 = Q(\widetilde{v}) + dq(\widetilde{v}).$$

Clearly $\widetilde{v} > 0$ and $\widetilde{x} > 0$ if $q(\widetilde{v}) = \dfrac{(d_2 - d_1)\widetilde{v}}{1 + d\widetilde{v}}$, has a positive root.

Equilibrium occurring in $u - v$ plane is $\hat{E}(0,\hat{u},\hat{v})$. Here the solution can be obtained by solving equations

$$g(0) + d_1 + Q(\hat{v}) = R(\hat{u}) + cp(\hat{u}),$$

$$d_2 - d_1 + cp(\hat{u}) = Q(\hat{v}) + dq(\hat{v}).$$

Thus $\hat{E}(0,\hat{u},\hat{v})$ exists if $g(0) + d_2 = R(\hat{u}) + dq(\widetilde{v})$, has a positive solution.

Interior planner equilibrium occurring in the $x - u - v$ plane is $E^*(x^*,u^*,v^*)$. Here x^*, u^* and v^* can be obtained by solving

$$g(x^*) = R(u^*),$$

$$g(x^*) + d_1 + Q(v^*) = R(u^*) + cp(u^*),$$

$$d_2 - d_1 + cp(u^*) = Q(v^*) + dq(v^*).$$

Then $p(u^*) = \frac{d_1 + Q(v^*)}{c}$, $g(x^*) = R(u^*) = \frac{d_1 + Q(v^*)}{cu^*}$ and $q(v^*) = \frac{d_2}{d}$. Hence E^* exist if and only if $\frac{d_1 + Q(v^*)}{c}$ is in the range of $p(u^*)$ and $g(0) > \frac{d_1 + Q(v^*)}{cu^*}$ also $\frac{d_2}{d}$ is in the range of $q(v^*)$.

The equilibrium $E_0(0,0,0)$ has variational matrix $M(E_0)$ given by

$$M(E_0) = \begin{bmatrix} g(0) - p'(0) & 0 & 0 \\ 0 & g(0) - p'(0) + d_1 + q'(0) & 0 \\ 0 & 0 & d_2 - d_1 - q'(0) \end{bmatrix}.$$

Which has all the three eigen values positive whenever $g(0) - p'(0) > 0$ and $d_2 > d_1 + q'(0)$. This implies that E_0 is completely unstable whenever E_1 exists.

The equilibrium $E_1(x_1,0,0)$ has variational matrix $M(E_1)$ given by

$$M(E_1) = \begin{bmatrix} x_1 g'(x_1) & x_1 R'(0) & 0 \\ 0 & d_1 + q'(0) & 0 \\ 0 & 0 & d_2 - d_1 - q'(0) \end{bmatrix}.$$

For $M(E_1)$ two of the eigen values are positive and one is negative giving a point with nonempty unstable manifolds and stable manifold.

The equilibrium point $E_2(0,u_2,0)$ has variational matrix $M(E_2)$ given by

$$M(E_2) = \begin{bmatrix} g(0) - R(u_2) & 0 & 0 \\ u_2 g'(0) & -u_2[R'(u_2) + cp'(u_2)] & u_2 Q'(0) \\ 0 & 0 & d_2 + g(0) - R(u_2) \end{bmatrix}.$$

For $M(E_2)$ one of the eigen value is negative and two are positive whenever $R'(u_2) + cp'(u_2) > 0$ and $g(0) - R(u_2) > 0$, again has a nonempty stable manifold and unstable manifolds.

The equilibrium point $E_3(0,0,v_3)$ has variational matrix $M(E_3)$ given by

$$M(E_3) = \begin{bmatrix} g(0) - p'(0) & 0 & 0 \\ 0 & g(0) - p'(0) + d_1 + Q(v_3) & 0 \\ 0 & v_3 cp'(0) & -v_3[Q'(v_3) + dq'(v_3)] \end{bmatrix}.$$

For $M(E_3)$ two of the eigen values are positive and one is negative whenever $Q'(v_3) + dq'(v_3) > 0$, again giving a point with nonempty unstable manifolds and stable manifold.

Now the variational matrix $M(\overline{E})$ about equilibrium point $\overline{E}(\overline{x},\overline{u},0)$ is given by

$$M(\overline{E}) = \begin{bmatrix} \overline{x}g'(\overline{x}) & -\overline{x}R'(\overline{u}) & 0 \\ \overline{u}g'(\overline{x}) & -\overline{u}[R'(\overline{u}) + cp'(\overline{u})] & \overline{u}Q'(0) \\ 0 & 0 & d_2 - d_1 + cp(\overline{u}) - q'(0) \end{bmatrix}.$$

The eigen values of $M(\overline{E})$ are $d_2 - d_1 + cp(\overline{u}) - q'(0)$ and $\overline{\lambda}_\pm$ where

$$\overline{\lambda}_\pm = \frac{1}{2}[\overline{x}g'(\overline{x}) - \overline{u}\{R'(\overline{u}) + cp'(\overline{u})\}] \pm \sqrt{[\overline{x}g'(\overline{x}) - \overline{u}\{R'(\overline{u}) + cp'(\overline{u})\}]^2 + 4\overline{x}c\overline{u}p'(\overline{u})g'(\overline{x})}\ .$$

Since $R'(\overline{u}) + cp'(\overline{u}) > 0$ therefore the signs of the real part of $\overline{\lambda}_+$ and $\overline{\lambda}_-$ are negative. This implies that $\overline{E}$ is locally asymptotically stable in $x - u$ plane and unstable in the v direction.

The equilibrium $\widetilde{E}(\widetilde{x},0,\widetilde{v})$ has variational matrix $M(\widetilde{E})$ given by

$$M(\widetilde{E}) = \begin{bmatrix} \widetilde{x}g'(\widetilde{x}) & \widetilde{x}R'(0) & 0 \\ 0 & d_1 + Q(\widetilde{v}) & 0 \\ 0 & \widetilde{v}cp'(0) & -\widetilde{v}[Q'(\widetilde{v}) + dq'(\widetilde{v})] \end{bmatrix}.$$

For $M(\widetilde{E})$ two of the eigen values are negative and one is positive whenever $Q'(\widetilde{v}) + dq'(\widetilde{v}) > 0$ hold. Hence $\widetilde{E}$ again has non-empty stable and unstable manifolds.

The equilibrium point $\hat{E}(0,\hat{u},\hat{v})$ has variational matrix $M(\hat{E})$ given by,

$$M(\hat{E}) = \begin{bmatrix} g(0) - R(\hat{u}) & 0 & 0 \\ \hat{u}g'(0) & -\hat{u}[R'(\hat{u}) + cp'(\hat{u})] & \hat{u}Q'(\hat{v}) \\ 0 & \hat{v}cp'(\hat{u}) & -\hat{v}[Q'(\hat{v}) + dq'(\hat{v})] \end{bmatrix}.$$

The eigen values of $M(\hat{E})$ are $g(0) - R(\hat{u})$ and $\hat{\lambda}_{\pm}$,

$$\hat{\lambda}_{\pm} = -\frac{1}{2}[\hat{u}\{R'(\hat{u}) + cp'(\hat{u})\} + \hat{v}\{Q'(\hat{v}) + dq'(\hat{v})\}]$$
$$\pm \sqrt{[\hat{u}\{R'(\hat{u}) + cp'(\hat{u})\} + \hat{v}\{Q'(\hat{v}) + dq'(\hat{v})\}]^2 - 4[\hat{u}\hat{v}\{R'(\hat{u}) + cp'(\hat{u})\}\{Q'(\hat{v}) + dq'(\hat{v})\} - \hat{u}\hat{v}cp'(\hat{u})Q'(\hat{v})]}.$$

The signs of real part of $\hat{\lambda}_+$ and $\hat{\lambda}_-$ are negative. This implies that $\hat{E}$ is locally asymptotic stable in $u - v$ plane and unstable in x direction if and only if $g(0) - R(\hat{u}) > 0$.

Now the equilibrium point $E^*(x^*, u^*, v^*)$ has variational matrix $M(E^*)$ given by;

$$M(E^*) = \begin{bmatrix} x^*g'(x^*) & -x^*R'(u^*) & 0 \\ u^*g'(x^*) & -u^*[R'(u^*) + cp'(u^*)] & u^*Q'(v^*) \\ 0 & v^*cp'(u^*) & -v^*[Q'(v^*) + dq'(v^*)] \end{bmatrix}.$$

The characteristic equation for the variational matrix $M(E^*)$ is given by,

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0, \tag{5}$$

where,

$$B_1 = u^*\{R'(u^*) + cp'(u^*)\} + v^*\{Q'(v^*) + dq'(v^*)\} - x^*g'(x^*),$$

$$B_2 = u^*v^*\{R'(u^*)(Q'(v^*) + dq'(v^*) + cdp'(u^*)q'(v^*))\} - x^*u^*cg'(x^*)p'(u^*) - x^*v^*g'(x^*)(Q'(v^*) + dq'(v^*)),$$

$$B_3 = -cdx^*u^*v^*g'(x^*)p'(u^*)q'(v^*).$$

Evidently, $B_1 > 0$, $B_2 > 0$ and $B_3 > 0$ if $Q'(v^*) + dq'(v^*) > 0$ and $R'(u^*) + cp'(u^*) > 0$. Therefore, by Routh-Hurwitz criterion, if $B_1B_2 - B_3 > 0$, then all roots of equation (5) have negative real parts and E^* is locally asymptotically stable equilibrium point.

Now we find the assumptions required to prove the persistence of the system. First and foremost we deduce the conditions for existence of positive interior equilibrium point, whose details are as given below:

Note that if $p(u) < \dfrac{d_1 + Q(v)}{c}$ and $q(v) < \dfrac{d_2}{d}$, (so that E^* cannot exist) for $u \geq 0$ and $v \geq 0$, then from system (1)

$$\dot{y}(t) = y[-d_1 + cp(u) - Q(v)] < 0, \qquad \text{for } y > 0,$$

And

$\dot{z}(t) = z[-d_2 + dq(v)] < 0,$ for $z > 0$.

So that $\lim_{t\to\infty} y(t) = 0$ and then either $x \to 0$ or $u \to \infty$ or both and $\lim_{t\to\infty} z(t) = 0$ then either $y \to 0$ or $v \to \infty$ or both. Hence we assume the following hold.

(A1): $u^* > 0$ exists such that $p(u^*) = \dfrac{d_1 + Q(v^*)}{c}$.

(A2): $v^* > 0,$ exist such that $q(v^*) = \dfrac{d_2}{d}$.

The next assumption required in order to obtain persistence of solution is

(A3): $g(0) - R(u) > 0.$

Our next assumption is one that is difficult to justify biologically, but seems to be required in order to obtain boundedness and persistence of solutions.

(A4): $R'(u) + cp'(u) > 0$ and $Q'(v) + dq'(v) > 0.$

Next we make the assumption for E_1 exist, for if not, then the unbounded solutions can occur.

(A5): E_1 exists if $g(0) - p'(0) > 0$ and $d_2 - (d_1 + q'(0)) > 0.$

With the above assumptions, we can prove that the solutions of system (4) and hence of (1) are bounded.

Theorem (3.1): Let (H1) – (H4) and (A1) – (A5) holds. Then solutions of system (4) are bounded.

Proof: From the first equation of system (4), we have

$$\dot{x}(t) = x[g(x) - R(u)],$$

$$\dot{x}(t) \le x(t)g(x(t)),$$

By (H2), we have $g(0) > 0$ and $g'(x) < 0,$ for $x \ge 0,$ then there exist $K > 0$ such that $g(K) = 0.$ Hence if $x > K,$ $g(x) < 0$ and so $\dot{x}(t) < 0$. This proves the boundedness of $x(t).$

Next, we prove the boundedness of $u(t)$.

Consider the second equation of system (4), we have

$$\begin{aligned}\dot{u}(t) &= u[g(x) - R(u) + d_1 - cp(u) + Q(v)],\\ &\le u[g(0) - R(u) + d_1 - cp(u) + Q(0)],\\ &= u[g(0) - R(u) + d_1 - cp(u) + q'(0)],\end{aligned}$$

Let, $F(u) = g(0) - R(u) + d_1 - cp(u) + q'(0),$

and, $F(0) = g(0) - R(0) + d_1 - cp(0) + q'(0),$

$= g(0) - p'(0) + d_1 + q'(0) > 0,$ Using (A5).

and

$$F(u_2) = g(0) - R(u_2) + d_1 - cp(u_2) + q'(0) = 0,$$

Using existence of $E_2(0,u_2,0)$, $F'(u) = -R'(u) - cp'(u)$.

$= -\{R'(u) + cp'(u)\} < 0$, Using (A4), Hence if $u > u_2$, then $F(u) < 0$ and so $\dot{u}(t) < 0$ proving the boundedness of $u(t)$.

Now we prove the boundedness of $v(t)$. From the third equation of system (4), we have

$$\dot{v}(t) = v[(d_2 - d_1) + cp(u) - Q(v) - dq(v)],$$

$$\dot{v}(t) \le v[(d_2 - d_1) - Q(v) - dq(v)],$$

Let, $G(v) = d_2 - d_1 - Q(v) - dq(v),$

and $G(0) = d_2 - d_1 - Q(0) - dq(0),$

$= d_2 - (d_1 + q'(0)) > 0,$ Using (A5).

By existence of E_3,

$$G(v_3) = d_2 - d_1 - Q(v_3) - dq(v_3) = 0,$$

And $G'(v) = -\{Q'(v) - dq'(v)\} < 0,$ Using (A4),

Now if $v > v_3$, $G(v) < 0$ and so $\dot{v}(t) < 0$. This proves the boundedness of $v(t)$.

Hence each equations of system (4) are bounded and so system (4) is bounded.

4. Persistence

From biological point of view, persistence means the survival of all populations of in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of non-negative cone. Butler et al (1986), Freedman & Waltman (1984&1985) developed following definition of persistence.

Definition: A population $N(t)$ is said to persist (sometimes called strongly persist) if $N(0) > 0 \Rightarrow N(t) > 0$ and $\liminf_{t\to\infty} N(t) > 0$. Further, a population $N(t)$ is said to persist uniformly (also known as permanence) if $N(t)$ persists and there exist $\delta > 0$ independent of $N(0) > 0$, such that $\liminf_{t\to\infty} N(t) \ge \delta$. Finally, we say that a system persists (uniformly) whenever each component persists (uniformly).

Theorem (4.1): Let (H1) - (H4) and (A 1) – (A 5) holds, then the system (4) and hence system (1) persists (does not persist) if E^* exist (does not exist).

Proof: To prove this theorem, we have to show that there are no omega limit points on the axes of orbits initiating in the interior of positive octant.

Let $\theta(x)$ be the orbit through point $X = (x, y, z)$ and $\Omega(x)$ be the omega limit set of the orbit through X and X is the point in the positive octant. Note that, $\Omega(x)$ is bounded.

We claim that E_0 does not belong to $\Omega(x)$. If E_1 exists, then E_0 is completely unstable and so $E_0 \notin \Omega(x)$. Now let E_1 does not exist and $E_0 \in \Omega(x)$, then by Butler-McGehee lemma, there exist a point P in $\Omega(x) \cap W^s(E_0)$, where $W^s(E_0)$ denotes the stable manifold of E_0, since $\theta(P)$ lies in $\Omega(x)$ and $W^s(E_0)$ is the *x*-axis, we conclude that $\theta(P)$ is unbounded. However by theorem (3.1) all orbits are bounded in the positive time and so $\Omega(x)$ is bounded. This contradiction shows that $E_0 \notin \Omega(x)$, in all cases.

Next if E_1 does not belong to $\Omega(x)$, for otherwise, since E_1 is a saddle point which follows from the condition $d_2 > d_1 + q'(0) > 0,$ by Butler-McGehee lemma there exist a point P in $\Omega(x) \cap W^s(E_1)$. Now $W^s(E_1)$ is the *x*-axis implies that an unbounded orbit lies in $\Omega(x)$, which is contrary to the boundedness of the system.

Now we show that $E_2 \notin \Omega(x)$. If $E_2 \in \Omega(x)$ then the condition $g(0) - R(u_2) > 0$ and $R'(u_2) + cp'(u_2) > 0,$ implies that E_2 is a saddle point, $W^s(E_2)$ is the *u*-axis implies that an unbounded orbit lies in $\Omega(x)$, again a contradiction.

Next if $E_3 \notin \Omega(x)$, since E_3 is a saddle point, which follows from the condition $Q'(v_3)+dq'(v_3)>0$, here $W^s(E_3)$ is the v-axis implies that an unbounded orbit lies in $\Omega(x)$, a contradiction.

Next $\overline{E} \notin \Omega(x)$. If $\overline{E} \in \Omega(x)$, since $\overline{E}$ is a saddle point, then by Butler-McGehee lemma, there exist a point P in $\Omega(x) \cap W^s(\overline{E})$, $W^s(\overline{E})$ is x- u plane implies that an unbounded orbit lies in $\Omega(x)$, a contradiction . Similarly, we can show that $\widetilde{E} \notin \Omega(x)$..

Also if E^* exist then $\hat{E}$ is a saddle point and so $\hat{E} \notin \Omega(x)$ as before.

Thus the non persistence when E^* does not exist follows readily, since then $\hat{E}$ is locally asymptotically stable. This completes the proof of the theorem.

5. Numerical Example

We conclude this section with an example to illustrate our results. Consider the system

$$\dot{x}(t) = x(a-bx) - y\left(\frac{a_1\frac{x}{y}}{b_1+\frac{x}{y}}\right), \qquad x(0)=x_0>0,$$

$$\dot{y}(t) = y\left[-d_1 + c\left(\frac{a_1\frac{x}{y}}{b_1+\frac{x}{y}}\right)\right] - z\left(\frac{a_2\frac{y}{z}}{b_2+\frac{y}{z}}\right), \qquad y(0)=y_0>0,$$

$$\dot{z}(t) = z\left[-d_2 + d\left(\frac{a_2\frac{y}{z}}{b_2+\frac{y}{z}}\right)\right], \qquad z(0)=z_0>0. \tag{6}$$

If $u=\frac{x}{y}$ and $v=\frac{y}{z}$, then transformed system becomes,

$$\dot{x}(t) = x\left[(a-bx) - \frac{a_1}{b_1+u}\right], \qquad x(0)=x_0>0,$$

$$\dot{u}(t) = u\left[(a-bx) - \frac{a_1}{b_1+u} + d_1 - \frac{ca_1u}{b_1+u} + \frac{a_2}{b_2+v}\right], \qquad u(0)=\frac{x_0}{y_0}>0,$$

$$\dot{v}(t) = v\left[d_2 - d_1 + \frac{ca_1u}{b_1+u} - \frac{a_2}{b_2+v} - \frac{da_2v}{b_2+v}\right], \qquad v(0)=\frac{y_0}{z_0}>0. \tag{7}$$

Here $g(x)=(a-bx)$, $p(u)=\frac{a_1u}{b_1+u}$, $R(u)=\frac{a_1}{b_1+u}$, $q(v)=\frac{a_2v}{b_2+v}$, and $Q(v)=\frac{a_2}{b_2+v}$.

For assumption (A1) to hold, we will require $u^*>0$ to exist. This is turn requires $a_1c(b_2+v^*)>d_1(b_2+v^*)+a_2$, where

$$u^* = \frac{d_1 b_1 (b_2 + v^*) + a_2 b_1}{a_1 c (b_2 + v^*) - \{d_1 (b_2 + v^*) + a_2\}},$$

For (A2) to hold, we will require $v^* > 0$ exist. i.e. $a_2 d > d_2$ must hold, where

$$v^* = \frac{d_2 b_2}{a_2 d - d_2},$$

For (A3) to be valid, the inequality $a(b_1 + u^*) > a_1$, must hold. Where

$$x^* = \frac{a(b_1 + u^*) - a_1}{b(b_1 + u^*)}.$$

For (A 4) to hold, we also require $b_1 c > 1$ and $db_2 > 1$.

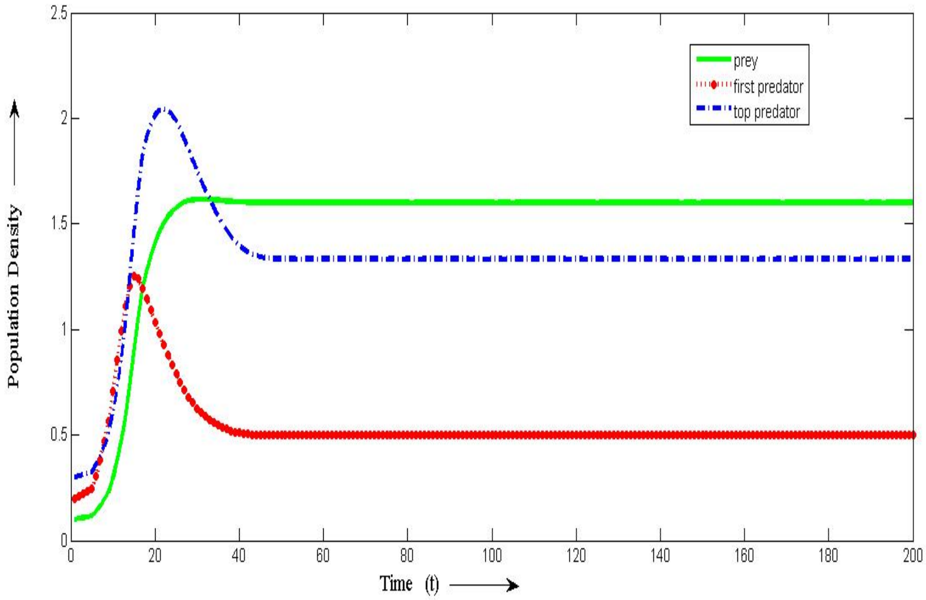
For (A 5) to be valid, the inequalities $ab_1 > a_1$ and $d_2 b_2 - a_2 > d_1 b_2$, must hold.

Numerical simulation and Discussion:

Now, we observe that all the above inequalities hold when,

$$a = 2,\quad b = 1,\quad a_1 = 1,\quad b_1 = 2,\quad a_2 = 1,\quad b_2 = 2,\quad c = 4,\quad d = 5,\quad d_1 = 0.5 \quad \text{and} \quad d_2 = 2\,. \tag{8}$$

We find that all the equilibrium points for the system (7) exist and given by, $E_0(0,0,0)$, $E_1(1,0,0)$, $E_2(0,5,0)$, $E_3(0,0,0.3333)$, $\overline{E}(1.375,0.6,0)$, $\widetilde{E}(1,0,0.3333)$, $\hat{E}(0,1,4)$ and $E^*(1.6,0.5,1.3333)$. For E^* the characteristic equation (5) becomes $\lambda^3 + 3.4\lambda^2 + 3.4336\lambda + 1.2288 = 0$. Roots of this equation are -1.48517313 and $-0.8774134 \pm 0.23984152$, this implies that E^* is locally asymptotically stable equilibrium point. Now, $x^* = 1.6$, $u^* = 0.5$, $v^* = 1.3333$ and hence $x^* = 1.6$, $y^* = 3.2$ and $z^* = 2.4006$. The results of numerical simulation are displayed graphically. In figure (1) the prey, first predator and super predator population are plotted against time, from this figure it is noted that for given initial values both the populations tend to their corresponding value of equilibrium point E^* and hence coexist in the form of stable steady state, assuring the local stability of E^*.



Fig(1), Graph of *x*, *u* and *v* with time.

In figure 2(A), (B) and (C) we have plotted x, u and v against time for different values of conversion rate (c). From these figures it is found that x and u are decreasing functions of c, but v increases with increase in the value of c. Here we also note that $y\left(=\frac{x}{u}\right)$, increases with c and $z\left(=\frac{y}{v}\right)$, decreases with c.

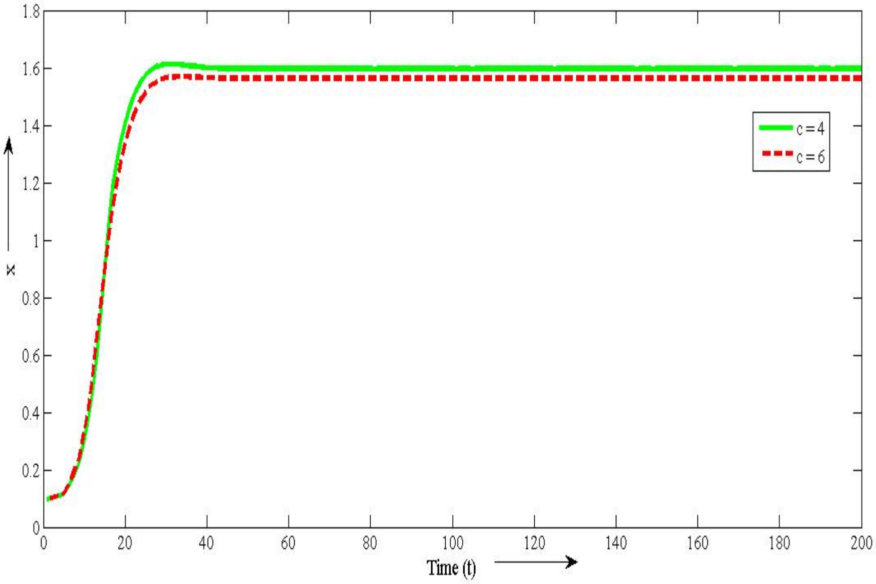


Fig. 2(A): Graph of x verses time t for different values of c and other values of parameters are same as (8).

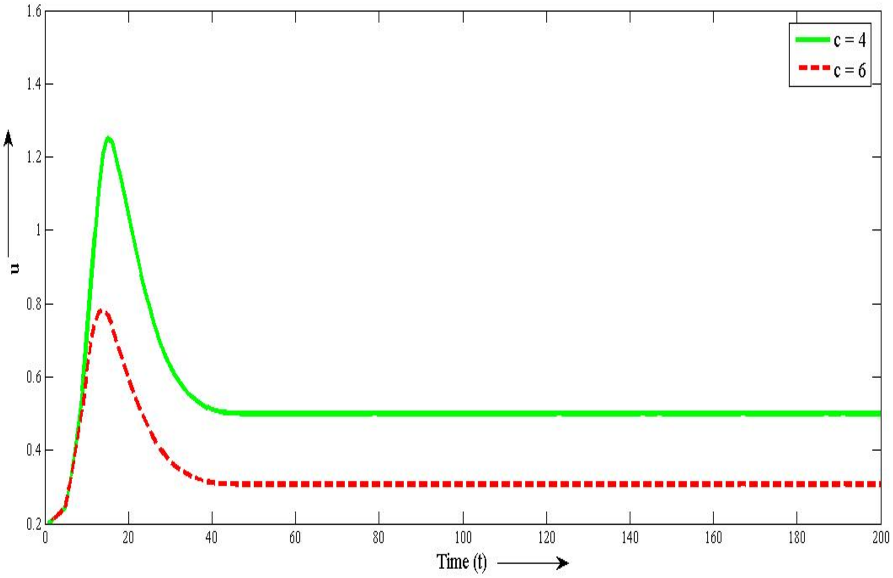


Fig. 2(B): Graph of u verses time t for different values of c and other values of parameters are same as (8).

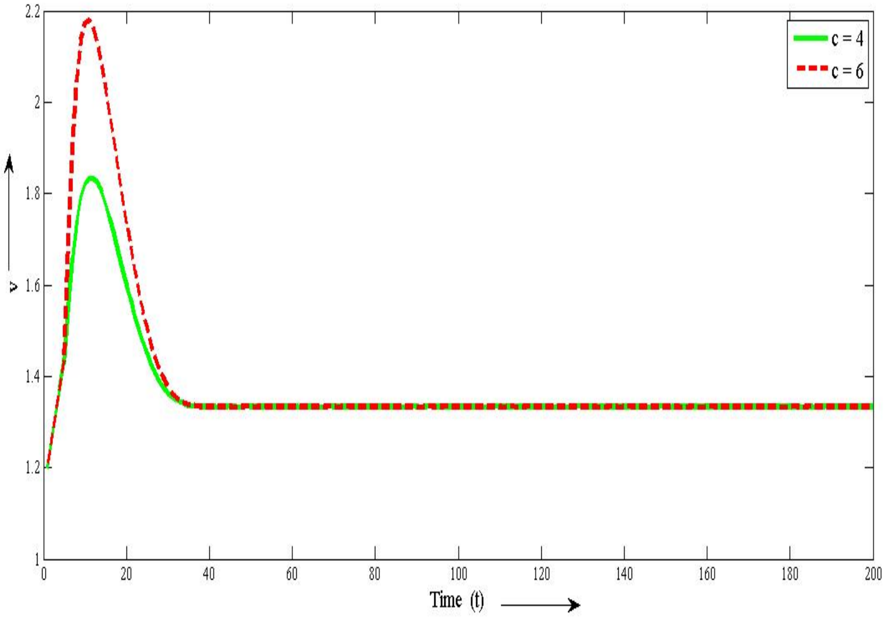


Fig. 2(C): Graph of *v* verses time t for different values of *c* and other values of parameters are same as(8).

Figures 3 – 5 are the plots of *x* versus *u*, *u* versus *v* and *v* versus *x*, respectively for different initial starts 1, 2, 3 and 4. From the graphs, we obtained that solutions converge to equilibrium point E^* for different initial starts indicating the global stability of E^*.

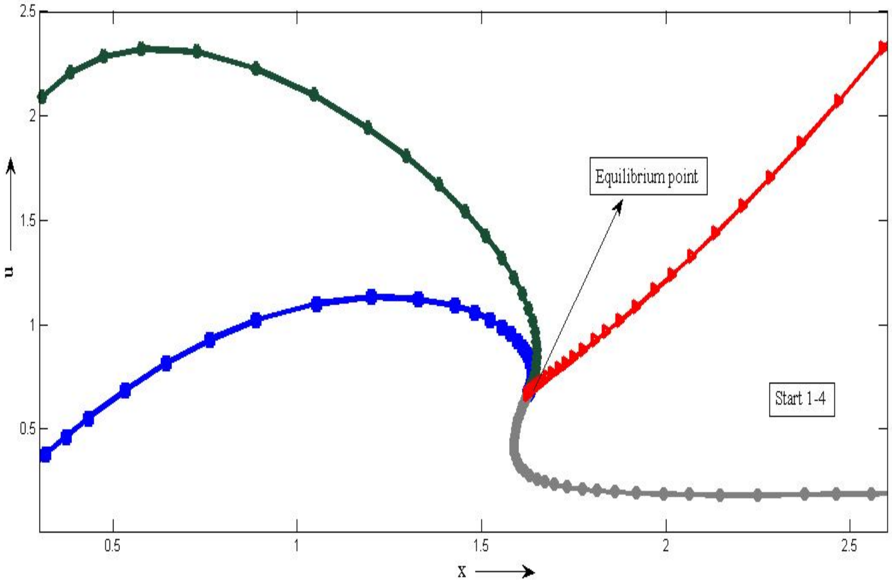


Fig. 3: Graph of *x* verses *u* for different initial starts and other parameters are same as (8).

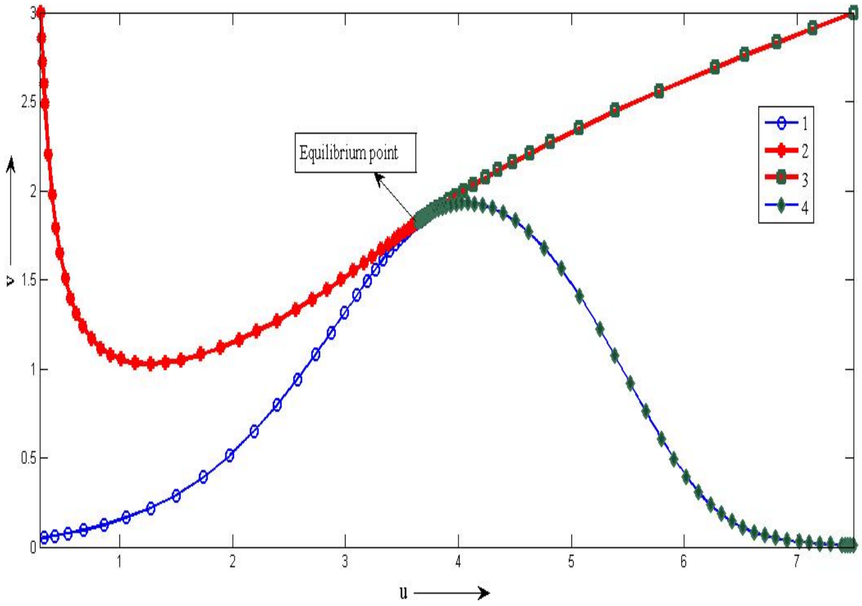


Fig. 4: Graph of *u* verses *v* for different initial starts and other parameters are same as (8).

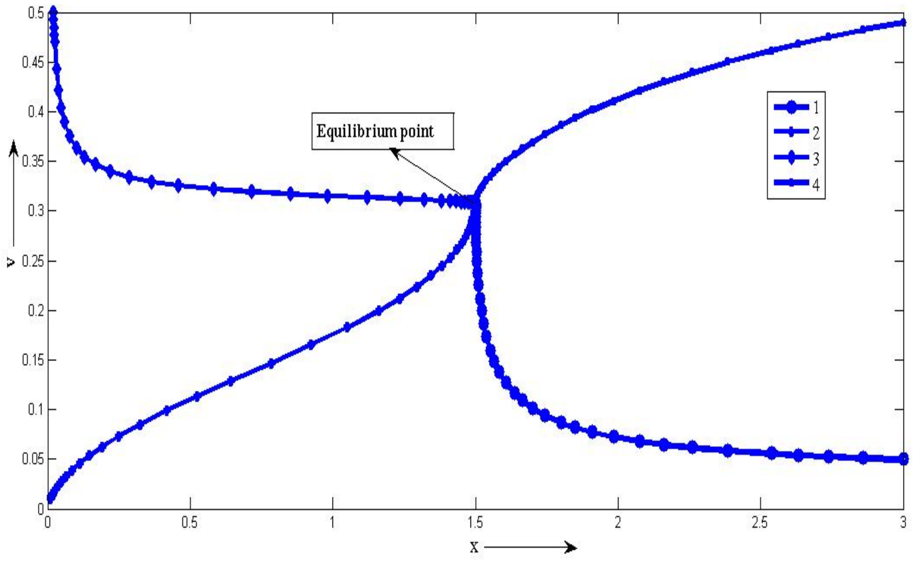


Fig. 5: Graph of *x* verses *v* for different initial starts and other parameters are same as (8).

6. Vanishing Top predator

When the top predator vanishes, then in model (6) only first predator and prey occurs. Consider the system

$$\dot{x}(t) = x(a-bx) - y\left(\frac{a_1\frac{x}{y}}{b_1+\frac{x}{y}}\right), \qquad x(0)=x_0>0,$$

$$\dot{y}(t) = y\left[-d_1 + \left(\frac{ca_1\frac{x}{y}}{b_1+\frac{x}{y}}\right)\right], \qquad y(0)=y_0>0, \tag{9}$$

Which is obtained by deleting *z* from system (6), if we take $u=\frac{x}{y}$, then system (9) is transformed and becomes

$$\dot{x}(t) = x\left[(a-bx) - \frac{a_1}{b_1+u}\right], \qquad x(0)=x_0>0,$$

$$\dot{u}(t) = u\left[(a-bx) - \frac{a_1}{b_1+u} + d_1 - \frac{ca_1u}{b_1+u}\right], \qquad u(0)=\frac{x_0}{y_0}>0. \tag{10}$$

Considering the non-negativeness of the parameter and unknowns, we get four equilibrium points $E_0(0,0)$, $E_1(x_1,0)$, $E_2(0,u_2)$, and $\overline{E}(\overline{x},\overline{u})$. The point $E_0(0,0)$ is always a positive equilibrium point. The second equilibrium point $E_1(x_1,0)$, where $x_1=\frac{ab_1-a_1}{bb_1}$ is non-negative if $ab_1>a_1$. Equilibrium point $E_2(0,u_2)$, with $u_2=\frac{b_1(a+d_1)-a_1}{ca_1-(a+d_1)}$, is positive if $(a+d_1)<ca_1$ and $b_1(a+d_1)>a_1$ or $(a+d_1)>ca_1$ and $b_1(a+d_1)<a_1$. At last the equilibrium point $\overline{E}(\overline{x},\overline{u})$, where $\overline{x}=\frac{a(b_1+\overline{u})-a_1}{b(b_1+\overline{u})}$ and $\overline{u}=\frac{b_1d_1}{ca_1-d_1}$, is non negative if $ca_1>d_1$ and $a(b_1+\overline{u})>a_1$.

Stability of equilibrium point:

The dynamical behaviour of equilibrium points is studied by computation of eigen values of variational matrix $V(E)$ at each equilibrium points,

$$V(E)=\begin{bmatrix} a-2bx-\frac{a_1}{b_1+u} & \frac{xa_1}{(b_1+u)^2} \\ -bu & a-bx+d_1-\frac{a_1}{b_1+u}-\frac{ca_1u}{b_1+u}+\frac{ua_1(1-cb_1)}{(b_1+u)^2} \end{bmatrix}.$$

The equilibrium point $E_0(0,0)$ has variational matrix $V(E_0)$, given by

$$V(E_0)=\begin{bmatrix} a-\frac{a_1}{b_1} & 0 \\ 0 & a+d_1-\frac{a_1}{b_1} \end{bmatrix}.$$

Clearly, $\lambda=\frac{ab_1-a_1}{b_1}$ and $\lambda=\frac{b_1(a+d_1)-a_1}{b_1}$ are positive eigenvalues if $ab_1>a_1$ and $b_1(a+d_1)>a_1$. Therefore equilibrium E_0 is unstable node whenever E_1 exists.

The equilibrium point $E_1(x_1,0)$ has variational matrix $V(E_1)$ given by

$$V(E_1)=\begin{bmatrix} -bx_1 & \dfrac{x_1a_1}{b_1^2} \\ 0 & d_1 \end{bmatrix}.$$

Clearly $\lambda=-bx_1$ is negative and $\lambda=d_1$ is a positive eigenvalue. Hence, equilibrium E_1 is a saddle point.

For the equilibrium point $E_2(0,u_2)$, corresponding variational matrix $V(E_2)$ becomes

$$V(E_2)=\begin{bmatrix} a-\dfrac{a_1}{b_1+u_2} & 0 \\ -bu_2 & \dfrac{u_2a_1(1-cb_1)}{(b_1+u_2)^2} \end{bmatrix}.$$

Clearly $\lambda=\dfrac{a(b_1+u_2)-a_1}{b_1+u_2}$ is positive and $\lambda=\dfrac{u_2a_1(1-cb_1)}{(b_1+u_2)^2}$ is a negative eigenvalue, whenever $cb_1>1$ and $a(b_1+u_2)>a_1$. Hence, equilibrium E_2 is a saddle point.

Now the equilibrium point $\overline{E}(\overline{x},\overline{u})$ has variational matrix $V(\overline{E})$ given by

$$V(\overline{E})=\begin{bmatrix} b\overline{x} & \dfrac{\overline{x}a_1}{(b_1+\overline{u})^2} \\ -b\overline{u} & \dfrac{\overline{u}a_1(1-cb_1)}{(b_1+\overline{u})^2} \end{bmatrix}.$$

The characteristic equation of equilibrium $\overline{E}$ is,

$$\overline{\lambda}^2+\overline{\lambda}\left(b\overline{x}+\frac{\overline{u}a_1(cb_1-1)}{(b_1+\overline{u})^2}\right)+\frac{bca_1b_1\overline{u}\overline{x}}{(b_1+\overline{u})^2}=0,$$ and corresponding eigen values are $\overline{\lambda}_+=\dfrac{-L+\sqrt{M}}{2}$ and $\overline{\lambda}_-=\dfrac{-L-\sqrt{M}}{2}$.

where

$$L=b\overline{x}+\frac{\overline{u}a_1(cb_1-1)}{(b_1+\overline{u})^2} \quad \text{and} \quad M=L^2-\frac{4bca_1b_1\overline{u}\overline{x}}{(b_1+\overline{u})^2}.$$

Here L is positive if $cb_1>1$. Therefore eigenvalues $\overline{\lambda}_\pm$ always has negative real part. Therefore equilibrium $\overline{E}$ is asymptotically stable.

Numerical simulations and discussion:

By using MATLAB software, result of numerical simulation are displayed if we set a parameter as,

$$a=2,\ b=1,\ a_1=1,\ b_1=2,\ c=4 \text{ and } d_1=0.5. \tag{11}$$

Then the coordinates of $\overline{E}(\overline{x},\overline{u})$ becomes (1.5625, 0.2857), therefore $\overline{E}(\overline{x},\overline{y})$ will be (1.5625, 5.4690). Such that eigen values are obtained as $\overline{\lambda}_+=-1.4849$ and $\overline{\lambda}_-=-0.460342$. Which shows that $\overline{E}(\overline{x},\overline{u})$ is asymptotically stable. For initial point, the solution shows stability as seen in fig 6. Now if we increase the value of carrying capacity c then prey population decreases shown in fig 7(A). In figure 7(B) we observe that the predator population with time also decreases with increase of *c*. Which shows that as conversion rate increases the population of prey and predator with time decreases.

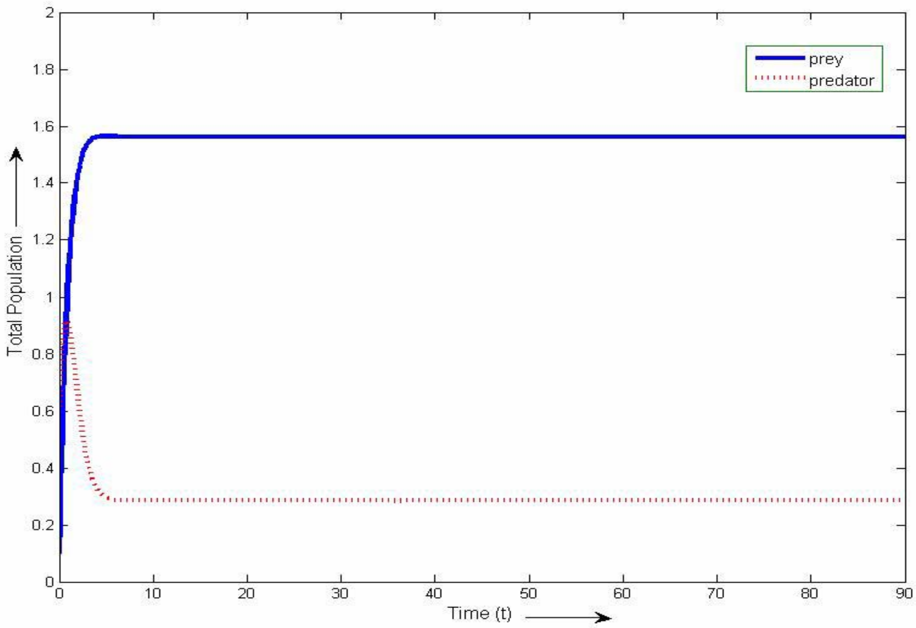


Fig 6: Graph of *x* and *u* with time in the absence of top predator.

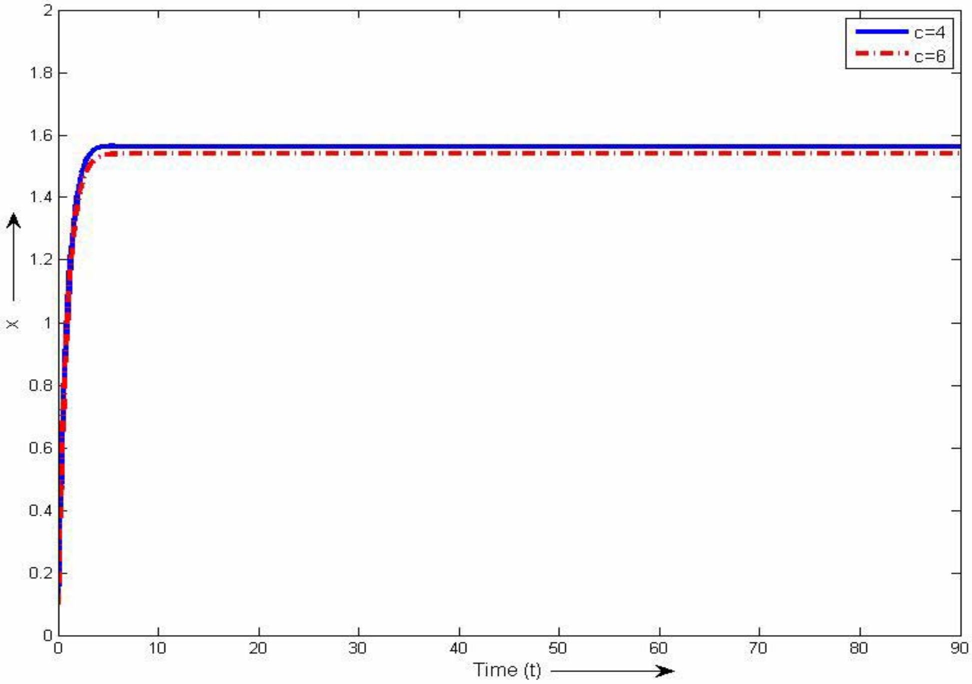


Fig.7(A): Graph of *x* verses time t for different values of *c* and other values of parameters are same as (11)

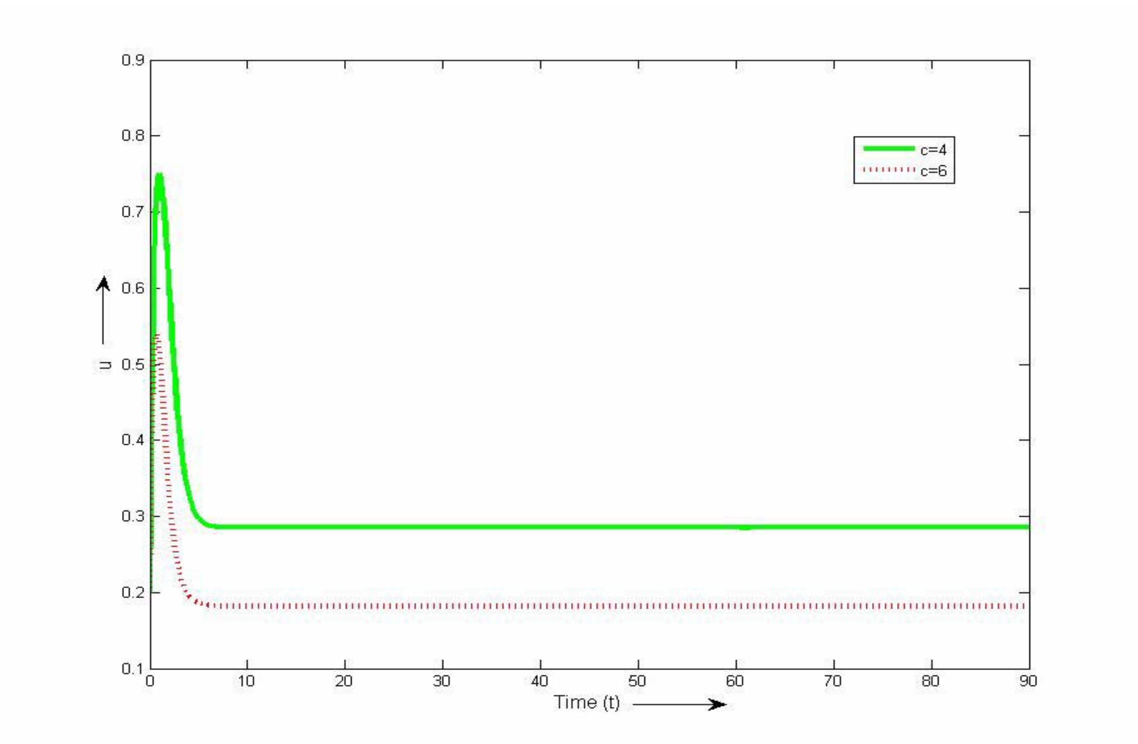


Fig.7(B): Graph of u verses time t for different values of c and other values of parameters are same as (11).

7. Conclusion

The main focus of this paper is to analyze the dynamical behavior of a tri-trophic ratio-dependent food chain model. We describe the above situation by means of a system of three non-linear ordinary differential equations. For solving the system, we transformed our model by means of change of variables. The analysis, after transformation, consisted of equilibrial stability and persistence criteria. It is shown (in theorem (3.1)) that system (1) is bounded, which in turn implies that the system is biologically well behaved, criteria for long time survival (persistence of the populations) of system is interpreted biologically and obtained that system persists if interior equilibrium point exist otherwise the system does not persists. We have obtained the stability of the most important equilibrium point E^* by using Routh - Hurwitz criteria, which gives the conditions for stable co-existence of the prey, the predator and the super predator. In the case of absence of top predator, the existence of equilibrium point and the stability of the solutions are studied and it is observed that when the top predator is absent, predator(y) and prey(x) coexist in the form of stable equilibrium. Our mathematical model (1) is biologically implemented in the case of plant – pest (e.g. caterpillar) – beneficial predator (natural enemy of the pest). We observe that the size of plant(x) in the absence and presence of beneficial predator (z) are $\bar{x}$ and x^* respectively, so that $x^* - \bar{x} > 0.$ Also $\bar{y} - y^* < 0,$ where $\bar{y}$ and y^* are the size of pest population in absence and presence of beneficial predator (z). This implies that due to attack of predators on pests, fitness of plants is enhanced and cause depression for the pests. This gives a strong theoretical support to the approach of ‘Bio control of pest’ to reduce the hazards of chemical pesticides. All mathematical findings are checked numerically by using MATLAB programming.

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