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Strong convergence to common fixed points of a finite family of Z-operators in normed spaces

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Abstract

In this paper, we consider an implicit iteration process for approximating common fixed points of a finite family of Z-operators and we prove strong convergence theorem for such mappings in normed spaces. Also, we give a few corollaries and conclusions for same mappings. Our process contains implicit iteration processes of Xu and Ori, Zhao et.al and Rafiq. Our results generalize and improve some results in contemporary literature.

Keywords: Implicit iteration process, Common fixed point, Strong convergence.
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1. Introduction and Preliminaries

We recall some definitions in a metric space (X,d). A mapping $T: X \to X$ is called an a-contraction if

$$d(Tx,Ty) \le ad(x,y), \qquad \forall\, x,y \in X \tag{1}$$

where $a \in (0,1)$.

The map T is called Kannan mapping (Kannan, 1968) if there exists $b \in \left(0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le b\left[d(x,Tx)+d(y,Ty)\right], \qquad \forall\, x,y \in X. \tag{2}$$

A similar definition is due to Chatterjea (Chatterjea 1972): there exists $c \in \left(0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le c\left[d(x,Ty)+d(y,Tx)\right], \qquad \forall\, x,y \in X. \tag{3}$$

Combining these three definitions, Zamfirescu proved the following important result (Zamfirescu, 1972).

Theorem 1.1. Let (X,d) be a complete metric space and $T: X \to X$ a mapping for which there exists the real numbers a,b and c satisfying $0 < a < 1$, $0 < b, c < 1/2$ such that for each pair x, y in X, at least one of the following conditions holds:
(z_1) $d(Tx,Ty) \le ad(x,y)$;
(z_2) $d(Tx,Ty) \le b\left[d(x,Tx)+d(y,Ty)\right]$;
(z_3) $d(Tx,Ty) \le c\left[d(x,Ty)+d(y,Tx)\right]$.
Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \qquad n \in \mathrm{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

An operator T satisfying the contractive conditions $(z_1)-(z_3)$ in the above theorem is called Zamfirescu operator (alternatively, we shall say that T satisfies condition Z, see (Rhoades, 1974).

In 2001, Xu and Ori introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in \mathrm{I}\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1 , \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2 , \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N , \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1 x_{N+1} , \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n \qquad \forall n \ge 1, \tag{4}$$

where $T_n = T_{n(mod\ N)}$ (here the $mod\ N$ function takes values in I) (Xu and Ori, 2001). They proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space.

Later on, Chidume and Shahzad studied the strong convergence of the this implicit process to a common fixed point for a finite family of nonexpansive mappings (Chidume and Shahzad, 2005).

Zhou and Chang introduced the convergence of modified implicit iteration process for a finite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces (Zhou and Chang, 2002). In 2006, Rafiq studied the following implicit iteration process for strong convergence to a common fixed point for a finite family of Z-operators in normed spaces,

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n + u_n \qquad \forall n \ge 1, \tag{5}$$

where $T_n = T_{n(mod\ N)}$, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{u_n\}$ is summable sequence in K (Rafiq, 2006). He proved the following results.

Theorem 1.2. Let K be a nonempty closed convex subset of a normed space E. Let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N Z-operators with $F = \bigcap_{i=1}^{N} F(T_i) \ne \emptyset$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (5) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\|u_n\| = 0(1-\alpha_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Theorem 1.3. Let K be a nonempty closed convex subset of a normed space E. Let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N operators satisfying condition Z with $F = \bigcap_{i=1}^{N} F(T_i) \ne \emptyset$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (4) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Recently, Zhao et.al [13] introduced the following two implicit iteration schemes as follows:

$$x_n = \alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n , \qquad \forall n \ge 1, \tag{6}$$

for fixed points of nonexpansive mapping T in Banach space. And, the other iteration scheme was introduced

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n , \qquad \forall n \ge 1. \tag{7}$$

where $T_n = T_{n(mod\ N)}$, for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{N}$ in Banach spaces. They also proved some strong convergence theorems for such mappings.

Motivated by the above works, in this paper, we introduce the following implicit iteration process for approximating the common fixed points of $\{T_1, T_2, \ldots, T_N\}$.

Let E be a normed space, K a nonempty closed convex subset of E and $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N Z-operators. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, we have the following iteration process: for arbitrary chosen $x_0 \in K$,

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n + u_n, \qquad \forall n \ge 1 \tag{8}$$

where $T_n = T_{n(mod\ N)}$ and $\{u_n\}$ is summable sequence in K.

Remaks 1.4. (1) If $\beta_n = 0$, then we see that (8) reduces to (5). Also, this iteration process contains the process (4) as its special case.

(2) The implicit iteration process (8) are a generalization of the implicit iteration processes (6) and (7).

The purpose of this paper is to study the strong convergence of implicit iteration process (8) to a common fixed point for a finite family of Z-operators in normed spaces. Our results improve and extend the corresponding results of Rafiq [6].

We need the following lemma in order to prove our main result.

Lemma 1.5. Let $\{r_n\},\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative numbers satisfying

$$r_{n+1} \le (1-s_n)r_n + s_n t_n \qquad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$ and $lim_{n\to\infty} t_n = 0$, then $lim_{n\to\infty} r_n = 0$ (Chang, 1997).

2. Main results

Theorem 2.1. Let K be a nonempty closed convex subset of a normed space E. Let $\{T_1, T_2, \ldots, T_N\}: K \to K$ be N Z-operators with $F = \bigcap_{i=1}^{N} F(T_i) \ne \varnothing$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (8) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\|u_n\| = 0(1-\alpha_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Since $F = \bigcap_{i=1}^{N} F(T_i) \ne \varnothing$, then Z-operators $\{T_1, T_2, \ldots, T_N\}$ have a common fixed point in K. We suppose that $p \in F$. From each T_i is a Z-operator for all $i \in \mathrm{I}$, at least one of the conditions $(z_1),(z_2)$ and (z_3) is satisfied. If (z_2) holds, then for $x, y \in K$

$$\begin{aligned}\|T_i x - T_i y\| &\le b\left[\|x - T_i x\| + \|y - T_i y\|\right] \\ &\le b\left[\|x - T_i x\| + \|y - x\| + \|x - T_i x\| + \|T_i x - T_i y\|\right]\end{aligned} \tag{9}$$

implies

$$(1-b)\|T_i x - T_i y\| \le b\|x - y\| + 2b\|x - T_i x\|. \tag{10}$$

From $0 \le b < 1/2$ we obtain

$$\|T_i x - T_i y\| \le \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - T_i x\|. \tag{11}$$

If (z_3) holds, then similarly we obtain

$$\|T_i x - T_i y\| \le \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - T_i x\|. \tag{12}$$

Denote

$$\delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{13}$$

Then we have $0 \le \delta < 1$ and in view of (z_1), (11)-(13) it results that the inequality

$$\|T_i x - T_i y\| \le \delta\|x - y\| + 2\delta\|x - T_i x\| \tag{14}$$

holds for all $x, y \in K.$

By using (8), we have

$$\begin{aligned}\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n + u_n - p\| \\ &\le \alpha_n\|x_{n-1} - p\| + \beta_n\|T_n x_{n-1} - p\| + \gamma_n\|T_n x_n - p\| + \|u_n\|\end{aligned} \tag{15}$$

Using (14) with $y = x_{n-1}$ and $x = p$, we get

$$\|T_n x_{n-1} - p\| \le \delta\|x_{n-1} - p\|, \tag{16}$$

and also, for $y = x_n$ by using the same method, we have

$$\|T_n x_n - p\| \le \delta \|x_n - p\|. \tag{17}$$

Substituting (16) and (17) into (15), we obtain

$$\|x_n - p\| \le \alpha_n \|x_{n-1} - p\| + \beta_n \delta \|x_{n-1} - p\| + \gamma_n \delta \|x_n - p\| + \|u_n\|$$

which leads to

$$(1-\gamma_n\delta)\|x_n - p\| \le (\alpha_n + \beta_n\delta)\|x_{n-1} - p\| + \|u_n\|.$$

Thus, it is implies that

$$\|x_n - p\| \le \frac{\alpha_n + \beta_n\delta}{1-\gamma_n\delta}\|x_{n-1} - p\| + \frac{1}{1-\gamma_n\delta}\|u_n\|. \tag{18}$$

Let

$$A_n = \alpha_n + \beta_n\delta$$
$$B_n = 1-\gamma_n\delta,$$

and consider

$$\begin{aligned}\theta_n &= 1-\frac{A_n}{B_n} = 1-\frac{\alpha_n+\beta_n\delta}{1-\gamma_n\delta} = \frac{1-\gamma_n\delta-\alpha_n-\beta_n\delta}{1-\gamma_n\delta}\\ &= \frac{(1-\alpha_n)-\delta(\beta_n+\gamma_n)}{1-\gamma_n\delta} = \frac{(1-\alpha_n)(1-\delta)}{1-\gamma_n\delta}\\ &\ge (1-\delta)(1-\alpha_n).\end{aligned}$$

Since

$$1-\delta \le 1-\gamma_n\delta \le 1,$$

so we have

$$\frac{A_n}{B_n} \le 1-(1-\delta)(1-\alpha_n).$$

It follows from (18) that

$$\|x_n - p\| \le \left[1-(1-\delta)(1-\alpha_n)\right]\|x_{n-1} - p\| + \frac{1}{1-\delta}\|u_n\|. \tag{19}$$

From (19) and Lemma 1.5, we have

$$\lim_{n\to\infty}\|x_n - p\| = 0.$$

That is, $x_n \to p \in F$ and this completes the proof.

Theorem 2.1 lead to the following Corollary:

Corollary 2.2. Let K be a nonempty closed convex subset of a normed space E. Let $\{T_1, T_2, \ldots, T_N\}: K \to K$ be N operators satisfying condition Z with $F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (7) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

If $N = 1$, implicit iteration process (7) reduces to (6). And, we obtain that the Corollary as follows:

Corollary 2.3. Let K be a nonempty closed convex subset of a normed space E. Let $T: K \to K$ be Z-operator with $F = F(T) \neq \varnothing$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (6) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

3. Conclusions

The following are the conclusions that could be drawn from the study:

(1) The contractive condition (1) makes T a continuous condition on X while this is not the case with the contractive

conditions (2)-(3) and (14).

(2) Our results generalize Theorem 1.2 and Theorem 1.3 of Rafiq (Rafiq, 2006).

(3) Chatterjea's and Kannan's contractive conditions (2) and (3) are both included in the class of Zamfirescu operators.

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