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Analytical solution of one dimensional temporally dependent advection-dispersion equation in homogeneous porous media

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Abstract

In present paper, advection-dispersion equation is considered one dimensional longitudinal semi-infinite domain. The solute dispersion parameter is considered temporally dependent and flow velocity is considered uniform. Nature of pollutant and porous medium are considered chemically non-reactive. The first order decay term which is inversely proportional to the dispersion coefficient is also considered. Initially the porous domain is considered solute free. Analytical solutions are obtained by using Laplace transform technique for continuous uniform and increasing input source concentration.

Keywords: Advection; dispersion; first order decay; groundwater; pollutant.

1. Introduction

Advection-dispersion equation is applicable in many disciplines like groundwater hydrology, chemical engineering bio sciences, environmental sciences and petroleum engineering. It helps understand the contaminant or pollutants concentration distribution behavior through an open medium like air, rivers, lakes or porous medium like aquifers, underground oil reservoirs. A list of previous work in which analytical solutions have been obtained for advection-dispersion equation by Bastian and Lapidus (1956), Banks and Ali (1964), Ogata (1970), Marino (1974) and Al-Niami and Rushton (1977). In most of works, porous parameter s are taken adsorption, first order decay, zero order production. Such solutions have been compiled by Lindstrom and Boersma (1989). Coming nearer to real problems, Lin (1977), considered the layered porous media and non linear adsorption, Banks and Jerasate (1962), Kumar (1983) considered the porous media flow unsteady/ non-uniform flow in homogeneous domain.

van Genuchten and Alves (1982) complied the analytical solutions with first order decay and zero order production term. Yates (1990, 1992) obtain the analytical solution for one dimensional advective dispersion equation with linearly or exponentially increasing dispersion coefficient. Toride et al. (1993) presented a comprehensive set of analytical solutions for one-dimensional non-equilibrium solute transport through semi-infinite soil systems. The models involve the one-site, two-site, and two-region transport models, and include provisions for first-order decay and zero-order production. Zhuo and Selim (2002) proposed two methods for describing the scale dependent dispersivity. Pang et al. (2003) obtained the temporal moment solutions for one–dimensional advection–dispersion solute transport with linear equilibrium sorption and first-order degradation for time pulse sources to analyze soil column experimental data. Su et al. (2005) presented an analytical solution to advection-dispersion equation with spatially and temporally varying dispersion coefficient for predicting solute transport in a steady, saturated sub-surface flow through homogeneous porous media. De Smedt (2006) presented analytical solutions for solute transport in rivers including the effects of transient storage and first order decay. Advection-dispersion phenomena occur in many physical situations including the transfer of heat in fluids, flow through porous media, and the spread of contaminants in fluids and in chemical separation processes (Najafi and Hajinezhad, 2008). Jaiswal et al. (2009) and Kumar et al. (2010) obtained analytical solutions for temporally and spatially dependent solute dispersion in one dimensional semi-infinite media.

Decay coefficient, which represent the production or decay of solute concentration within the porous medium. The decay coefficient is the rate coefficient that represents increasing concentration when it is negative and decreasing when it is positive (Hurst, 1991). In present paper, advection-dispersion equation is considered one dimensional longitudinal initially solute free semi-infinite domain. The solute dispersion parameter is considered temporally dependent and flow velocity is considered uniform. The first order decay term, the value of the quantity is dependent upon the concentration in the solutions which is inversely proportional to the dispersion coefficient is also considered. The input condition is assumed at the origin of the domain. The second condition is considered at the end of the domain.

2. Advection-dispersion equation

The linear advection-dispersion equation with first order decay in one dimension may be written as follows

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x} - u(x,t)c \right) - \gamma c \tag{1}$$

Let us write $D(x,t) = D_0 f_1(x,t)$ and $u(x,t) = u_0 f_2(x,t)$ and the first order decay term which is inversely proportional to the dispersion coefficient i.e., $\gamma(x,t) = \gamma_0 / f_1(x,t)$ in eq. (1), we may get

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f_1(x,t)\frac{\partial c}{\partial x} - u_0 f_2(x,t)c \right) - \gamma_0 c / f_1(x,t) \tag{2}$$

where D_0, u_0 and γ_0 are constants.

Let us introduce a new independent variable X by a transformation (Jaiswal et al., 2009; Kumar et al., 2010)

$$X = \int \frac{dx}{f_1(x,t)} \quad \text{or} \quad \frac{dX}{dx} = \frac{1}{f_1(x,t)} \tag{3}$$

Eq. (2) becomes

$$f_1(x,t)\frac{\partial c}{\partial t} = \frac{\partial}{\partial X}\left(D_0 \frac{\partial c}{\partial X} - u_0 f_2(x,t)c \right) - \gamma_0 c \tag{4}$$

3. Unsteady dispersion along uniform flow

Let $f_1(x,t) = f(mt)$ and $f_2(x,t) = 1$ where m is a resistive coefficient whose dimension is inverse of that the time variable t. $f(mt)$ is chosen such that for $m = 0$ or $t = 0$, $f(mt) = 1$. Thus $f(mt)$ is an expression in non-dimensional variable (mt). Then from eq. (3), we have

$$X = \frac{x}{f(mt)} \tag{5}$$

Eq. (4) becomes

$$f(mt)\frac{\partial c}{\partial t} = \frac{\partial}{\partial X}\left(D_0 \frac{\partial c}{\partial X} - u_0 c \right) - \gamma_0 c \tag{6}$$

Let us introduce a new time variable using the following transformation (Crank, 1975),

$$T = \int_0^t \frac{dt}{f(mt)} \tag{7}$$

The partial differential eq. (6) reduces into that with constant coefficients as

$$\frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial c}{\partial X} - \gamma_0 c \tag{8}$$

3.1. Uniform input condition

Let the domain is initially solute free. Thus initial and boundary conditions for eq. (1) in a semi-infinite longitudinal domain are as follows:

$$c(x,t) = 0,\ x \ge 0,\ t = 0 \tag{9}$$

$$c(x,t) = C_0,\ x = 0,\ t > 0 \tag{10a}$$

$$c(x,t) = 0,\ x \to \infty,\ t \ge 0 \tag{10b}$$

These conditions in terms of new space and time variable may be written as

$$c(X,T) = 0,\ X \ge 0,\ T = 0 \tag{11}$$

$$c(X,T) = C_0,\ X = 0,\ T > 0 \tag{12a}$$

$$c(X,T)=0\,,\ X\to\infty\,,\ T\geq 0 \tag{12b}$$

Now introducing a new dependent variable by following transformation

$$c(X,T)=K(X,T)exp\left\{\frac{u_0}{2D_0}X-\left(\frac{u_0^2}{4D_0}+\gamma_0\right)T\right\} \tag{13}$$

the set of eqs. (8), (11) and (12) reduced into

$$\frac{\partial K}{\partial T}=D_0\frac{\partial^2 K}{\partial X^2} \tag{14}$$

$$K(X,T)=0\,,\ X\geq 0\ ,\ T=0 \tag{15}$$

$$K(X,T)=C_0 exp(\alpha^2 T)\,,\ X=0\,,\ \ T>0\,,\ \alpha^2=\left(\frac{u_0^2}{4D_0}+\gamma_0\right) \tag{16a}$$

$$K(X,T)=0\,,\ X\to\infty\,,\ T\geq 0 \tag{16b}$$

Applying Laplace transformation on eqs. (14) – (16), we have

$$p\overline{K}=D_0\frac{d^2\overline{K}}{dX^2} \tag{17}$$

$$\overline{K}(X,p)=\frac{C_0}{(p-\alpha^2)}\,,\ X=0 \tag{18a}$$

$$\overline{K}(X,p)=0\ ,\ X\to\infty \tag{18b}$$

Thus the general solution of eq. (17) may be written as

$$\overline{K}(X,p)=C_1 exp\left(-X\sqrt{p/D_0}\right)+C_2 exp\left(X\sqrt{p/D_0}\right) \tag{19}$$

Using condition (18a,b) on the above solution we get, $C_1=\dfrac{C_0}{(p-\alpha^2)}$ and $C_2=0$.

Thus the particular solution in the Laplacian domain may be written as

$$\overline{K}(X,p)=\frac{C_0}{(p-\alpha^2)}exp\left(-X\sqrt{p/D_0}\right) \tag{20}$$

Taking inverse Laplace transform, the solution of advection-dispersion solute transport for continuous uniform input condition may be written in terms of $c(x,T)$ by using transformations (13), (7) and (5) as,

$$c(x,T)=\frac{C_0}{2}\left[exp\left\{\frac{\beta-\left(\beta^2+\gamma_0\right)^{1/2}x}{f(mt)\sqrt{D_0}}\right\}erfc\left\{\frac{\frac{x}{f(mt)}-\left(u_0^2+4\gamma_0 D_0\right)^{1/2}T}{2\sqrt{D_0 T}}\right\}\right]$$
$$+\frac{C_0}{2}\left[exp\left\{\frac{\beta+\left(\beta^2+\gamma_0\right)^{1/2}x}{f(mt)\sqrt{D_0}}\right\}erfc\left\{\frac{\frac{x}{f(mt)}+\left(u_0^2+4\gamma_0 D_0\right)^{1/2}T}{2\sqrt{D_0 T}}\right\}\right], \tag{21}$$

where $\beta^2=\dfrac{u_0^2}{4D_0}$ and $T=\displaystyle\int_0^t\frac{dt}{f(mt)}$.

3.2. Input condition of increasing nature

The pollutant concentration may not be uniform. It may increases due to human and other responsible activities. This type of condition is taken to be of flux type or mixed type i.e.,

$$-D(x,t)\frac{\partial c}{\partial x}+u(x,t)c=u_0C_0\ ,\ x=0\,,\ t>0 \tag{22}$$

Eq. (22) reduces by applying the previous transformations, into

$$-D_0\frac{\partial K}{\partial X}+\frac{u_0}{2}K=u_0C_0 exp(\alpha^2 T)\ ,\ X=0\,,\ T>0 \tag{23}$$

Applying Laplace Transformation on eq. (23), we may get

$$-D_0\frac{\partial\overline{K}}{\partial X}+\frac{u_0}{2}\overline{K}=\frac{u_0C_0}{(p-\alpha^2)}\ ,\ X=0 \tag{24}$$

Now using input condition (24) in place of (18a) in general solution (19), we get

$$C_1=\frac{u_0C_0}{\sqrt{D_0}\,(p-\alpha^2)(\sqrt{p}+\beta)} \tag{25}$$

Thus the particular solution in the Laplacian domain may be written as

$$\overline{K}(X,p)=\frac{u_0C_0}{\sqrt{D_0}\,(p-\alpha^2)(\sqrt{p}+\beta)}exp\left(-X\sqrt{p/D_0}\right) \tag{26}$$

Taking inverse Laplace transform, the solution of advection-dispersion solute transport for varying input condition may be written in terms of $c(x,T)$ by using transformations (13), (7) and (5) as,

$$c(x,T)=\frac{u_0C_0}{2\sqrt{D_0}\left\{\beta+\left(\beta^2+\gamma_0\right)^{1/2}\right\}}exp\left\{\frac{\beta-\left(\beta^2+\gamma_0\right)^{1/2}x}{f(mt)\sqrt{D_0}}\right\}erfc\left\{\frac{\frac{x}{f(mt)}-\left(u_0^2+4\gamma_0D_0\right)^{1/2}T}{2\sqrt{D_0T}}\right\}$$
$$+\frac{u_0C_0}{2\sqrt{D_0}\left\{\beta-\left(\beta^2+\gamma_0\right)^{1/2}\right\}}exp\left\{\frac{\beta+\left(\beta^2+\gamma_0\right)^{1/2}x}{f(mt)\sqrt{D_0}}\right\}erfc\left\{\frac{\frac{x}{f(mt)}+\left(u_0^2+4\gamma_0D_0\right)^{1/2}T}{2\sqrt{D_0T}}\right\}$$
$$+\frac{u_0^2C_0}{2\gamma_0D_0}exp\left\{\frac{u_0x}{D_0f(mt)}-\gamma_0T\right\}erfc\left\{\frac{\frac{x}{f(mt)}+u_0T}{2\sqrt{D_0T}}\right\}, \tag{27}$$

where $\beta^2=\frac{u_0^2}{4D_0}$ and $T=\int_0^t\frac{dt}{f(mt)}$.

4. Illustration and discussion

The concentration values are evaluated from the analytical solutions described by eq. (21) for uniform input and eq. (27) for varying input in a finite domain $0\le x\le 10$ (m). The other input values are considered as: $C_0=1.0$, $D_0=1.25$ (m^2/day), $u_0=1.15$ (m/day). In addition to these, $m=0.1$ (day^{-1}), $\gamma_0=0.04$ have been considered. The solutions are computed at times $t=2.5$ (day), 3.0 and 3.5. In Figure 1, curves represent the solution for an expression $f(mt)=(1+mt)^2$ which is of increasing nature.

In Figure 1, the concentration values are uniform at different time which is shows the condition of problem.

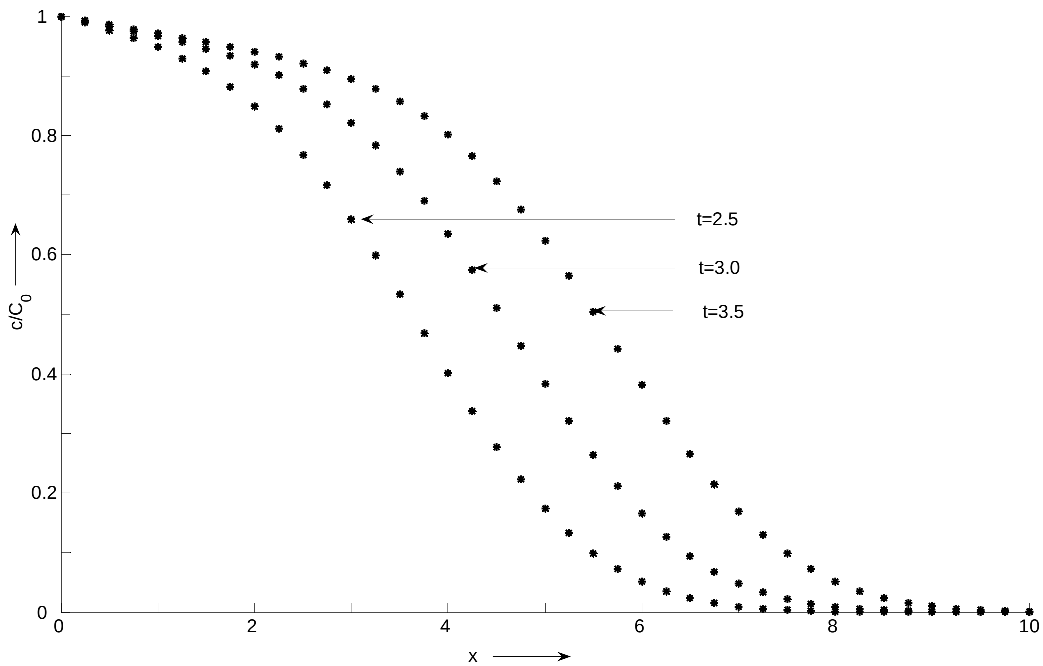

Figure 1. Distribution of solute concentrations of $f(mt) = (1+mt)^2$ at time $t = 2.5$, 3.0 , 3.5 for solution (21).

Comparison have been done between concentration distribution patterns for an decreasing $f(mt) = exp(-mt)$ and increasing $f(mt) = (1+mt)^2$, at $t = 3.0$ (day), in Figure 2.

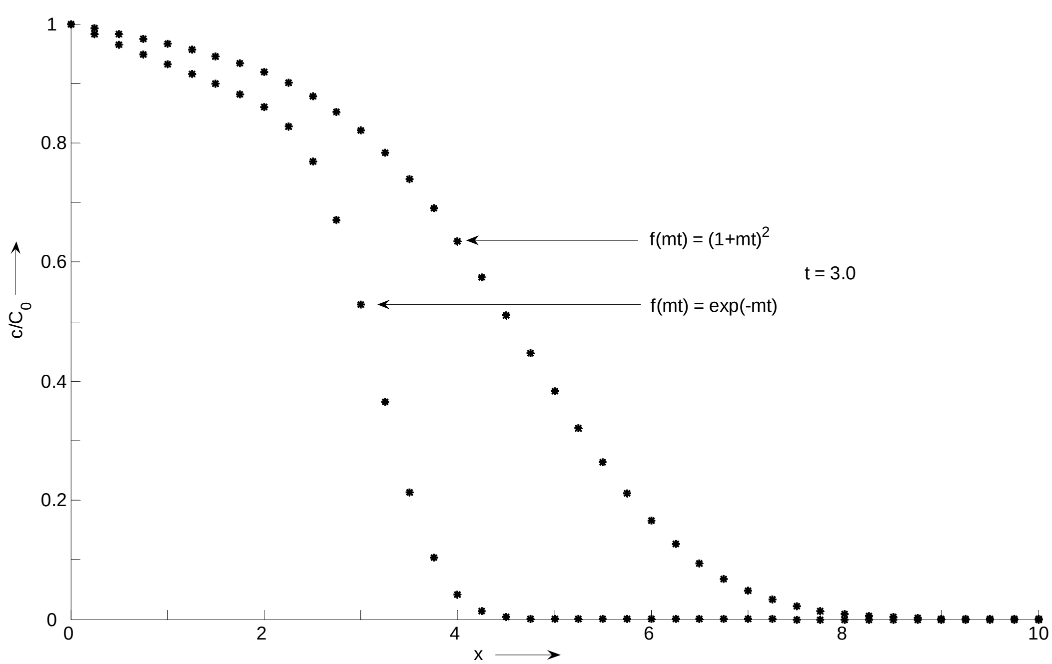

Figure 2. Comparison of solute concentration of increasing and decreasing function at time $t = 3.0$ for solution (21)

For increasing input concentration, the Figure 3 is drawn for same input data and increasing function of time. In Figure 3, the concentration values are increases, when the time increases.

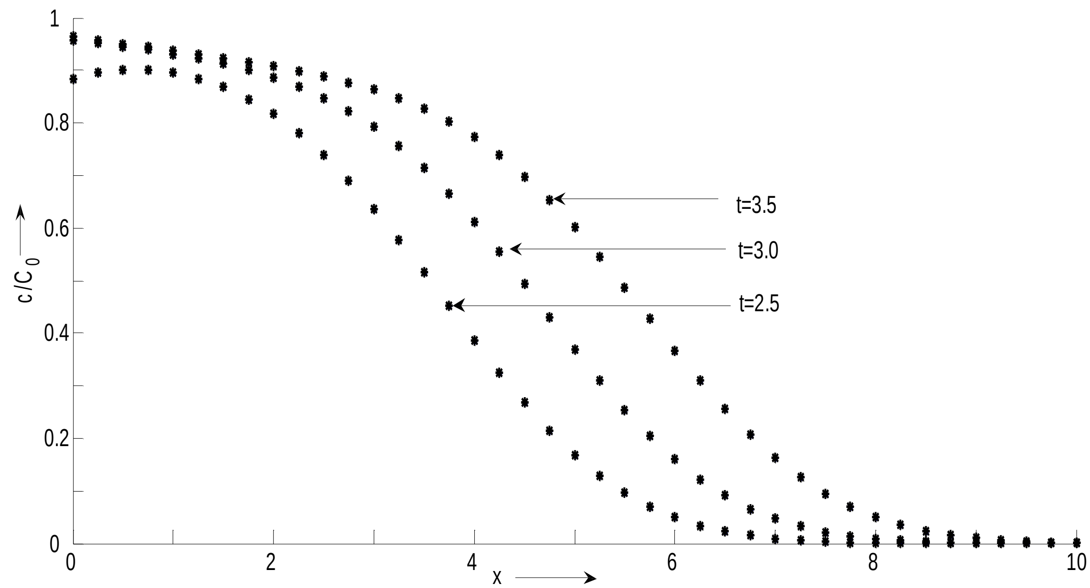

Figure 3. Distribution of solute concentrations of $f(mt) = (1+mt)^2$ at time $t = 2.5$, 3.0, 3.5 for solution (27).

Comparison have been done between concentration distribution patterns for an decreasing $f(mt) = exp(-mt)$ and increasing $f(mt) = (1+mt)^2$, at $t = 3.0$ (day), in Figure 4.

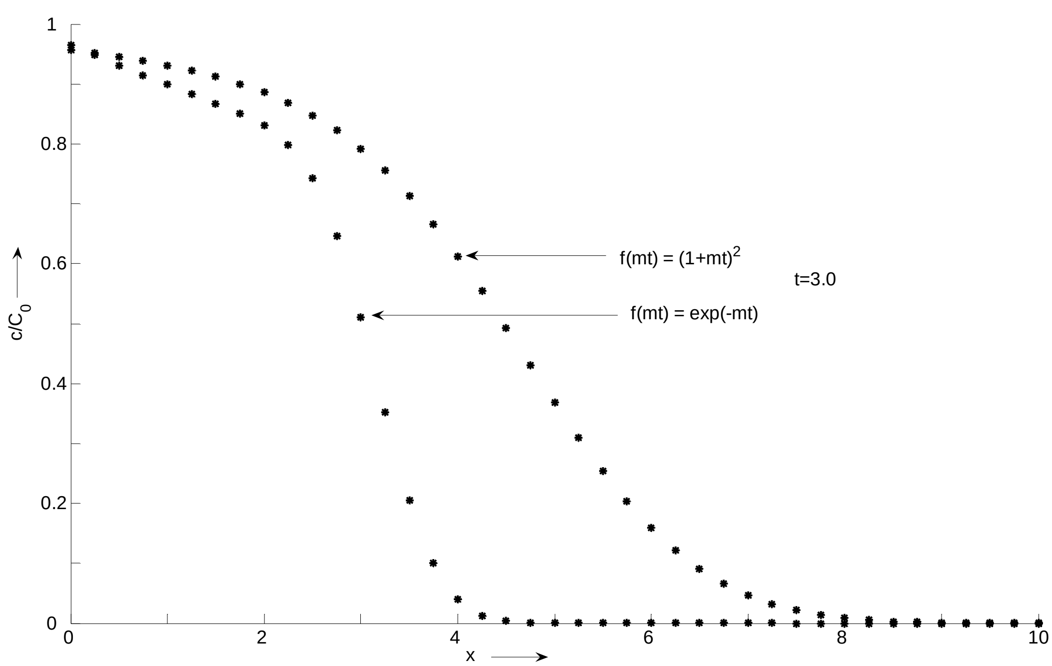

Figure 4. Comparison of solute concentration of increasing and decreasing function at time $t = 3.0$ for solution (27).

Solute concentrations are higher for increasing function than decreasing function at same time and decreasing function rapidly decreases correspond to increasing function at same position in Figures 2 and 4. The values are coincided far from the origin in all figures. Since our assumption is decay term is inversely proportional to dispersion coefficient which correlate the accepted concept that the decay term vary with dispersion coefficient, i.e., when dispersion increases then decay term decreases and vice-versa. The time dependent behavior of solutes in subsurface is of interest for many practical problems where the concentration is observed or needs to be predicted at fixed positions. Problems of solute transport involving sequential first order decay reactions frequently occurs in soil and groundwater systems, for example the migration of simultaneous movement of interacting nitrogen species, organic phosphate transport and the transport of pesticides and their metabolites. The accuracy of the numerical method is validated by direct comparisons with the analytical results given in eqs. (21) and (27).

5. Conclusion

Advection-dispersion equation is considered one dimensional semi-infinite domain. The solute dispersion parameter is considered temporally dependent with uniform velocity. First order decay term is also considered. Analytical solutions are obtained for uniform and increasing input source. At the origin of the domain the source concentration is uniform. For uniform input, concentrations values are increasing for different time at same position. The solution of the problem may help to determine the position and time to reach the minimum / maximum or harmless concentration. It may be used as the preliminary predictive tools in groundwater management.

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