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Nonlinear analysis of shear deformable beam-columns partially supported on tensionless Winkler foundation

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Abstract

In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on tensionless Winkler foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to the two stress functions and solved using the Analog Equation Method, a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy and the range of applications of the developed method.

Keywords: Nonlinear Analysis, Large Deflections, Timoshenko Beam, Shear center, Shear deformation coefficients; Boundary element method, Winkler foundation, Tensionless foundation

1. Introduction

In most investigations concerning beams supported on elastic foundation it is assumed that the bodies in contact (beam and subgrade) are bonded to each other and, consequently, compressive as well as tensile reactions are considered to be admissible. However, for most foundation materials, the admission of tensile stresses across the interface separating the beam from the foundation is not realistic. In this case, where no bonding between beam and subgrade occurs, regions of no contact develop beneath the beam. These regions are unknown and the change of the transverse displacement sign provides the condition for the determination of the contact region.

Besides, the study of nonlinear effects on the analysis of structural elements is essential in civil engineering applications, wherein weight saving is of paramount importance. This non-linearity results from retaining the square of the slope in the strain–displacement relations (intermediate non-linear theory), avoiding in this way the inaccuracies arising from a linearized second – order analysis. Moreover, due to the intensive use of materials having relatively high transverse shear modulus, the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading.

Over the past thirty years, many researchers have developed and validated various methods of performing an analysis for beam-columns, partially supported on Winkler foundation but only few of them took into account the realistic tensionless character of the subgrade reaction. To begin with, Sharma and Dasgupta (1975) employed an iteration method using Green's functions for the analysis of uniformly loaded such Bernoulli beams, followed by Kaschiev and Mikhajlov (1995), who presented a finite element solution for beams subjected to arbitrary loading. Later Zhang and Murphy (2004) presented for the same problem an analytical/numerical solution making no assumption about either the contact area or the kinematics associated with the transverse deflection of the beam. Maheshwari (2007) employed the finite difference method with the help of appropriate boundary and continuity conditions for the analysis of beams on tensionless reinforced granular fill-soil system, while Ma *et. al*. (2008) used the transfer displacement function method (TDFM) to present the response of an infinite beam resting on a tensionless elastic foundation subjected to arbitrarily complex transverse loads. Finally, Tullini and Tralli (2009) presented a finite element solution for the static analysis of a foundation Timoshenko beam resting on elastic half-plane by employing locking-free Hermite polynomials. Nevertheless, in all of the aforementioned research efforts only a linear analysis is performed.

As the deflections become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. Recently, Silveira *et.al*. (2008) presented a nonlinear analysis of Bernoulli structural elements under unilateral contact constraints employing a Ritz type approach, while Tsiatas (2009) demonstrated a boundary integral equation solution to the nonlinear problem of non-uniform Bernoulli beams resting on a nonlinear triparametric elastic foundation. Also, in these research efforts the shear deformation effect is ignored.

In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on tensionless Winkler foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to two stress functions and solved using the Analog Equation Method (Katsikadelis, 2002), a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

1) Shear deformation effect is taken into account on the nonlinear analysis of beam-columns subjected to arbitrary loading (distributed or concentrated transverse loading and bending moments in both directions, as well as axial loading).
2) The homogeneous linear half-space is approximated by a tensionless Winkler foundation.
3) The beam-column is supported by the most general boundary conditions including elastic support or restrain, while its cross section is an arbitrary doubly symmetric one.
4) The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as shear forces along the span induced by the applied axial loading.
5) The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's (Timoshenko and Goodier, 1984) and Cowper's (Cowper, 1966) definitions, for which several authors (Schramm *et al.*, 1994; Schramm *et al.*, 1997) have pointed out that one obtains unsatisfactory results or definitions given by other researchers (Stephen, 1980; Hutchinson, 2001), for which these factors take negative values.
6) The effect of the material's Poisson ratio v is taken into account.
7) The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy and the range of applications of the developed method.

2. Statement of the problem

Let us consider a prismatic beam-column of length l (Figure 1), of constant arbitrary doubly symmetric cross-section of area A. The homogeneous isotropic and linearly elastic material of the beam-column cross-section, with modulus of elasticity E, shear modulus G and Poisson's ratio v occupies the two dimensional multiply connected region Ω of the y,z plane and is bounded by the $\Gamma_j\left(j=1,2,...,K\right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Figure 1b Cyz is the principal bending coordinate system through the cross section's centroid. The beam-column is partially supported on a tensionless homogeneous elastic soil with k_x, k_y and k_z the moduli of subgrade reaction for the *x, y, z* directions, respectively (Winkler spring stiffness). Taking into account the unbonded contact between beam and subgrade, the interaction pressure at the interface is compressive and can be represented for the longitudinal and transverse directions by the following relations

$$p_{sx}=U_u\left(x\right)k_x u \tag{1a}$$

$$p_{sy} = U_v\left(x\right)k_y v \tag{1b}$$

$$p_{sz} = U_w\left(x\right)k_z w \tag{1c}$$

where $U_i\left(x\right)$ is the unit step function defined as

$$U_i\left(x\right) = \begin{cases} 1 & if \;\; i > 0 \\ 0 & if \;\; i \leq 0 \end{cases} \qquad i = u, v, w \tag{2}$$

The beam is subjected to the combined action of the arbitrarily distributed or concentrated axial loading $p_x = p_x\left(x\right)$, transverse loading $p_y = p_y\left(x\right)$, $p_z = p_z\left(x\right)$ acting in the y and z directions, respectively and bending moments $m_y = m_y\left(x\right)$, $m_z = m_z\left(x\right)$ along y and z axes, respectively (Figure 1a).

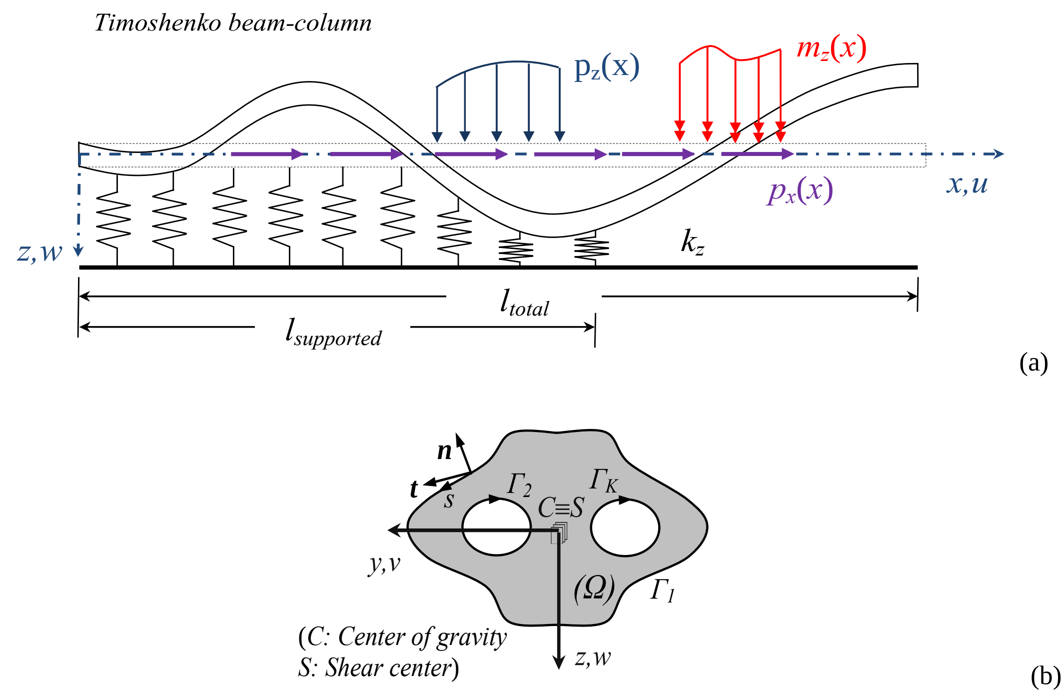

Figure 1. x-z plane of a prismatic beam-column in axial - flexural loading (a) with an arbitrary doubly symmetric cross-section occupying the two dimensional region Ω (b)

Under the action of the aforementioned loading, the displacement field of the beam taking into account shear deformation effect is given as

$$\bar{u}\left(x, y, z\right) = u\left(x\right) - y\theta_z\left(x\right) + z\theta_y\left(x\right) \tag{3a}$$

$$\bar{v}\left(x\right) = v\left(x\right) \tag{3b}$$

$$\bar{w}\left(x\right) = w\left(x\right) \tag{3c}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u\left(x\right)$, $v\left(x\right)$, $w\left(x\right)$ are the corresponding components of the centroid C and $\theta_y\left(x\right)$, $\theta_z\left(x\right)$ are the angles of rotation due to bending of the cross-section with respect to its centroid.

Employing the strain-displacement relations of the three - dimensional elasticity for moderate displacements (Ramm and Hofmann, 1995; Rothert and Gensichen, 1987), the following strain components can be easily obtained

$$\varepsilon_{xx} = \frac{\partial \overline{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \overline{v}}{\partial x}\right)^2 + \left(\frac{\partial \overline{w}}{\partial x}\right)^2\right] \tag{4a}$$

$$\gamma_{xz} = \frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{u}}{\partial z} + \left(\frac{\partial \overline{v}}{\partial x}\frac{\partial \overline{v}}{\partial z} + \frac{\partial \overline{w}}{\partial x}\frac{\partial \overline{w}}{\partial z}\right) \tag{4b}$$

$$\gamma_{xy} = \frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y} + \left(\frac{\partial \overline{v}}{\partial x}\frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial x}\frac{\partial \overline{w}}{\partial y}\right) \tag{4c}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \tag{4d}$$

where it has been assumed that for moderate displacements $\left(\partial \overline{u}/\partial x\right)^2 << \partial \overline{u}/\partial x$, $\left(\partial \overline{u}/\partial x\right)\left(\partial \overline{u}/\partial z\right) << \left(\partial \overline{u}/\partial x\right) + \left(\partial \overline{u}/\partial z\right)$, $\left(\partial \overline{u}/\partial x\right)\left(\partial \overline{u}/\partial y\right) << \left(\partial \overline{u}/\partial x\right) + \left(\partial \overline{u}/\partial y\right)$. Substituting the displacement components (3) to the strain-displacement relations (4), the strain components can be written as

$$\varepsilon_{xx}(x,y,z) = u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right) \tag{5a}$$

$$\gamma_{xy} = v' - \theta_z \tag{5b}$$

$$\gamma_{xz} = w' + \theta_y \tag{5c}$$

where prime $(')$ denotes the derivative with respect to x, while γ_{xy}, γ_{xz} are the additional angles of rotation of the cross-section due to shear deformation.

Considering strains to be small, employing the second Piola – Kirchhoff stress tensor and assuming an isotropic and homogeneous material, the stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \tag{6}$$

or employing eqns. (5) as

$$S_{xx} = E\left[u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right)\right] \tag{7a}$$

$$S_{xy} = G\cdot\left(v' - \theta_z\right) \tag{7b}$$

$$S_{xz} = G\cdot\left(w' + \theta_z\right) \tag{7c}$$

Considering a beam-column element of length dx at its deformed shape and equating the external loads with the internal reaction, the equations of equilibrium are written as

$$-EA\left(u'' + w'w'' + v'v''\right) + U_u k_x u = p_x \tag{8a}$$

$$-\left(Nv'\right)' - GA_y\left(v'' - \theta_z'\right) + U_v k_y v = p_y \tag{8b}$$

$$-EI_z\theta_z'' - GA_y\left(v' - \theta_z\right) = m_z \tag{8c}$$

$$-\left(Nw'\right)' - GA_z\left(w'' + \theta_y'\right) + U_w k_z w = p_z \tag{8d}$$

$$-EI_y \theta_y'' + GA_z \left(w' + \theta_y \right) = m_y \tag{8e}$$

where A is the cross section area, I_y, I_z the moments of inertia with respect to the principle bending axes and GA_y, GA_z are its shear rigidities of the Timoshenko's beam theory, where

$$A_y = \kappa_y A = \frac{1}{a_y} A \qquad\qquad A_z = \kappa_z A = \frac{1}{a_z} A \tag{9a,b}$$

are the shear areas with respect to y, z axes, respectively with κ_y, κ_z the shear correction factors and a_y, a_z the shear deformation coefficients.

Combining equations (8b,c) and (8d,e), the governing differential equations with respect to u, v, w of a geometrically nonlinear Timoshenko beam-column, partially supported on a tensionless Winkler foundation, subjected to the combined action of axial and transverse loading are obtained as

$$-EA\left(u'' + w'w'' + v'v''\right) + U_u k_x u = p_x \tag{10a}$$

$$EI_z v'''' + \frac{EI_z}{GA_y}\left(Nv'\right)''' - \left(Nv'\right)' + \left(k_y v - \frac{EI_z}{GA_y}\left(k_y v''\right) \right) U_v = p_y - \frac{EI_z}{GA_y}\left(p_y''\right) - m_z' \tag{10b}$$

$$EI_y w'''' + \frac{EI_y}{GA_z}\left(Nw'\right)''' - \left(Nw'\right)' + \left(k_z w - \frac{EI_y}{GA_z}\left(k_z w''\right) \right) U_w = p_z - \frac{EI_y}{GA_z}\left(p_z''\right) + m_y' \tag{10c}$$

These equations are also subjected to the pertinent boundary conditions, which are given as

$$a_1 u(x) + \alpha_2 N(x) = \alpha_3 \tag{11}$$

$$\beta_1 v(x) + \beta_2 V_y(x) = \beta_3 \qquad\qquad \bar{\beta}_1 \theta_z(x) + \bar{\beta}_2 M_z(x) = \bar{\beta}_3 \tag{12a,b}$$

$$\gamma_1 w(x) + \gamma_2 V_z(x) = \gamma_3 \qquad\qquad \bar{\gamma}_1 \theta_y(x) + \bar{\gamma}_2 M_y(x) = \bar{\gamma}_3 \tag{13a,b}$$

at the beam ends $x = 0,l$. In eqns. (12), (13) V_y, V_z and M_y, M_z are the reactions and bending moments with respect to y, z axes, respectively, which together with the angles of rotation due to bending θ_y, θ_z are given as

$$V_y = -EI_z v''' - \frac{EI_z}{GA_y}\left[Nv''' - U_v k_y v' \right] + Nv' \tag{14a}$$

$$V_z = -EI_y w''' - \frac{EI_y}{GA_z}\left[Nw''' - U_w k_z w' \right] + Nw' \tag{14b}$$

$$M_z = EI_z v'' + \frac{EI_z}{GA_y}\left[Nv'' - U_v k_y v \right] \tag{14c}$$

$$M_y = -EI_y w'' - \frac{EI_y}{GA_z}\left[Nw'' - U_w k_z w \right] \tag{14d}$$

$$\theta_y = \frac{EI_y}{G^2 A_z^2}\left(U_w k_z w' - \left(Nw'\right)'' \right) - \frac{1}{GA_z}\left(EI_y w''' + GA_z w' \right) \tag{14e}$$

$$\theta_z = \frac{EI_z}{G^2 A_y^2}\left(\left(Nv'\right)'' - U_v k_y v' \right) + \frac{1}{GA_y}\left(EI_z v''' + GA_y v' \right) \tag{14f}$$

Finally, $\alpha_j, \beta_j, \bar{\beta}_j, \gamma_j, \bar{\gamma}_j$ ($j = 1,2,3$) are functions specified at the beam-column ends $x = 0,l$. Eqns. (11)-(13) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\bar{\beta}_1 = \bar{\gamma}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = 0$).

The solution of the boundary value problem given from eqns. (10), subjected to the boundary conditions (11)-(13), which represents the nonlinear flexural analysis of a Timoshenko beam-column, partially supported on a tensionless Winkler foundation, presumes the evaluation of the shear deformation coefficients a_y, a_z, corresponding to the principal coordinate system Cyz. These coefficients are established equating the approximate formula of the shear strain energy per unit length (Stephen, 1980)

$$U_{appr.} = \frac{a_y Q_y^2}{2AG} + \frac{a_z Q_z^2}{2AG} \tag{15}$$

with the exact one given from

$$U_{exact} = \int_\Omega \frac{(\tau_{xz})^2 + (\tau_{xy})^2}{2G} d\Omega \tag{16}$$

and are obtained as (Sapountzakis and Mokos, 2005)

$$a_y = \frac{1}{\kappa_y} = \frac{A}{\Delta^2} \int_\Omega \left[(\nabla \Theta) - \boldsymbol{e} \right] \cdot \left[(\nabla \Theta) - \boldsymbol{e} \right] d\Omega \tag{17a}$$

$$a_z = \frac{1}{\kappa_z} = \frac{A}{\Delta^2} \int_\Omega \left[(\nabla \Phi) - \boldsymbol{d} \right] \cdot \left[(\nabla \Phi) - \boldsymbol{d} \right] d\Omega \tag{17b}$$

where $(\tau_{xz})_j, (\tau_{xy})_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv \boldsymbol{i_y} (\partial/\partial y) + \boldsymbol{i_z} (\partial/\partial z)$ is a symbolic vector with $\boldsymbol{i_y}, \boldsymbol{i_z}$ the unit vectors along y and z axes, respectively, Δ is given from

$$\Delta = 2(1+\nu) I_y I_z \tag{18}$$

ν is the Poisson ratio of the cross section material, $\boldsymbol{e}$ and $\boldsymbol{d}$ are vectors defined as

$$\boldsymbol{e} = \left(\nu I_y \frac{y^2 - z^2}{2} \right) \boldsymbol{i_y} + \nu I_y yz \boldsymbol{i_z} \tag{19a}$$

$$\boldsymbol{d} = \nu I_z yz \boldsymbol{i_y} - \left(\nu I_z \frac{y^2 - z^2}{2} \right) \boldsymbol{i_z} \tag{19b}$$

and $\Theta(y,z)$, $\Phi(y,z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems (Sapountzakis and Mokos, 2005)

$$\nabla^2 \Theta = -2 I_y y \quad \text{in} \quad \Omega \tag{20a}$$

$$\frac{\partial \Theta}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{e} \quad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{20b}$$

$$\nabla^2 \Phi = -2 I_z z \quad \text{in} \quad \Omega \tag{21a}$$

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{d} \quad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{21b}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations $a_y = a_z = 0$. It is also worth here noting that the boundary conditions (20b), (21b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ vanishes on the free surface of the beam.

3. Integral Representations – Numerical Solution

According to the precedent analysis, the nonlinear flexural analysis of a Timoshenko beam-column, partially supported on a tensionless Winkler foundation, undergoing moderate large deflections reduces in establishing the displacement components $u(x)$ and $v(x)$, $w(x)$ having continuous derivatives up to the second and up to the fourth order with respect to x, respectively. Moreover, these displacement components must satisfy the coupled governing differential equations (10) inside the beam and the boundary conditions (11)-(13) at the beam ends $x = 0,l$. Eqns. (10) are solved using the Analog Equation Method (Katsikadelis, 2002) as it is developed for hyperbolic differential equations (Sapountzakis and Katsikadelis, 2000).

3.1 For the transverse displacements v, w

Let $v(x)$, $w(x)$ be the sought solution of the aforementioned boundary value problem. Setting as $u_2(x) = v(x)$, $u_3(x) = w(x)$ and differentiating these functions four times with respect to x yields

$$\frac{\partial^4 u_i}{\partial x^4} = q_i(x) \qquad (i = 2,3) \tag{22}$$

Eqns. (22) indicate that the solution of eqns. (10b), (10c) can be established by solving eqns. (22) under the same boundary conditions (12)-(13), provided that the fictitious load distributions $q_i(x)$ $(i = 2,3)$ are first established. These distributions can be determined using BEM as follows.

Following the procedure presented in (Sapountzakis and Katsikadelis, 2000) and employing the constant element assumption for the load distributions q_i along the L internal beam elements (as the numerical implementation becomes very simple and the obtained results are of high accuracy), the integral representations of the displacement components u_i $(i = 2,3)$ and their first derivatives with respect to x when applied for the beam-column ends ($0,l$), together with the boundary conditions (12)-(13) are employed to express the unknown boundary quantities $u_i(\zeta)$, $u_{i,x}(\zeta)$, $u_{i,xx}(\zeta)$ and $u_{i,xxx}(\zeta)$ ($\zeta = 0,l$) in terms of q_i as

$$\begin{bmatrix} \mathbf{D}_{11} & \mathbf{0} & \mathbf{D}_{13} & \mathbf{D}_{14} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} & \mathbf{0} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{E}_{41} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_{2,xxx} \\ \hat{\mathbf{u}}_{2,xx} \\ \hat{\mathbf{u}}_{2,x} \\ \hat{\mathbf{u}}_{2} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\beta}_3 \\ \overline{\boldsymbol{\beta}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{Bmatrix} \mathbf{q}_2 \tag{23a}$$

$$\begin{bmatrix} \mathbf{G}_{11} & \mathbf{0} & \mathbf{G}_{13} & \mathbf{G}_{14} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} & \mathbf{0} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{E}_{41} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_{3,xxx} \\ \hat{\mathbf{u}}_{3,xx} \\ \hat{\mathbf{u}}_{3,x} \\ \hat{\mathbf{u}}_{3} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\gamma}_3 \\ \overline{\boldsymbol{\gamma}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{Bmatrix} \mathbf{q}_3 \tag{23b}$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{13}$, $\mathbf{D}_{14}$, $\mathbf{D}_{21}$, $\mathbf{D}_{22}$, $\mathbf{D}_{23}$, $\mathbf{G}_{11}$, $\mathbf{G}_{13}$, $\mathbf{G}_{14}$, $\mathbf{G}_{21}$, $\mathbf{G}_{22}$, $\mathbf{G}_{23}$ are 2×2 known square matrices including the values of the functions $\beta_j, \overline{\beta}_j, \gamma_j, \overline{\gamma}_j$ ($j = 1,2$) of eqns. (12)-(13); $\boldsymbol{\beta}_3$, $\overline{\boldsymbol{\beta}}_3$, $\boldsymbol{\gamma}_3$, $\overline{\boldsymbol{\gamma}}_3$ are 2×1 known column matrices including the boundary values of the functions $\beta_3, \overline{\beta}_3, \gamma_3, \overline{\gamma}_3$ of eqns. (12)-(13); $\mathbf{E}_{jk}$, ($j = 3,4$, $k = 1,2,3,4$) are square 2×2 known coefficient matrices and $\mathbf{F}_j$ $(j = 3,4)$ are $2\times L$ rectangular known matrices originating from the integration of kernels on the axis of the beam. Moreover,

$$\hat{\mathbf{u}}_i = \{u_i(0) \quad u_i(l)\}^T \tag{24a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{ \frac{\partial u_i(0)}{\partial x} \quad \frac{\partial u_i(l)}{\partial x} \right\}^T \tag{24b}$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{ \frac{\partial^2 u_i(0)}{\partial x^2} \quad \frac{\partial^2 u_i(l)}{\partial x^2} \right\}^T \tag{24c}$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{ \frac{\partial^3 u_i(0)}{\partial x^3} \quad \frac{\partial^3 u_i(l)}{\partial x^3} \right\}^T \tag{24d}$$

are vectors including the two unknown boundary values of the respective boundary quantities and $\mathbf{q}_i = \left\{ q_1^i \; q_2^i \; \ldots \; q_L^i \right\}^T$ $(i = 2,3)$ is the vector including the L unknown nodal values of the fictitious load.

Discretization of the integral representations of the displacement components u_i $(i = 2,3)$ and their derivatives with respect to x, after elimination of the boundary quantities employing eqns. (23), gives

$$\mathbf{u}_i = \mathbf{T}_i \mathbf{q}_i + \mathbf{t}_i \qquad (i = 2,3) \tag{25a}$$

$$\mathbf{u}_{i,x} = \mathbf{T}_{ix} \mathbf{q}_i + \mathbf{t}_{ix} \qquad (i = 2,3) \tag{25b}$$

$$\mathbf{u}_{i,xx} = \mathbf{T}_{ixx} \mathbf{q}_i + \mathbf{t}_{ixx} \qquad (i = 2,3) \tag{25c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{T}_{ixxx} \mathbf{q}_i + \mathbf{t}_{ixxx} \qquad (i = 2,3) \tag{25d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{q}_i \qquad (i = 2,3) \tag{25e}$$

where $\mathbf{u}_i$, $\mathbf{u}_{i,x}$, $\mathbf{u}_{i,xx}$, $\mathbf{u}_{i,xxx}$, $\mathbf{u}_{i,xxxx}$ are vectors including the values of $u_i(x)$ and their derivatives at the L nodal points, $\mathbf{T}_i$, $\mathbf{T}_{ix}$, $\mathbf{T}_{ixx}$, $\mathbf{T}_{ixxx}$ are known $L \times L$ matrices and $\mathbf{t}_i$, $\mathbf{t}_{ix}$, $\mathbf{t}_{ixx}$, $\mathbf{t}_{ixxx}$ are known $L \times 1$ matrices.

In the conventional BEM, the load vectors $\mathbf{q}_i$ are known and eqns. (25) are used to evaluate $u_i(x)$ and their derivatives at the L nodal points. This, however, can not be done here since $\mathbf{q}_i = \left\{ q_1^i \; q_2^i \; \ldots \; q_L^i \right\}^T$ $(i = 2,3)$ are unknown. For this purpose, $2L$ additional equations are derived, which permit the establishment of $\mathbf{q}_i$. These equations result by applying eqns. (10b), (10c) to the L collocation points, leading to the formulation of the following set of $2L$ simultaneous equations

$$[\mathbf{C_2}]\{\mathbf{q_2}\} = \{\mathbf{b_2}\} + \{\mathbf{d_2}\} \tag{26a}$$

$$[\mathbf{C_3}]\{\mathbf{q_3}\} = \{\mathbf{b_3}\} + \{\mathbf{d_3}\} \tag{26b}$$

where the $\mathbf{C_2}$, $\mathbf{C_3}$ $L \times L$ matrices and the $\mathbf{b_2}$, $\mathbf{d_2}$, $\mathbf{b_3}$, $\mathbf{d_3}$ $L \times 1$ vectors are given as

$$[\mathbf{C_2}] = \Big[[\mathbf{Y''''}]_{\mathbf{dg.}} - [\mathbf{Y'''}]_{\mathbf{dg.}} [\mathbf{T_{2,xxx}}] - \big([\mathbf{Y''}]_{\mathbf{dg.}} + \mathbf{Y}[\mathbf{K_y}] \big) [\mathbf{T_{2,xx}}] - [\mathbf{Y'}]_{\mathbf{dg.}} [\mathbf{T_{2,x}}] + [\mathbf{T_2}][\mathbf{K_y}] \Big] \tag{27a}$$

$$[\mathbf{C_3}] = \Big[[\mathbf{Z''''}]_{\mathbf{dg.}} - [\mathbf{Z'''}]_{\mathbf{dg.}} [\mathbf{T_{3,xxx}}] - \big([\mathbf{Z''}]_{\mathbf{dg.}} + \mathbf{Z}[\mathbf{K_z}] \big) [\mathbf{T_{3,xx}}] - [\mathbf{Z'}]_{\mathbf{dg.}} [\mathbf{T_{3,x}}] + [\mathbf{T_3}][\mathbf{K_z}] \Big] \tag{27b}$$

$$\{\mathbf{b_2}\} = [\mathbf{Y'''}]_{\mathbf{dg.}} \{\mathbf{t_{2,xxx}}\} + \big([\mathbf{Y''}]_{\mathbf{dg.}} + \mathbf{Y}[\mathbf{K_y}] \big) \{\mathbf{t_{2,xx}}\} + [\mathbf{Y'}]_{\mathbf{dg.}} \{\mathbf{t_{2,x}}\} - [\mathbf{K_y}]\{\mathbf{t_2}\} \tag{27c}$$

$$\{\mathbf{b_3}\} = [\mathbf{Z'''}]_{\mathbf{dg.}} \{\mathbf{t_{3,xxx}}\} + \big([\mathbf{Z''}]_{\mathbf{dg.}} + \mathbf{Z}[\mathbf{K_z}] \big) \{\mathbf{t_{3,xx}}\} + [\mathbf{Z'}]_{\mathbf{dg.}} \{\mathbf{t_{3,x}}\} - [\mathbf{K_z}]\{\mathbf{t_3}\} \tag{27d}$$

$$\{\mathbf{d_2}\} = \{\mathbf{p_y}\} - \{\mathbf{m'_z}\} - [\mathbf{Y}]_{\mathbf{dg.}} \{\mathbf{p''_y}\} \tag{27e}$$

$$\{\mathbf{d_3}\} = \{\mathbf{p_z}\} + \{\mathbf{m'_y}\} - [\mathbf{Z}]_{\mathbf{dg.}} \{\mathbf{p''_z}\} \tag{27f}$$

where $[\mathbf{Z''''}]_{\mathbf{dg.}}$, $[\mathbf{Z'''}]_{\mathbf{dg.}}$, $[\mathbf{Z''}]_{\mathbf{dg.}}$, $[\mathbf{Z'}]_{\mathbf{dg.}}$, $[\mathbf{Z}]_{\mathbf{dg.}}$ and $[\mathbf{Y''''}]_{\mathbf{dg.}}$, $[\mathbf{Y'''}]_{\mathbf{dg.}}$, $[\mathbf{Y''}]_{\mathbf{dg.}}$, $[\mathbf{Y'}]_{\mathbf{dg.}}$, $[\mathbf{Y}]_{\mathbf{dg.}}$ are $L \times L$ diagonal matrices whose elements at the i-th nodal point are given as

$$\left(Z''''\right)_{ii} = EI_y \left[1 + \alpha_z \frac{1}{GA}\left(N\right)_i\right] \tag{28a}$$

$$\left(Z'''\right)_{ii} = 3\alpha_z \frac{EI_y}{GA}\left(p_x\right)_i \tag{28b}$$

$$\left(Z''\right)_{ii} = 3\alpha_z \frac{EI_y}{GA}\left(\frac{dp_x}{dx}\right)_i + \left(N\right)_i \tag{28c}$$

$$\left(Z'\right)_{ii} = \alpha_z \frac{EI_y}{GA}\left(\frac{d^2 p_x}{dx^2}\right)_i - \left(p_x\right)_i \tag{28d}$$

$$\left(Z\right)_{ii} = \alpha_z \frac{EI_y}{GA} \tag{28e}$$

$$\left(Y''''\right)_{ii} = EI_z \left[1 + \alpha_y \frac{1}{GA}\left(N\right)_i\right] \tag{28f}$$

$$\left(Y'''\right)_{ii} = 3\alpha_y \frac{EI_z}{GA}\left(p_x\right)_i \tag{28g}$$

$$\left(Y''\right)_{ii} = 3\alpha_y \frac{EI_z}{GA}\left(\frac{dp_x}{dx}\right)_i + \left(N\right)_i \tag{28h}$$

$$\left(Y'\right)_{ii} = \alpha_y \frac{EI_z}{GA}\left(\frac{d^2 p_x}{dx^2}\right)_i - \left(p_x\right)_i \tag{28i}$$

$$\left(Y\right)_{ii} = \alpha_y \frac{EI_z}{GA} \tag{28j}$$

Moreover, in eqns. (27) $\{\mathbf{p_y}\}$, $\{\mathbf{p_z}\}$, $\{\mathbf{p''_y}\}$, $\{\mathbf{p''_z}\}$, $\{\mathbf{m'_y}\}$ and $\{\mathbf{m'_z}\}$ are $L \times 1$ vectors containing the values of the external loading and its derivatives at these points and $[\mathbf{K_y}]$, $[\mathbf{K_z}]$ are diagonal matrices whose diagonal elements are given as

$$\left(\mathbf{K}_y\right)_{ii} = \left(k_y\right)_i \left(U_v\right)_i \tag{29a}$$

$$\left(\mathbf{K}_z\right)_{ii} = \left(k_z\right)_i \left(U_w\right)_i \tag{29b}$$

where $\left(k_y\right)_i$, $\left(k_z\right)_i$, $\left(U_v\right)_i$, $\left(U_w\right)_i$ are the values of the corresponding moduli of subgrade and the unit step function at the i-th nodal point.

3.2 For the axial displacement u

Let $u_1 = u(x)$ be the sought solution of the boundary value problem described by eqns. (10a) and (11). Differentiating this function two times yields

$$\frac{\partial^2 u_1}{\partial x^2} = q_1(x) \tag{30}$$

Eqn. (30) indicates that the solution of the original problem can be obtained as the axial displacement of a beam with unit axial rigidity subjected to an axial fictitious load $q_1(x)$ under the same boundary conditions. The fictitious load is unknown. Following the same procedure as in 3.1, the discretized counterpart of the integral representations of the displacement component u_1 and its first derivative with respect to x when applied to all nodal points in the interior of the beam yields

$$\mathbf{u}_1 = \mathbf{T}_1 \mathbf{q}_1 + \mathbf{t}_1 \tag{31a}$$

$$\mathbf{u}_{1,x} = \mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x} \tag{31b}$$

where $\mathbf{T}_1$, $\mathbf{T}_{1x}$ are known $L \times L$ matrices, similar with those mentioned before for the displacements u_2, u_3. Application of eqn. (10a) to the L collocation points, after employing eqns. (25), (31) leads to the formulation of the following system of L equations with respect to $\mathbf{q_1}$, $\mathbf{q_2}$ and $\mathbf{q_3}$ fictitious load vectors

$$\begin{aligned}\left(\mathbf{EA} - \mathbf{K}_x\mathbf{T}_1\right)\mathbf{q_1} = & -\mathbf{p_x} - \mathbf{EA}\left[\left(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx}\right)\right]_{dg.}\left(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}\right) - \\ & - \mathbf{EA}\left[\left(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx}\right)\right]_{dg.}\left(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}\right) + \mathbf{K}_x\mathbf{t}_1\end{aligned} \tag{32}$$

where $\mathbf{EA}$, is an $L \times L$ diagonal matrices including the values of the corresponding quantities at the L nodal points and $\mathbf{K}_x$ is a diagonal matrix similar with those of equations (29) whose diagonal elements are given as

$$\left(\mathbf{K}_x\right)_{ii} = \left(k_x\right)_i \left(U_u\right)_i \tag{33}$$

Moreover, substituting eqns. (25), (31) in the axial stress resultant arising from the integration of the stress component (7a), the discretized counterpart of the axial force at the neutral axis of the beam is given as

$$\begin{aligned}\mathbf{N} = & \mathbf{EA}\left(\mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x}\right) + \frac{1}{2}\mathbf{EA}\left[\left(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}\right)\right]_{dg}\left(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}\right) + \\ & + \frac{1}{2}\mathbf{EA}\left[\left(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}\right)\right]_{dg}\left(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}\right)\end{aligned} \tag{34}$$

Eqns. (26a), (26b), (32) and (34) constitute a nonlinear coupled system of equations with respect to $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$ and $\mathbf{N}$ quantities. The solution of this system is accomplished iteratively by employing iterative numerical methods, such as the two term acceleration method (Sapountzakis and Katsikadelis, 2000; Isaacson and Keller, 1966).

3.3 For the stress functions $\Theta(y,z)$ and $\Phi(y,z)$

The evaluation of the stress functions $\Theta(y,z)$ and $\Phi(y,z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos (2005).

Moreover, since the nonlinear flexural problem of Timoshenko beam-columns is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia and the shear deformation coefficients (eqns. (17)) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments of inertia and the cross section area can be written as

$$I_y = \int_\Gamma \left(yz^2 n_y\right) ds \tag{35a}$$

$$I_z = \int_\Gamma \left(zy^2 n_z\right) ds \tag{35b}$$

$$A = \frac{1}{2}\int_\Gamma \left(yn_y + zn_z\right) ds \tag{35c}$$

while the shear deformation coefficients a_y and a_z are obtained from the relations

$$a_y = \frac{A}{\Delta^2}\left(\left(4v+2\right)I_y I_{\Theta y} + \frac{1}{4}v^2 I_{yy}^2 I_{ed} - I_{\Theta e}\right) \tag{36a}$$

$$a_z = \frac{A}{\Delta^2}\left(\left(4v+2\right)I_z I_{\Phi z} + \frac{1}{4}v^2 I_z^2 I_{ed} - I_{\Phi d}\right) \tag{36b}$$

where

$$I_{\Theta e} = \int_{\Gamma} \Theta (\boldsymbol{n} \cdot \boldsymbol{e}) ds \tag{37a}$$

$$I_{\Phi d} = \int_{\Gamma} \Phi (\boldsymbol{n} \cdot \boldsymbol{d}) ds \tag{37b}$$

$$I_{ed} = \int_{\Gamma} \left(y^4 z n_z + z^4 y n_y + \frac{2}{3} y^2 z^3 n_z \right) ds \tag{37c}$$

$$I_{\Theta y} = \frac{1}{6} \int_{\Gamma} \left[-2 I_{yy} y^4 z n_z + \left(3 \Theta n_y - y (\boldsymbol{n} \cdot \boldsymbol{e}) \right) y^2 \right] ds \tag{37d}$$

$$I_{\Phi z} = \frac{1}{6} \int_{\Gamma} \left[-2 I_{zz} z^4 y n_y + \left(3 \Phi n_z - z (\boldsymbol{n} \cdot \boldsymbol{d}) \right) z^2 \right] ds \tag{37e}$$

4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method. In all the examples treated, the numerical results have been obtained employing 41 nodal points (longitudinal discretization) and 300 boundary elements (cross section discretization).

4.1Example 1

For comparison reasons a linear analysis of a simply supported beam-column has been studied for three different load and geometry cases. Although displacements are considered small the problem is strongly non linear as the contact area of the beam and the soil is unknown.

A beam-column of length $l = 5$ and $EI = 10^3$ subjected to concentrated moments $M_1 = M_2 = -10^2$ at its ends and resting on a homogeneous elastic foundation with modulus of subgrade reaction k_z, as this is shown in Figure 2 (case i), has been studied. In Figures 3, 4 the beam-column deflections for the cases of conventional bilateral and unilateral (tensionless) Winkler springs, respectively are presented as compared with those obtained from analytical (Hetenyi, 1946), FEM (Pereira, 2003) and Ritz type (Silveira *et al.*, 2008) solutions for various values of the dimensionless foundation parameter $k = k_z l^4 / EI$ (Silveira *et al.*, 2008). Moreover, in Table 1 the extreme values of the beam-column deflection and of the soil reaction are presented for both cases of bilateral and unilateral foundation and for various values of the aforementioned dimensionless parameter k. From these figures and table the accuracy of the obtained results is remarkable, while the influence of both the foundation stiffness and the unilateral character of the soil reaction are easily verified. Moreover, the discrepancy in the deflections between the bilateral and the unilateral foundation model especially for a stiff soil is remarked.

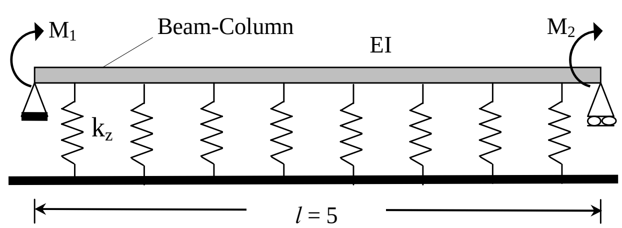

Figure 2. Prismatic beam-column on elastic foundation subjected to concentrated moments at its ends (case i)

As a variant of this example, the beam-column of length $l = 10$ and $EI = 10^3$ subjected to concentrated moments $M_1 = -M_2 = 10^2$ at its ends and a concentrated force $P(l/2) = 150$ at the midpoint of the beam, as this is shown in Figure 5 (case ii), has also been studied. In Figures 6, 7 the beam-column deflections for the cases of conventional bilateral and unilateral Winkler springs, respectively are presented as compared with those obtained from analytical (Hetenyi, 1946), FEM (Pereira, 2003) and Ritz type (Silveira *et al.*, 2008) solutions for various values of the dimensionless foundation parameter k, while in Table 2 the extreme values of the beam-column deflection and of the soil reaction are presented for both cases of bilateral and unilateral foundation reaction, leading to the same conclusions drawn from the previous beam-column case.

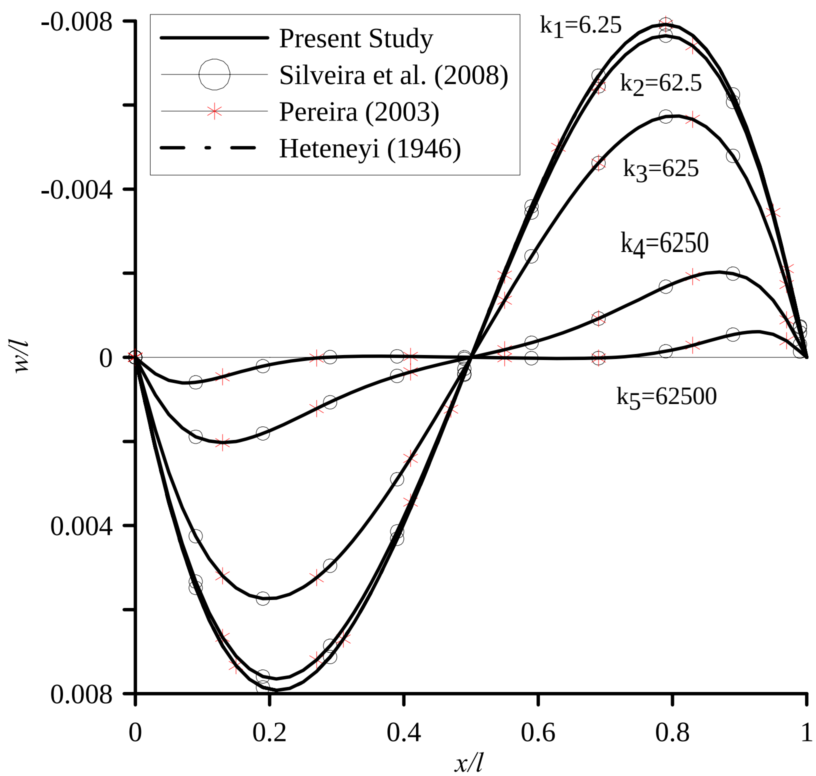

Figure 3. Deflections for various values of the soil parameter k, of the beam-column of example 1 (case i) resting on a bilateral elastic foundation

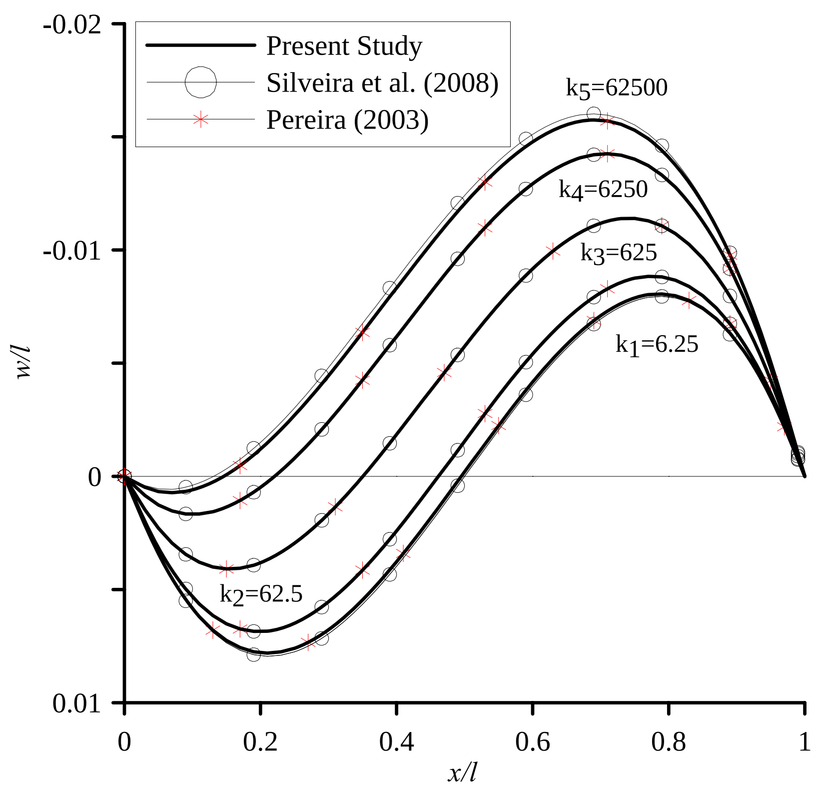

Figure 4. Deflections for various values of the soil parameter k, of the beam-column of example 1 (case i) resting on a unilateral elastic foundation

Table 1. Extreme values of the deflections and the foundation reaction of the beam-column of example 1 (case i)

k	Bilateral Winkler			Unilateral Winkler		
	Min w (10^{-2})	Max w (10^{-2})	$p_{sz}l^3/EI$	Min w (10^{-2})	Max w (10^{-2})	$p_{sz}l^3/EI$
6.25	-3.960	3.960	0.049	-4.030	3.905	0.048
62.5	-3.826	3.826	0.478	-4.418	3.425	0.428
625	-2.869	2.869	3.586	-5.697	2.046	2.557
6250	-1.015	1.015	12.688	-7.125	0.833	10.415
62500	-0.304	0.304	37.965	-8.008	0.287	35.854

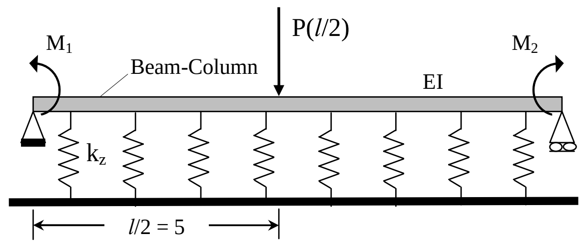

Figure 5. Prismatic beam-column on elastic foundation subjected to concentrated moments at its ends and force at its midpoint (case ii)

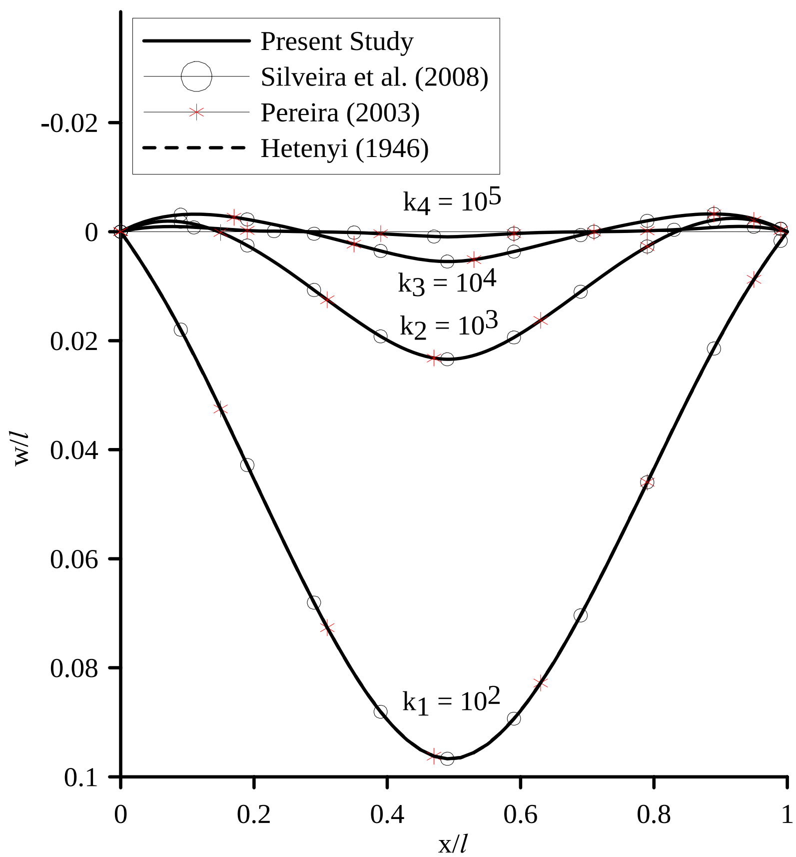

Figure 6. Deflections for various values of the soil parameter k, of the beam-column of example 1 (case ii) resting on a bilateral elastic foundation

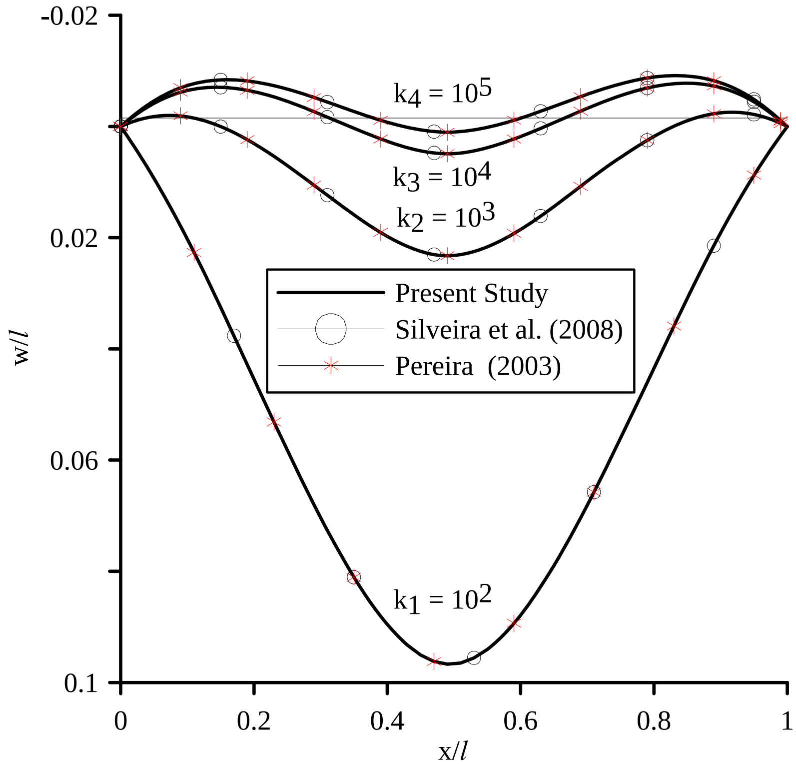

Figure 7. Deflections for various values of the soil parameter k, of the beam-column of example 1 (case ii) resting on a unilateral elastic foundation

Table 2. Extreme values of the deflections and the foundation reaction of the beam-column of example 1 (case ii)

k	Bilateral Winkler			Unilateral Winkler		
	Min w (10^{-2})	Max w (10^{-2})	$p_{sz}l^3/EI$	Min w (10^{-2})	Max w (10^{-2})	$p_{sz}l^3/EI$
10^2	0	9.671	9.670	0	9.671	9.671
10^3	-0.239	2.341	23.41	-0.255	2.326	23.26
10^4	-0.326	0.549	54.90	-0.778	0.490	49.04
10^5	-0.095	0.0941	94.06	-0.9139	0.104	104.36

Finally, as a second variant of this example, the beam-column of Figure 5 subjected to concentrated moments $M_1 = -M_2 = -10^2$ at its ends and a concentrated force $P(l/2) = -50$ at the midpoint of the beam, has also been studied (case iii). In Figure 8 the deflections of the beam-column resting on a tensionless subgrade are presented as compared with those obtained from a FEM (Pereira, 2003) and a Ritz type (Silveira *et al.*, 2008) solution for various values of the dimensionless foundation parameter k. Moreover, in Figure 9 the deflections of the beam-column resting either on a unilateral or a bilateral subgrade ($k = 10^4$) as well as for no soil at all are presented as compared with those obtained from a Ritz type solution (Silveira *et al.*, 2008) demonstrating once again the paramount importance of the tensionless character of the Winkler foundation. Finally, in Table 3 the extreme values of the beam-column deflection and of the soil reaction are presented for both cases of bilateral and unilateral foundation reaction, leading to the conclusions already drawn and noting the significant influence of the unilateral character of the soil reaction in both the deflections and the soil reaction especially in the case of a stiff soil.

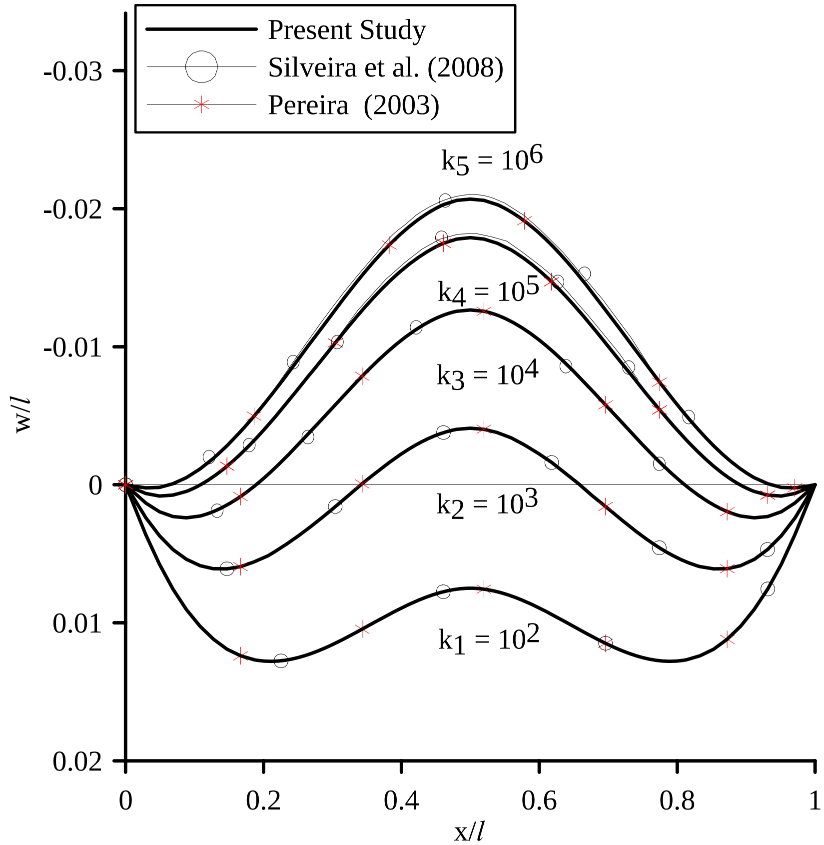

Figure 8. Deflections for various values of the soil parameter k, of the beam-column of example 1 (case iii) resting on a unilateral elastic foundation

Table 3. Extreme values of the deflections and the foundation reaction of the beam-column of example 1 (case iii)

k	Bilateral Winkler			Unilateral Winkler		
	Min w (10^{-2})	Max w (10^{-2})	$p_{sz}l^3/EI$	Min w (10^{-2})	Max w (10^{-2})	$p_{sz}l^3/EI$
10^2	-0.248	0.661	0.66	0	1.286	1.28
10^3	-0.197	0.318	3.18	-0.41	0.609	6.09
10^4	-0.031	0.095	9.53	-1.266	0.239	23.95
10^5	-0.006	0.026	26.09	-1.789	0.082	81.47
10^6	0	0.005	54.05	-2.070	0.023	231.4

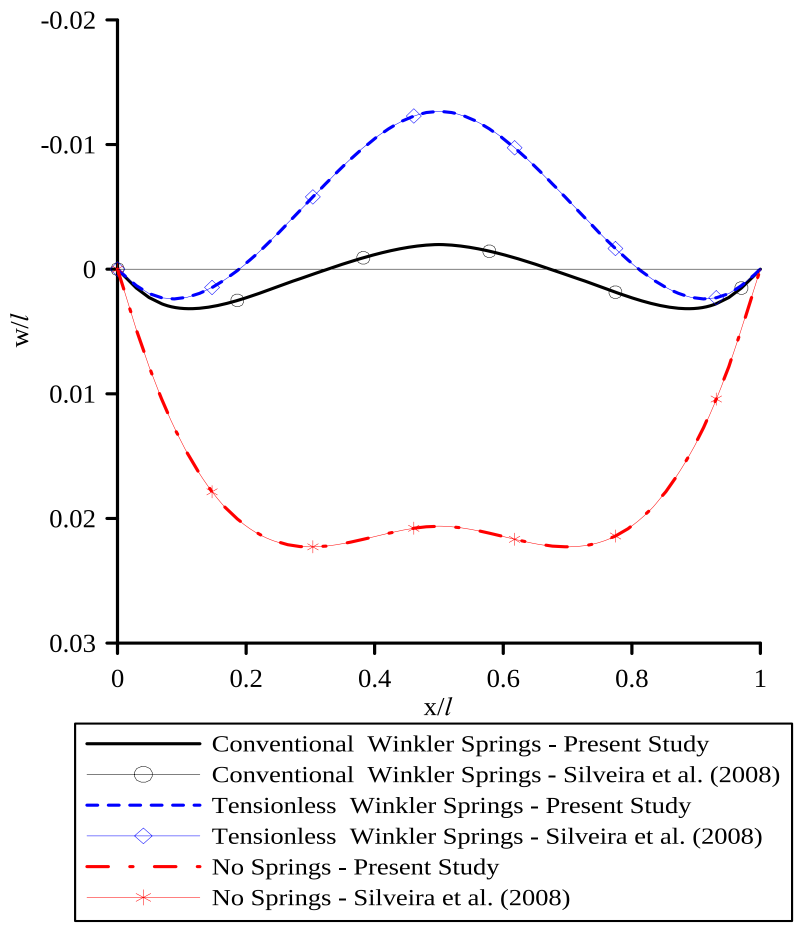

Figure 9. Deflections of the beam-column of example 1 (case iii) resting on either a unilateral or a bilateral elastic foundation for $k = 10^4$

4.2 Example 2

In order to illustrate the importance of the nonlinear analysis and the influence of the shear deformation effect, a clamped beam-column of length $l = 5m$, having a hollow rectangular cross section ($E = 210\,GPa$, $v = 0.3$, $a_z = 3.664$, $a_y = 1.766$) and resting on a homogeneous (either bilateral or unilateral) elastic foundation of stiffness k_z, as this is shown in Figure 10, is examined.

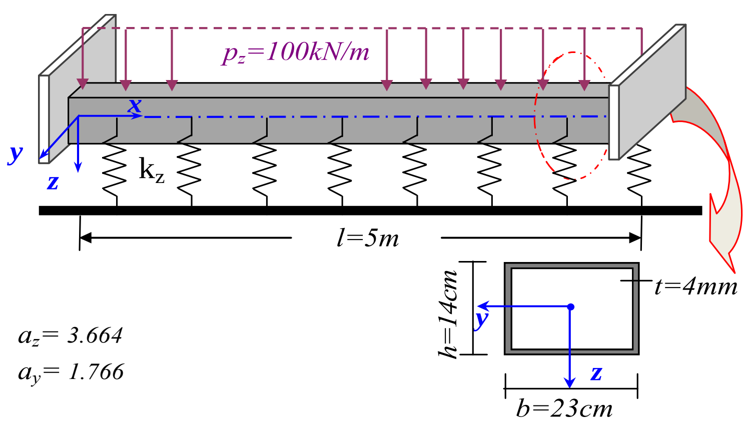

Figure 10. Clamped beam-column of hollow rectangular cross section subjected to the uniformly distributed load p_z (case i)

In Figure 11 the deflection w along the beam-column resting on a tensionless foundation with $k_z = 50kN/m^2$ and subjected to a uniformly distributed load $p_z = 100kN/m$ (case i) are presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. From this figure, the influence of the nonlinearity to the performed analysis is remarked, while the discrepancy of the obtained results due to the shear deformation effect justifies its inclusion even in thin walled sections. Moreover, in Table 4 the deflections and the bending moments at the midpoint $x = l/2$ and at the ends $x = 0, l$, respectively of the beam-column are presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. Finally, in Figure 12 the deflection curves of the beam-column resting on a tensionless foundation are presented for various values of the modulus k_z of the subgrade reaction, performing a nonlinear analysis, taking into account shear deformation effect and demonstrating the importance of the soil stiffness in the obtained results.

Table 4. Deflection (cm) and moment (kNm) at the midpoint and the ends of the clamped beam-column, respectively of example 2 (case i), for $k_z = 50kN/m^2$

	Without Shear Deformation		With Shear Deformation	
	Linear Analysis	Nonlinear Analysis	Linear Analysis	Nonlinear Analysis
$w_{(l/2)}$	7.49	6.93	7.95	7.28
$M_{y(0,l)}$	-202.98	-192.85	-199.31	-187.54

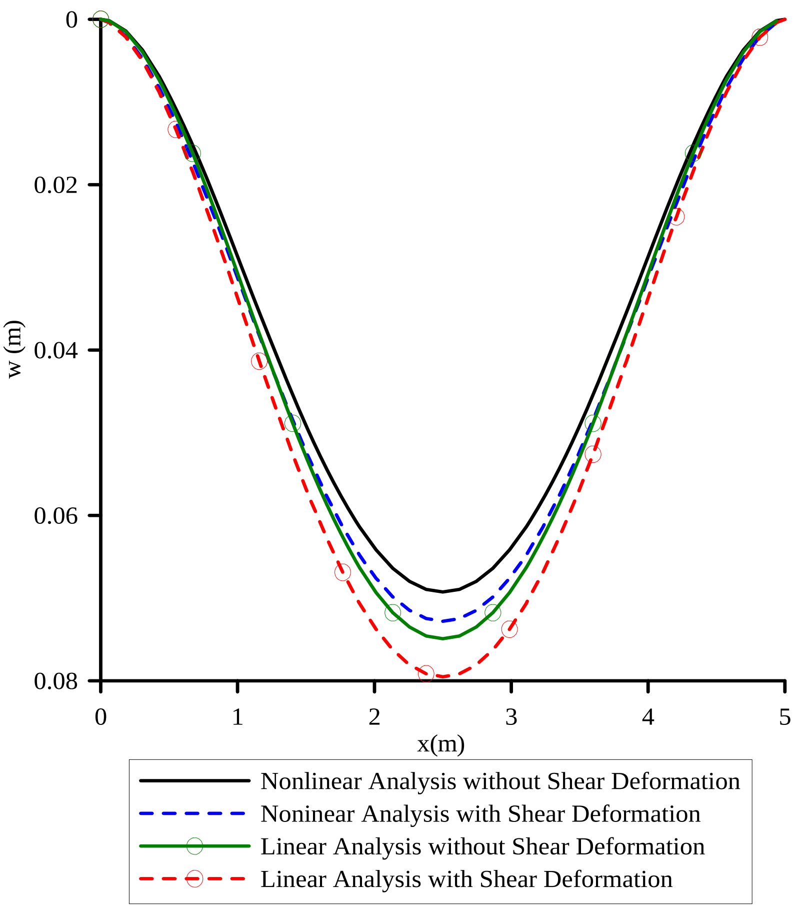

Figure 11. Deflection w along the beam-column of example 2(case i), for soil stiffness $k_z = 50kN/m^2$

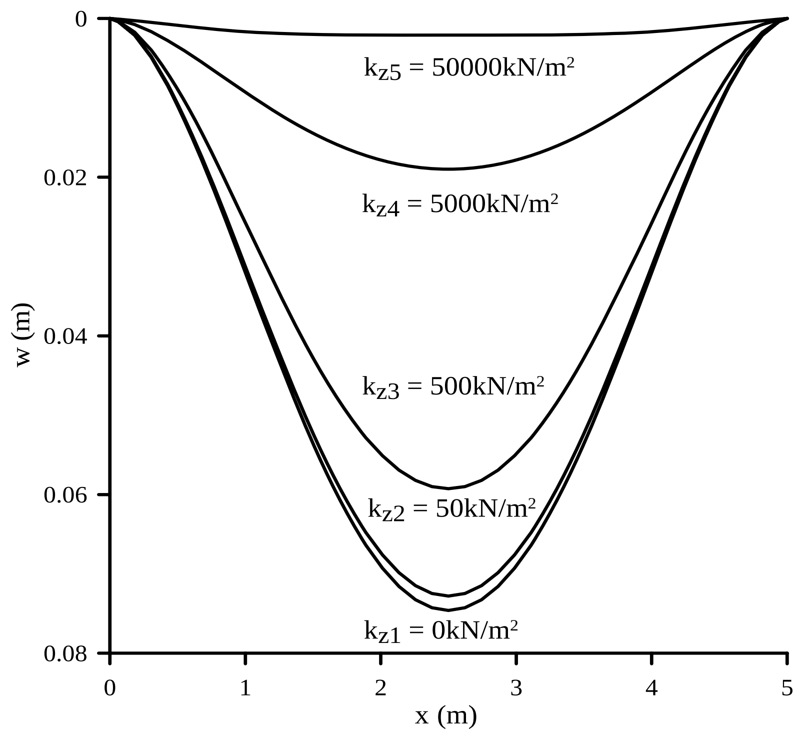

Figure 12. Deflection curves of the beam-column of example 2(case i) for various values of the modulus k_z of the subgrade reaction

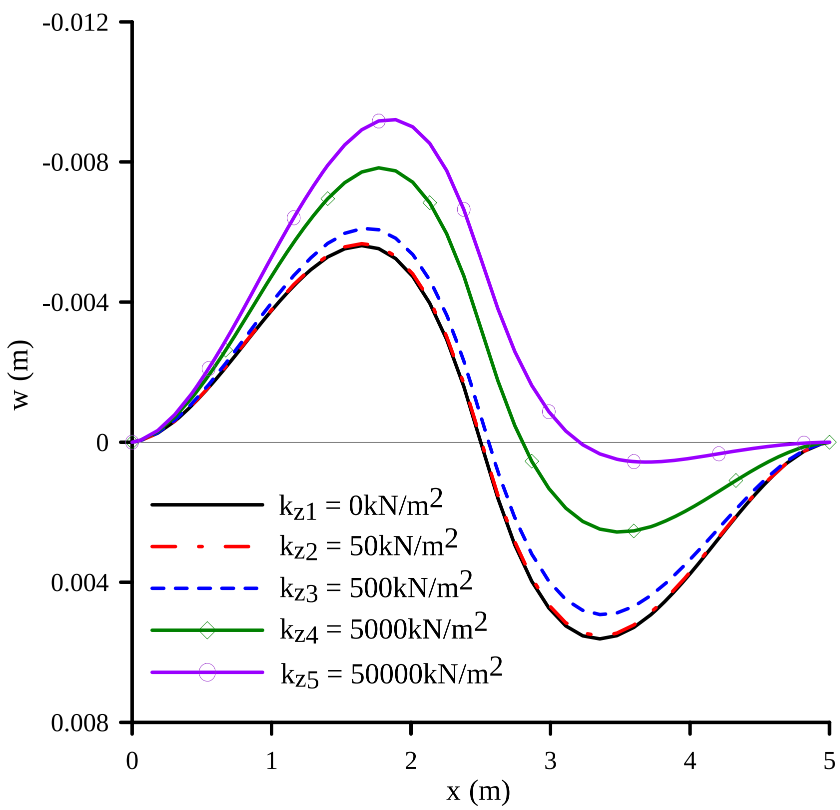

Figure 13. Deflection curves of the beam-column of example 2 (case ii) for various values of the modulus k_z of the tensionless subgrade reaction

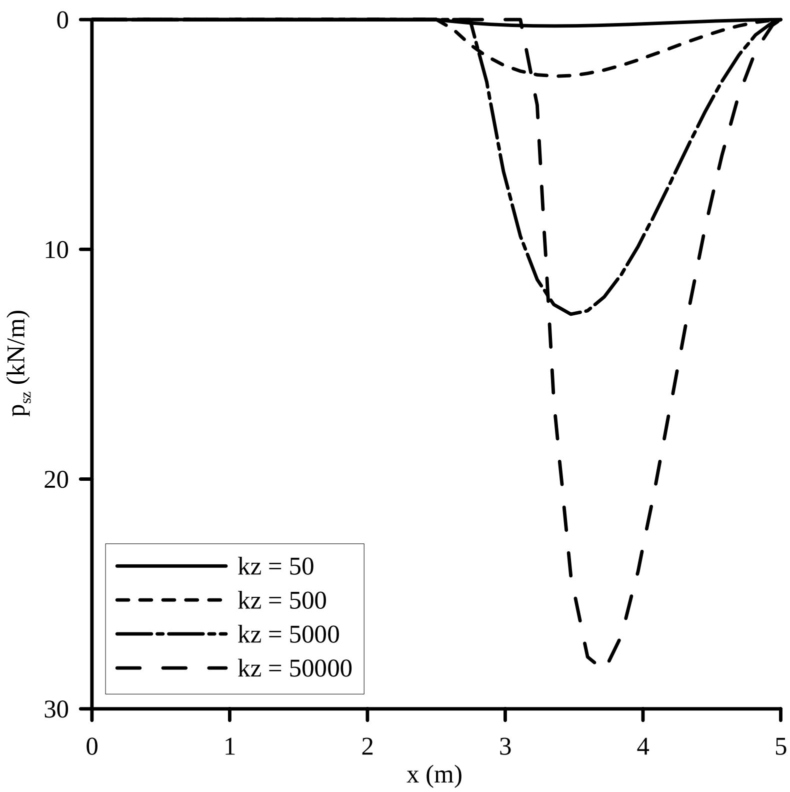

Figure 14. Elastic foundation reaction of example 2 (case ii) for various values of the modulus k_z

Table 5. Extreme values of the displacements and the foundation reaction of the beam-column of example 2 (case ii)

k_z (kN/m^2)	Bilateral Winkler			Unilateral Winkler		
	Min w (mm)	Max w (mm)	Max p_{sz} (kN/m)	Min w (mm)	Max w (mm)	Max p_{sz} (kN/m)
$5{\cdot}10^1$	-5.59	5.59	0.279	-5.67	5.53	0.276
$5{\cdot}10^2$	-5.39	5.39	2.694	-6.10	4.92	2.459
$5{\cdot}10^3$	-4.01	4.01	20.007	-7.84	2.56	12.822
$5{\cdot}10^4$	-1.45	1.45	72.675	-9.21	0.57	28.327

To illustrate the importance of the tensionless character of the subgrade reaction, the same beam-column subjected to a concentrated moment $M_y = -100kNm$ at its midpoint (case ii) is also studied. In Figures 13, 14 the deflection curves of the beam-column resting on a tensionless foundation and the foundation reaction are presented, respectively for various values of the modulus k_z of the subgrade reaction, performing a nonlinear analysis and taking into account shear deformation effect. Additionally, in Table 5 the extreme values of the displacements and the soil reaction are presented for both cases of bilateral and unilateral soil reaction for various values of the modulus k_z performing a geometrical nonlinear analysis and taking into account shear deformation effect. From the aforementioned figure and table, it is concluded that the unilateral character of the foundation is of paramount importance and the error occurred from the ignorance of this behavior is considerable.

4.3 Example 3

In this example, results obtained from the proposed method are compared with the experimental data obtained by Kerisel and Adam (1967) and the BEM-FEM coupling formulation presented by Filho *et al*. (2005). More specifically, a pile of length $l = 4.65m$ ($E = 2 \cdot 10^7 kN/m^2$, $d = 0.3573m$) is driven into clay soil ($E = 9233kN/m^2$, $\nu = 0.3$) (measured experimentally, taking the mean over the first three metres) and is subjected to a concentrated horizontal force $P_z(0) = 60kN$ and to bending moment $M_y(0) = 69kNm$ at its head. In Figure 15 the displacements w along the pile are presented as compared with those obtained from the aforementioned research efforts. Good agreement can be verified between the experimental data, the BEM-FEM coupling formulation and the proposed model.

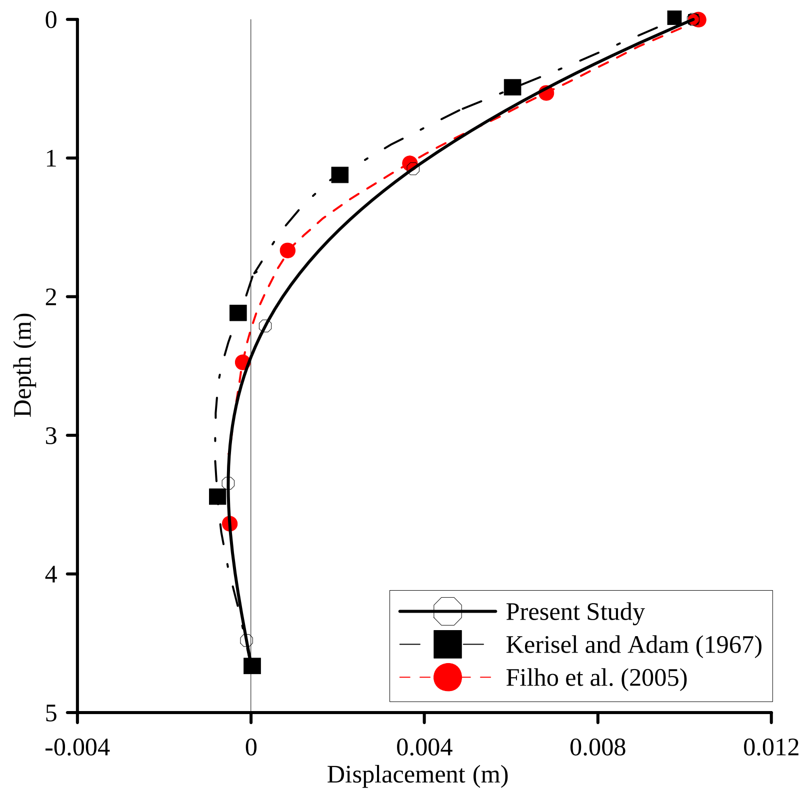

Figure 15. Displacement w along the pile of example 3

4.4 Example 4

To demonstrate the range of applications of the proposed method, a HEM 120 free-free beam-column of length $l = 6m$ ($E = 210\,GPa$, $a_y = 1.409$, $a_z = 4.026$, $v = 0.3$) resting on a constant stiffness soil of $k_z = 1500kN/m^2$ is considered. The beam-column is subjected to a uniformly distributed load $p_z = 1.0kN/m$ and to a concentrated force $P_z(l/2) = 50kN$ at its midpoint. The beam-column is also subjected to concentrated axial forces at its ends $P_x(l) = -P_x(0) = 150kN$.

In Figure 16 the displacement along the beam-column, performing either a linear or a nonlinear analysis, taking into account or ignoring the tensionless character of the Winkler springs, is presented. Finally, in Table 6 the minimum displacement values w_{min} and the maximum bending moment ones $M_{y\,max}$, for all cases of analysis are also presented.

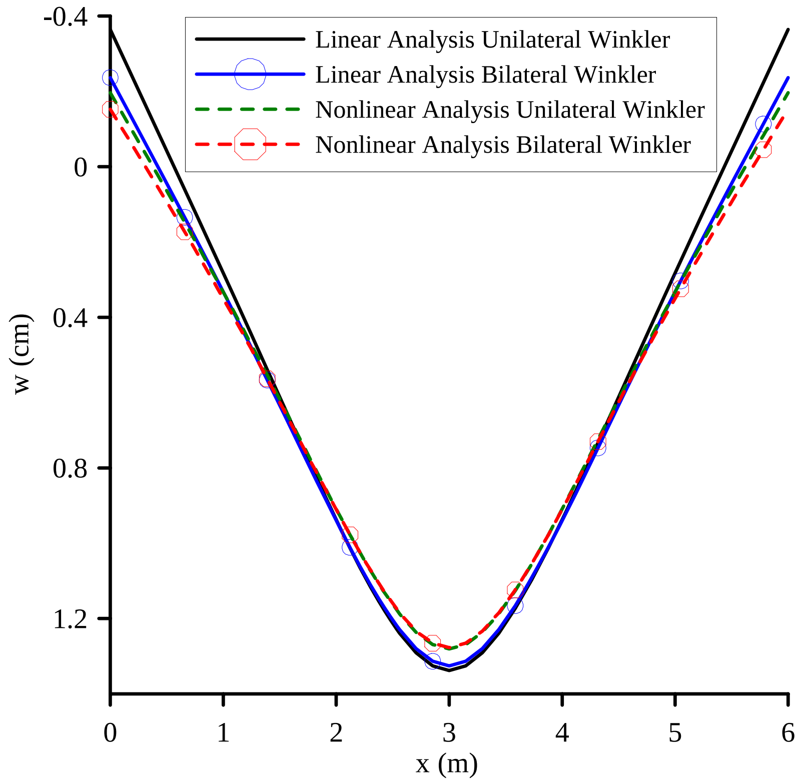

Figure16. Displacement w along beam-column of example 4

Table 6. Minimum values of the displacement $w\ (cm)$ and maximum values of the bending moment $M_y\ (kNm)$ of the beam-column of example 4

	Unilateral Winkler			
	Without Shear Deformation		With Shear Deformation	
	Linear Analysis	Nonlinear Analysis	Linear Analysis	Nonlinear Analysis
w_{min}	-0.380	-0.208	-0.364	-0.196
$M_{y\,max}$	17.8	16.9	15.4	14.5
	Bilateral Winkler			
	Without Shear Deformation		With Shear Deformation	
	Linear Analysis	Nonlinear Analysis	Linear Analysis	Nonlinear Analysis
w_{min}	-0.244	-0.159	-0.237	-0.152
$M_{y\,max}$	17.4	16.7	15.0	14.4

5. Concluding remarks

In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on tensionless Winkler foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. The main conclusions that can be drawn from this investigation are

1. The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary simply or multiply connected doubly symmetric cross section.
2. In some cases, the effect of shear deformation is significant, especially for low beam slenderness values.
3. The discrepancy of the obtained results performing a linear or a nonlinear analysis is remarkable.
4. The significant influence of the unilateral character of the foundation in both the deflections and the soil reaction, especially in the case of a stiff soil is demonstrated.
5. The importance of the soil stiffness to the response of the beam–column is verified.
6. The developed procedure retains most of the advantages of a BEM solution over a FEM approach, although it requires domain discretization.

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