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Cycle multiplicity of total graph of C_n, P_n, and $K_{1,n}$

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Abstract

Cycle multiplicity of a graph *G* is the maximum number of edge disjoint cycles in *G*. In this paper, we find the cycle multiplicity of total graph of cycles C_n, paths P_n, and star graph $K_{1,n}$ respectively.

Keywords: cycle multiplicity, total graph, cycle, path, star graph.

1. Introduction

Line partition number (Chatrand *et al*., 1971) of a graph *G* is the minimum number of subsets into which the edge-set of *G* can be partitioned so that the subgraph induced by each subset has property *P*. Dual to this concept of line partition number of graph is the maximum number of subsets into which the edge -set of *G* can be partitioned such that the subgraph induced by each subset does not have the property *P*. Define the property *P* such that a graph *G* has the property *P* if *G* contains no subgraph which is homeomorphic from the complete graph K_3. Now the line partition number and dual line partition number corresponding to the property *P* is referred to as arboricity and cycle multiplicity of *G* respectively. Equivalently the cycle multiplicity is the maximum number of line disjoint subgraphs contained in *G* so that each subgraph is not acyclic. This number is called the cycle multiplicity of *G* denoted by *CM(G)*. The formula for cycle multiplicity of a complete and complete bipartite graph is given in (Chatrand *et al*., 1971). In (Simões Pereira, 1972), the author found an upper bound for the line and middle graph of any graph. Also he proved that the bound becomes the formula for line and total graph of any forest.

We consider finite, simple, undirected graph $G(V(G), E(G))$ where $V(G)$ and $E(G)$ represent vertex set and edge set of *G* respectively. For any real number *r*, $[r]$ and $\lceil r \rceil$ denote the largest integer not exceeding *r* and the least integer not less than r, respectively. The other notations and terminology used in this paper can be found in (Harary, 1969).

Let *G* be a graph with vertex set *V(G)* and edge set *E(G)*. The total graph (Michalak, 1981) of *G*, denoted by *T(G)* is defined as follows. The vertex set of *T(G)* is $V(G) \cup E(G)$. Two vertices *x, y* in the vertex set of *T(G)* are adjacent in *T(G)* in case one of the following holds: (i) *x, y* are in *V(G)* and *x* is adjacent to *y* in *G*. (ii) *x, y* are in *E(G)* and *x, y* are adjacent in *G* (iii) *x* is in *V(G), y* is in *E(G)*, and *x, y* are incident in *G*

2. Cycle multiplicity of total graph of C_n

It is obvious that cycle multiplicity of any cycle is one. We obtain a formula to find the cycle multiplicity of the total graph of a cycle.

Theorem 2.1

Cycle multiplicity of Total Graph of *n*-Cycle, $CM[T(C_n)] = n+1$

Proof

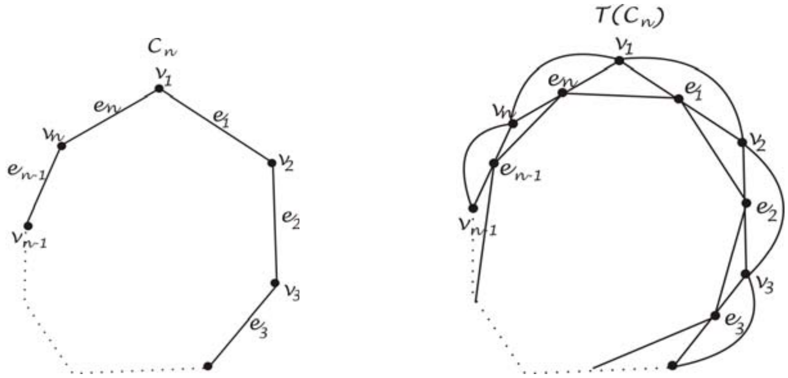

Figure 1. n-cycle and its Total Graph

Let $V(C_n) = \{v_1, v_2,\ldots,v_n\}$ and $E(C_n) = \{ e_1, e_2,\ldots,e_n, \}$ in which $e_i = v_iv_{i+1}$. By the definition of total graph, $V[T(C_n)] = \{v_1, v_2,\ldots,v_n\} \cup \{ e_1,e_2,\ldots,e_n\}$ and $E[T(C_n)] = \{ e_ie_{i+1} / (1 \le i \le n-1) \} \cup e_ne_1 \cup \{ v_iv_{i+1} / 1 \le i \le n-1\} \cup v_nv_1 \cup \{ e_iv_{i+1} /1 \le i \le n-1\} \cup e_nv_1 \cup \{v_ie_i / 1 \le i \le n \}$. The cycles of $T(C_n)$ are $C_i = e_iv_{i+1}e_{i+1}$ $(1 \le i \le n-1)$, $C_n = e_ne_1v_1$, $C_i^{|} = v_iv_{i+1}e_i$ $(1 \le i \le n-1)$, $C_n^{|} = v_nv_1e_n$. Let $C_{n+1} = v_1v_2,\ldots,v_nv_1$, $C_{n+2} = e_1e_2\ldots,e_ne_1$, $C_{n+3} = v_1e_1v_2e_2v_3,\ldots,v_ne_nv_1$. Now we collect set of line disjoint cycles, $\mathcal{C}_1 = \{C_i / (1 \le i \le n-1)\} \cup \{ C_n\} \cup \{C_{n+1}\}$, $\mathcal{C}_2 = \{C_i^{|} /1 \le i \le n-1\} \cup \{ C_n^{|} \} \cup \{ C_{n+2}\}$, $\mathcal{C}_3 = \{C_{n+1}, C_{n+2}, C_{n+3} \}$. Clearly $\mathcal{C}_i (1 \le i \le 3)$ is a set of line disjoint cycles in $T(C_n)$ and $|\mathcal{C}_1| = |\mathcal{C}_2| = n+1$. Since $n \ge 3$, $|\mathcal{C}_1|$ or $|\mathcal{C}_2| \ge |\mathcal{C}_3|$ and either $\mathcal{C}_1$ or $\mathcal{C}_2$ contains maximum number of line disjoint cycles of $T(C_n)$ and hence $CM[T(C_n)] = n+1$.

3. Cycle multiplicity of total graph of P_n

As P_n does not contain any cycle, its cycle multiplicity is zero. In the following theorem we states a formula to find the maximum number of line disjoint cycles in the total graph of a path.

Theorem 3.1

Cycle multiplicity of total graph of path, $CM[T(P_n)] = n$

.**Proof**

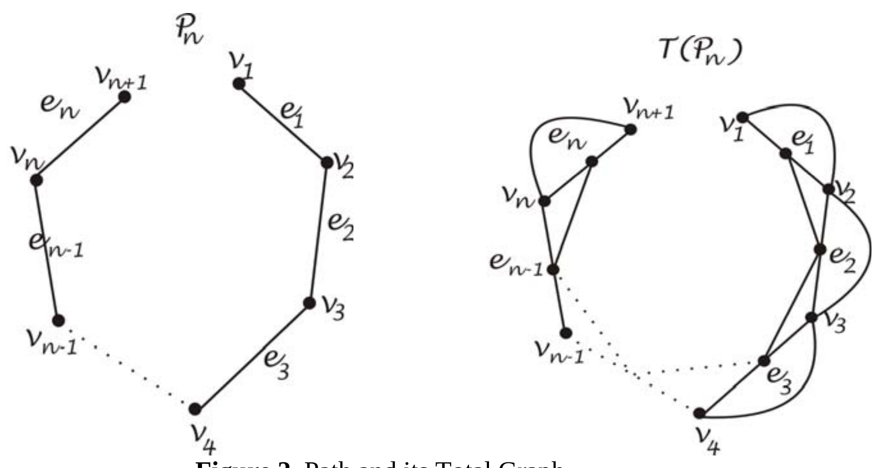

Figure 2. Path and its Total Graph

Let $V(P_n) = \{ v_1, v_2,\ldots,v_{n+1} \}$and $E(C_n) = \{ e_1, e_2,\ldots,e_n, \}$ By the definition of total graph, $V[T(P_n)] = V(P_n) \cup E(P_n)$, $E[T(P_n)] = \{v_ie_i / (1 \le i \le n) \} \cup \{e_iv_{i+1} / 1 \le i \le n \} \cup \{ v_iv_{i+1} /1 \le i \le n \} \cup \{ e_ie_{i+1} / 1 \le i \le n-1\}$ The cycles of $T(P_n)$ are

$C_i = v_i v_{i+1} e_i$ $(1 \le i \le n)$ and $C_i^{|} = e_i e_{i+1} v_{i+1}$ $(1 \le i \le n\text{-}1)$. Let $\mathcal{C}_1 = \{C_i /(1 \le i \le n)\}$ and $\mathcal{C}_2 = \{C_i^{|} / 1 \le i \le n-1\}$. The cycles in the set $\mathcal{C}_i$ ($i = 1,2$) are line disjoint cycles of $T(P_n)$. Also $|\mathcal{C}_2| < |\mathcal{C}_1| = n$ and hence $CM[T(P_n)] = n$

4. Cycle multiplicity of total graph of $K_{1,n}$

Since the star graphs are acyclic its cycle multiplicity is zero. We find a formula for the cycle multiplicity of total graph of a star graph

Theorem 4.1

Cycle multiplicity of total graph of $K_{m,n}$, $CM[T(K_{1,n})] = \begin{cases} \left[\dfrac{n^2+5n}{6}\right] \text{ if } n \text{ is odd} \\ \left[\dfrac{n(n+4)}{6}\right] \text{ if } n \text{ is even} \end{cases}$

Proof

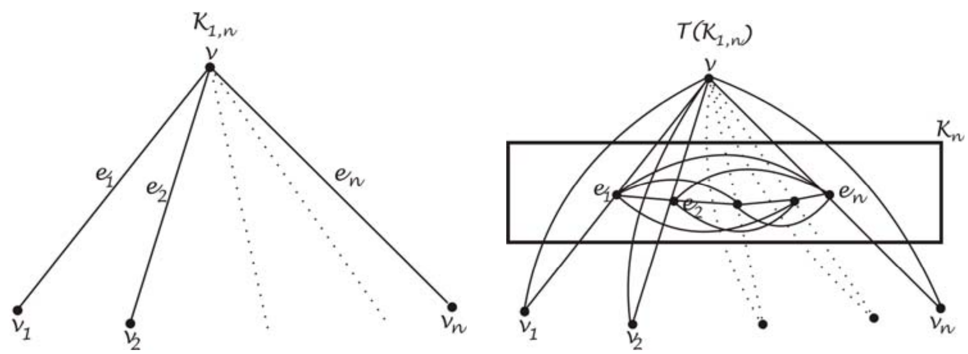

Figure 3. Star graph and its Total Graph

Let $V(K_{1,n}) = \{v_1, v_2, \ldots\ldots, v_n\}$ and $E(K_{1,n}) = \{ e_1, e_2, \ldots\ldots, e_n, \}$ By the definition of total graph, we have $V[T(K_{1,n})] = \{v\} \cup \{e_i / (1 \le i \le n)\} \cup \{v_i / (1 \le i \le n)\}$, in which the vertices $e_1, e_2, \ldots\ldots, e_n$ induces a cliques of order n (say K_n). Also the vertex v is adjacent with v_i $(1 \le i \le n)$.

Case (i)

If n is odd

We collect the set of line disjoint cycles of $T(K_{1,n})$ as below.

$\mathcal{C}_1 = \{v e_i e_{i+1} v / (i = 1, 3, \ldots\ldots, n-2)\}$, $\mathcal{C}_2 = \{v e_i e_{i+1} v / i = 2, 4, \ldots\ldots, n-1\}$, $\mathcal{C}_3$ = {set of line disjoint cycles in the clique K_n }. $\mathcal{C}_4 = \{v e_i v_i v / (1 \le i \le n) \}$, Clearly $|\mathcal{C}_1| = |\mathcal{C}_2| = \dfrac{n-1}{2}$.

To prove $|\mathcal{C}_3| = \left[\dfrac{n^2-n}{6}\right]$, i.e., we have to prove the number of line disjoint cycles in $\mathcal{C}_3$ is $\left[\dfrac{n^2-n}{6}\right]$ if n is odd. If $n = 1$, the number of line disjoint cycles in the clique K_1 is zero and $\left[\dfrac{n^2-n}{6}\right] = 0$ for $n = 1$. Similarly if $n = 3$, then the number of line disjoint cycles in the clique K_3 is one and $\left\lceil \frac{n^2-n}{6} \right\rceil = 1$ for $n = 2$. Therefore $|\mathcal{C}_3| = \left\lceil \frac{n^2-n}{6} \right\rceil$ if $n = 1, 3$. Assume that the result is true for $m = 2k-1$ for some k. i.e., $|\mathcal{C}_3| = \left\lceil \frac{2k^2-3k+1}{3} \right\rceil$, i.e., Number of line disjoint cycles in $K_m = |\mathcal{C}_3| = \left\lceil \frac{2k^2-3k+1}{3} \right\rceil$.

Now consider the clique K_n where $n = 2k + 1$. Consider $K_{n-2} = K_n - \{ e_{2k}, e_{2k+1}\} = K_{2k-1} = K_m$. Number of line disjoint cycles in K_{n-2} is $\left\lceil \frac{2k^2-3k+1}{3} \right\rceil$. Also the number of line disjoint cycles is decreased by $\left\lceil \frac{4k-1}{3} \right\rceil$. Therefore $|\mathcal{C}_3|$ in K_n is $\left\lceil \frac{2k^2-3k+1}{3} \right\rceil + \left\lceil \frac{4k-1}{3} \right\rceil$. i.e., $|\mathcal{C}_3| = \left\lceil \frac{2k^2+k}{3} \right\rceil$, i.e. $|\mathcal{C}_3| = \left\lceil \frac{n^2-n}{6} \right\rceil$, where $n = 2k+1$. Since n is odd there exist no edges in the clique which are left out in the extraction of line disjoint cycles. Since $n \geq 2$, $\frac{n-2}{2} \leq \left\lceil \frac{n^2-n}{6} \right\rceil$. Therefore $|\mathcal{C}_1| = |\mathcal{C}_2| \leq |\mathcal{C}_3|$. The cycles in $\mathcal{C}_3$ and $\mathcal{C}_4$ are line disjoint. Therefore maximum number of line disjoint cycles in $T(K_{1,n})$,

$CM[T(K_{1,n})] = |\mathcal{C}_3| + |\mathcal{C}_4| = \left\lceil \frac{n^2-n}{6} \right\rceil + n = \left\lceil \frac{n^2+5n}{6} \right\rceil$ if n is odd.

Case (ii)

If n is even

In this case we collect the set of line disjoint cycles as below.

$\mathcal{C}_1 = \{ve_ie_{i+1}v \,/\, i = 1, 3,\ldots\ldots,n-1)\}$, $\mathcal{C}_2 = \{ve_ie_{i+1}v \,/\, i = 2, 4,\ldots\ldots,n-2\}$, clearly $|\mathcal{C}_1| = \frac{n}{2}$ and $|\mathcal{C}_2| = \frac{n-2}{2}$ $\mathcal{C}_3$ = {set of line disjoint cycles in K_n }. $\mathcal{C}_4 = \{ve_iv_iv \,/\, (1 \leq i \leq n)\}$, We prove $|\mathcal{C}_3| = \frac{n(n-2)}{6}$. Maximum number of line disjoint cycles are extracted from K_n using the following steps.

Step 1:

Extract the line disjoint cycles $c_i = e_ie_{i+1}e_{i+2}e_i$ $(i = 1, 3, 5,\ldots\ldots n-1)$. Clearly $c_1, c_2,\ldots\ldots\ldots\ldots,c_{n-1}$ are line disjoint cycles. Thus we got $\frac{n}{2}$ line disjoint cycles.

Step 2:

Delete the edges $e_i e_{\left(\frac{n}{2}+i\right)}$ $(i = 1, 2,\ldots\ldots\ldots, \frac{n}{2})$ from K_n.

Step 3:

Extract $\left\lceil \frac{n^2-5n}{6} \right\rceil$ line disjoint 3–cycles from $K_n - \{ e_i e_{\left(\frac{n}{2}+i\right)}$ $(i = 1, 2,\ldots\ldots\ldots, \frac{n}{2})$ }.

Therefore $\frac{n}{2} + \frac{n^2-5n}{6} = \left\lfloor \frac{n(n-2)}{6} \right\rfloor$. Since $e_i e_{\left(\frac{n}{2}+i\right)}$ $(i = 1, 2, \ldots\ldots\ldots, \frac{n}{2})$ are mutually non adjacent edges in $T(K_{1,n})$. Let $\mathcal{C}_5$ $= \{v\, e_i e_{\left(\frac{n}{2}+i\right)} v \,/\, (i = 1, 2, \ldots\ldots\ldots, \frac{n}{2})\}$. The cycles in $\mathcal{C}_5$ are line disjoint. The cycles in $\mathcal{C}_3$ and $\mathcal{C}_5$ are line disjoint and also the cycles in $\mathcal{C}_3$ and $\mathcal{C}_4$ are line disjoint . Since $|\mathcal{C}_5| \leq |\mathcal{C}_4|$. Therefore maximum number of line disjoint cycles in $T(K_{1,n})$, $CM[T(K_{1,n})] = |\mathcal{C}_3| + |\mathcal{C}_4| = \left\lfloor \frac{n(n-2)}{6} \right\rfloor + n = \left\lfloor \frac{n(n+4)}{6} \right\rfloor$.

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Biographical notes

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Dr. S. Panayappan has 25 years of research experience and completed 5 research projects. He is presently serving as Principal investigator of University Grant Commission's Major project at Government Arts College Coimbatore. 45 M.Phil and 5 PhD scholars have been awarded the degree under his supervision and he is guiding 8 PhD scholars at present. He has Published 25 papers in National and International Journals/Proceedings. He has organized 3 conferences, including one graph theory conference. Area of interest includes Graph theory and Operator Theory.

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