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Additional chapter for evaluation indeterminate limits of functions and series in teaching mathematics for engineering education

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Abstract

Infinity divided by infinity, zero divided by zero, one divided by zero are the most important indeterminate forms obtained when evaluating limits for single variable functions and series. Well-known method; L' Hôpital rule has been employed to simplify and resolve these indeterminate forms' limits such that 0/0, ∞/∞ , 1/0 in terms of quotients of their derivatives. In some cases, L' Hôpital rule is applied more than once to solve indeterminate limits. Besides, it is so complicated to take the derivative for some functions of a single variable and series, so L' Hôpital rule is ineffective and not practical to solve limits with indeterminate forms for those functions and series. L' Hôpital rule is also impractical for the indeterminate limits in the form: -∞·0By considering all these facts, new approaches including Central Finite Difference (CFD), Forward Finite Difference (FFD), Backward Finite Difference (BFD), High Accurate Central Finite Difference (HACFD), High Accurate Forward Finite Difference (HAFFD), High Accurate Backward Finite Difference (HABFD) methods are presented that provides efficient ways to solve these limits. Taking derivative is not required in all approaches. HACFD with step size: 0.0001 is the most preferred technique here to obtain exact results among all other techniques.

Keywords: Indeterminate limits: 0/0, ∞/∞ , 1/0, -∞·0; L' Hôpital rule; Central, Forward, Backward Finite Differences; High Accurate Central, Forward, Backward Finite Differences.

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1. Introduction

In calculus, the most famous and well-known method is L' Hôpital rule for evaluating the limits of indeterminate forms: 0/0, ∞/∞ , 1/0. This method has been employed comprehensively in the literature for this purpose and also for various applications (Hassani 2024, Marinescu, Monea 2024, Rakhmatullayevich, Mirvaliyevna 2024, Boas 2018, Matveeva, Ryzhkova 2017, Arkhipov et al 1999, Gulmaro 2018, Huang 1988, Szabó 1989, Výborný Nester 1989, Hartig 1991, Sheldon 2017, Cooke, 1988, Vianello 1993, Tian 1993, Muntean 1993, Takeuchi 1995, Spigler, Vianello 1993, Durán 1992, Shishkina 2007, Popa 1999, Aczél 1990, Estrada, Pavlović 2017, Zlobec 2012, Chapra, Canale 2010.). L' Hôpital rule is a significant and only way to overcome the complexity of indeterminate limit forms: 0/0, ∞/∞, 1/0 by use of differentiation of both numerator and denominator. It is the only method which is always educated in mathematics lectures and also mathematics books.

Nevertheless, it is such a difficult and lengthy process that takes derivative for some functions and series. So applying L' Hôpital rule becomes inefficient and impractical method in these cases. Moreover, L' Hôpital rule does not work for indeterminate form: -

∞·0. Zlobec (Zlobec, 2012) used L' Hôpital rule without derivative by employing Lagrange multiplier for only 0/0 indeterminate limit form.

In this study, new approaches including FFD, BFD, HAFFD and HABFD techniques can be applied to only forms: 0/0, 1/0 of both functions and series. In other respects; new methods with use of CFD, HACFD techniques are proposed for all forms: 0/0, ∞/∞ , 1/0, -∞·0 of both functions and series that L'Hôpital rule is impractical for evaluations. These concepts beside L'Hôpital rule can be conveniently given in mathematics lectures. Taking derivative and using Lagrange multiplier are not required in all approaches. The approaches including CFD, HACFD here, can be applied to all indeterminate limits conveniently. After description of the methods proposed, numerical results from various complicated functions and series are presented to prove the applicability of them.

2. Proposed Methods

Well-known rule; L'Hôpital defines the limit of a quotient of functions:

$$\lim_{x\to x^*}\frac{f(x)}{g(x)} \tag{1}$$

Where $\lim_{x\to x^*} f(x) = 0 \ or \ \infty$ and $\lim_{x\to x^*} g(x) = 0 \ or \ \infty$ is equal to the limit of the quotient of their derivatives:

$$\lim_{x\to x^*}\frac{f'(x)}{g'(x)} \tag{2}$$

Provided that eq (2) exists. So eq (1) becomes

$$\lim_{x\to x^*}\frac{f(x)}{g(x)} = \frac{f'(x^*)}{g'(x^*)} \tag{3}$$

Eq (3) is applicable, if $g'(x^*)$ exists and it is not null, then $\lim_{x\to x^*}\frac{f(x)-f(x^*)}{g(x)-g(x^*)}$ exists if and only if $f'(x^*)$ exists. So one can obtain (Gulmaro, 2018)

$$\lim_{x\to x^*}\frac{f(x)-f(x^*)}{g(x)-g(x^*)} = \frac{f'}{g'}(x^*) \tag{4}$$

L'Hôpital rule confirms the rules of derivation. So this method can be considered as being one more rule of derivation. The substitution of the ratio of the derivatives by the limit of the ratio of the derivatives to the number where the limit will be calculated, actually necessitates that the derivative of the functions is defined in an interval contained in the domain of the functions and also necessitates that it is extended continuously to the number where the limit is calculated (Gulmaro, 2018). In some cases, taking derivative is ineffective and impractical. It is required to apply L'Hôpital rule more than once to overcome indeterminate limits. For this reason, new approaches including CFD and HACFD methods are proposed in this paper. The originating idea of CFD and HAFCD techniques is based on well-known Taylor series.

The form of the Taylor series by defining a step size $h = x_{i+1} - x_i$ and expressing as

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n \tag{5}$$

where the remainder term is defined as

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!}h^{n+1} \tag{6}$$

The term in eq (6) corresponds to $O((x_{i+1} - x_i)^{n+1})$ which is $O(h^{n+1})$ called as error.

For backward form, Taylor series in eq (5) can be rewritten as

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots \tag{7}$$

One of the ways to approximate the first derivative is to subtract eq (7) from the Taylor series expansion in eq (5) to obtain:

$$f(x_{i+1}) = f(x_{i-1}) + 2f'(x_i)h + \frac{2f^{(3)}(x_i)}{3!}h^3 + \cdots \tag{8}$$

which can be solved for

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - \frac{f^{(3)}(x_i)}{3!}h^2 - \cdots \tag{9}$$

Eq (9) can be also expressed as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2) \tag{10}$$

By employing eq (10), one of the method including CFD for solving indeterminate limit is formulated as

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \frac{\frac{f(x^*{}_{i+1}) - f(x^*{}_{i-1})}{2h} + O(h^2)}{\frac{g(x^*{}_{i+1}) - g(x^*{}_{i-1})}{2h} + O(h^2)} \tag{11}$$

Both numerator and denominator in terms of CFD have errors which are $O(h^2)$. This means that errors are proportional to the square of the same step size for both $f(x)$ and $g(x)$. Error is of the order of h^2 in spite of the forward and backward approximations that are of the order of h. Therefore, Taylor series approximations yield the practical information that the centered difference is the most accurate representation of the derivative (Chapra, Canale, 2010). Level of accuracy depends on both decreasing the step size and also the number of terms of the Taylor series during the derivation of these formulas. Hence, it is possible to develop more accurate formulas called as HACFD by withholding more terms. By substituting first order derivative in eq (10) into eq (5), centered finite difference representation of the second order derivative based on error $O(h^2)$ can be found as

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \tag{12}$$

Third order derivative based on error $O(h^2)$:

$$f^{(3)}(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3} \tag{13}$$

To find the high-accurate form of first derivative based on error $O(h^4)$, one should use both eq (12) and eq (13), and substitute them into eq (5):

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}}{2!}h^2 + \frac{\frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3})}{3!}h^3 + \cdots \tag{14}$$

So $f'(x_i)$ based on error $O(h^4)$ can be obtained from eq (14) which is

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + O(h^4) \tag{15}$$

By employing eq (15), another method including HACFD for solving indeterminate limit is formulated as

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \frac{\frac{-f(x^*{}_{i+2}) + 8f(x^*{}_{i+1}) - 8f(x^*{}_{i-1}) + f(x^*{}_{i-2})}{12h} + O(h^4)}{\frac{-g(x^*{}_{i+2}) + 8g(x^*{}_{i+1}) - 8g(x^*{}_{i-1}) + g(x^*{}_{i-2})}{12h} + O(h^4)} \tag{16}$$

With similar way, first derivative by FFD based on O(h) is (Chapra, Canale, 2010)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h) \tag{17}$$

By use of eq (17), FFD technique for solving indeterminate limit is obtained as

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \frac{\frac{f(x^*{}_{i+1}) - f(x^*{}_{i})}{h} + O(h)}{\frac{g(x^*{}_{i+1}) - g(x^*{}_{i})}{h} + O(h)} \tag{18}$$

More accurate form of first derivative by FFD based on $O(h^2)$ can be written as (Chapra, Canale, 2010)

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2) \tag{19}$$

By employing eq (19), HAFFD method for solving indeterminate limit become

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \frac{\frac{-f(x^*{}_{i+2}) + 4f(x^*{}_{i+1}) - 3f(x^*{}_{i})}{2h} + O(h^2)}{\frac{-g(x^*{}_{i+2}) + 4g(x^*{}_{i+1}) - 3g(x^*{}_{i})}{2h} + O(h^2)} \tag{20}$$

Similarly, first derivative by BFD based on O(h) is (Chapra, Canale, 2010)

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h) \tag{21}$$

By use of eq (19), BFD technique for solving indeterminate limit can be written as

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \frac{\frac{f(x^*{}_{i}) - f(x^*{}_{i-1})}{h} + O(h)}{\frac{g(x^*{}_{i}) - g(x^*{}_{i-1})}{h} + O(h)} \tag{22}$$

More accurate form of first derivative by BFD based on $O(h^2)$ can be obtained as (Chapra, Canale, 2010)

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + O(h^2) \tag{23}$$

By employing eq (23), BAFFD method for solving indeterminate limit become

$$\lim_{x\to x^*}\frac{f(x)}{g(x)}=\frac{\frac{3f(x^*_i)-4f(x^*_{i-1})+f(x^*_{i-2})}{2h}+O(h^2)}{\frac{3g(x^*_i)-4g(x^*_{i-1})+g(x^*_{i-2})}{2h}+O(h^2)} \quad (24)$$

Both numerator and denominator in terms of HACFD have errors which are (h^4). This means that errors are proportional to the fourth power of the same step size for both $f(x)$ and $g(x)$. Error is of the order of h^4 in spite of the forward and backward approximations that are of the order of h^2. So, HACFD is the most accurate identification of the derivative among all other methods.

3. Proposed Future Studies

Well-known rule; L'Hôpital rule for two variables functions has some steps and shown as follows

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)}{g(x,y)} \quad (25)$$

Where $\lim_{(x,y)\to(x_0,y_0)} \mathrm{f(x,y)} = 0$ *or* a number and $\lim_{(x,y)\to(x_0,y_0)} \mathrm{g(x,y)} = 0$. So one should take partial derivative with respect to x and y, at points $x_0,\ y_0$ respectively

$$\frac{f_x(x_0,y_0)}{g_x(x_0,y_0)} \quad (26a)$$

$$\frac{f_y(x_0,y_0)}{g_y(x_0,y_0)} \quad (26b)$$

Provided that eq (26a) should be equal to eq (26b) such that

$$\frac{f_x(x_0,y_0)}{g_x(x_0,y_0)}=\frac{f_y(x_0,y_0)}{g_y(x_0,y_0)}=k_1 \quad (27)$$

Currently, if indeterminate form is found again as a result of eq (27), second order partial derivatives should be taken:

$$\frac{f_{xx}(x_0,y_0)}{g_{xx}(x_0,y_0)}=\frac{f_{xy}(x_0,y_0)}{g_{xy}(x_0,y_0)}=\frac{f_{yx}(x_0,y_0)}{g_{yx}(x_0,y_0)}=\frac{f_{yy}(x_0,y_0)}{g_{yy}(x_0,y_0)}=k_2 \quad (28)$$

L'Hôpital rule for multivariable functions confirms the rules of partial derivation. So this method can be considered as being one more rule of partial derivation. Nevertheless, this is still not adequate for case of isolated and non isolated singularities. In some cases such as isolated and non isolated singularities, taking derivative is ineffective and impractical. Furthermore, sometimes, it is required to apply L'Hôpital rule more than once to overcome indeterminate limits. For this reason, proposed methods in this study can be applied to multivariate functions, correspondingly. The assumptions have been made by changing x variable and not changing y variable in Taylor series expansions for all proposed methods. On the contrary, changing y variable and not changing x variable in Taylor series expansions do not change the results. The originating idea of CFD, HACFD, FFD, HAFFD, BFD and HABFD techniques can be conveniently used which are based on well-known Taylor series.

Furthermore, as candidates for future work: indeterminate forms of complex functions can be elaborated on. The methods used in this study can be tested for complex functions. Other candidate can be various forms of indeterminate forms for functions can be considered. These forms can be ∞^∞, 0^∞, ∞^0. Each of the methods used in this study can be adjusted to be able to employ in form: ∞^∞, 0^∞, ∞^0. Similarly another candidate may be different forms of indeterminate forms for series can be studied. Those forms are ∞^∞, 0^∞, ∞^0. Each of the methods employed in this paper may be adjusted to be able to employ in form: ∞^∞, 0^∞, ∞^0 for series.

4. Results and Discussion

Examples for indeterminate limit forms: : 0/0, 1/0 are presented in Tables 1 and 2, respectively. Corresponding numerical results for these examples are displayed in Figures 1 and 2. Examples and numerical results for indeterminate form: ∞/∞ are also illustrated in Table 3. A general code is generated for application of the proposed methods in Matlab R2022a. All results are computed with use of step size: h=0.0001.

Table 1. Examples for indeterminate form: $\frac{0}{0}$

Example	Limit
1	$\lim_{x\to 5}\frac{\frac{1}{x}-\frac{1}{5}}{x-5}$

2	$\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}$
3	$\lim_{x\to 1}\frac{\ln x}{x-1}$
4	$\lim_{x\to 0}\frac{x-\text{atan}(x)}{x}$
5	$\lim_{x\to 1}\frac{\sin(x-1)-\tan(x-1)}{x-1}$
6	$\lim_{x\to 0}\frac{(x)}{\sin(x)}$

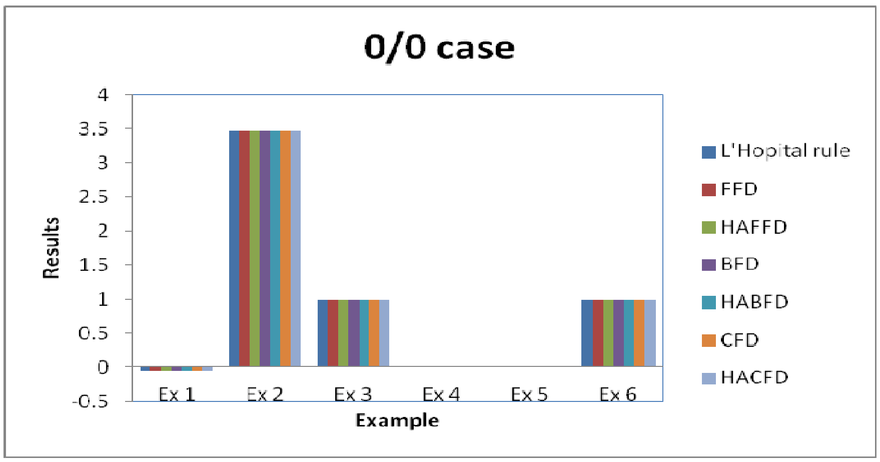

Figure 1. Numerical results for indeterminate form: 0/0 with h=0.0001

Table 2. Examples for indeterminate form: $\frac{1}{0}$

Example	Limit
1	$\lim_{x\to 0}\frac{1}{\sin(x)}$
2	$\lim_{x\to 0}\frac{1}{\tan(x)}$
3	$\lim_{x\to 1}\frac{e^{(x-1)}}{\sin(x-1)}$
4	$\lim_{x\to 2}\frac{e^{(x-2)}}{\tan(x-2)}$
5	$\lim_{x\to 3}\frac{(x-2)}{\ln(x-2)}$
6	$\lim_{x\to 3}\frac{(x-2)}{\tan(\sin((x-3)))}$

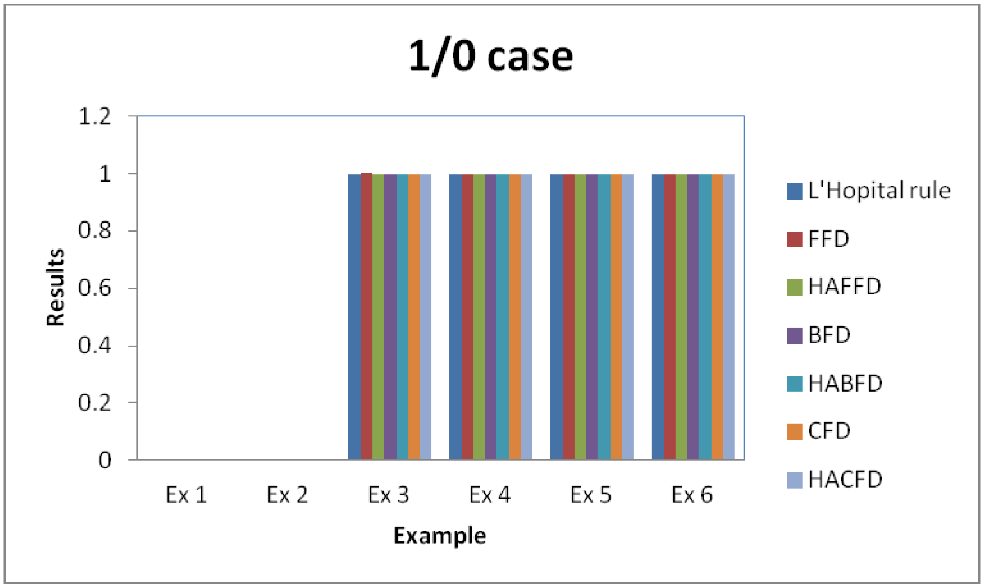

Figure 2. Numerical results for indeterminate form: 1/0 with h=0.0001

With use of h=0.0001, exact results for the limits can be obtained by all methods.

Table 3. Numerical results for indeterminate form: $\frac{\infty}{\infty}$

Example	Exact Result	L' Hôpital Rule	CFD	HACFD
$\lim_{x\to 0}\frac{-\ln(x)}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	0	0	0
$\lim_{x\to 0}\frac{-\log(x)}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	0	0	0
$\lim_{x\to 0}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	-	0	0
$\lim_{x\to 0}\frac{\tan\left(\sin\left(\frac{1}{x}\right)\right)}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	-	0	0
$\lim_{x\to 0}\frac{e^{\sin\left(\frac{1}{x}\right)}}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	-	0	0
$\lim_{x\to 0}\frac{e^{\sin\left(\cos\left(\frac{1}{x}\right)\right)}}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	-	0	0
$\lim_{x\to 0}\frac{e^{\cos\left(\sin\left(\frac{1}{x}\right)\right)}}{\frac{1}{x}}$	$\frac{\infty}{\infty}$	-	0	0
$\lim_{x\to 3}\frac{e^{\cos\left(\sin\left(\frac{1}{x-3}\right)\right)}}{\frac{1}{x-3}}$	$\frac{\infty}{\infty}$	-	0	0
$\lim_{x\to 0}\ln(x)\sin(x)$	$-\infty\cdot 0$	-	0	0

To Table 3; for indeterminate limit forms: ∞/∞, -∞·0 and inapplicability to L' Hôpital rule, only CFD and HACFD techniques can be used conveniently. Moreover, the effect of decreasing step size significantly causes accuracy for all methods proposed. This situation is proved by Figures 3, 4 and 5 for each indeterminate form: 0/0, 1/0 of functions and also series respectively.

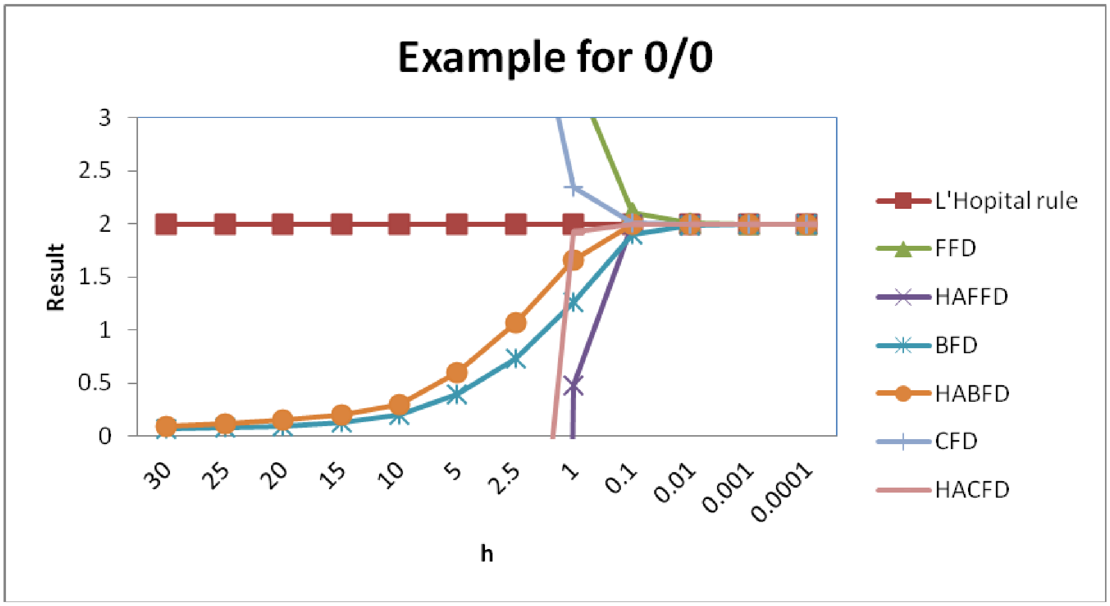

Figure 3. Comparison of all methods for example: $\lim_{x\to 1}\frac{2(e^{x-1}-1)}{x-1}$

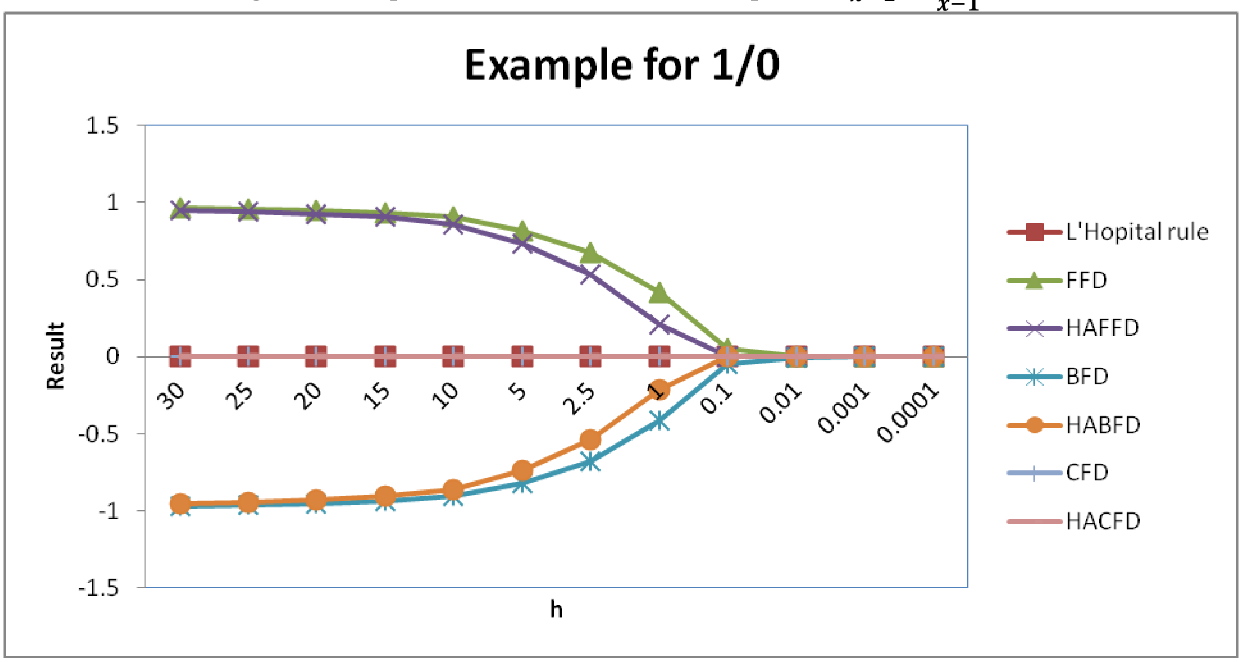

Figure 4. Comparison of all methods for example: $\lim_{x\to 0}\frac{\sqrt{1+x^2}}{x}$

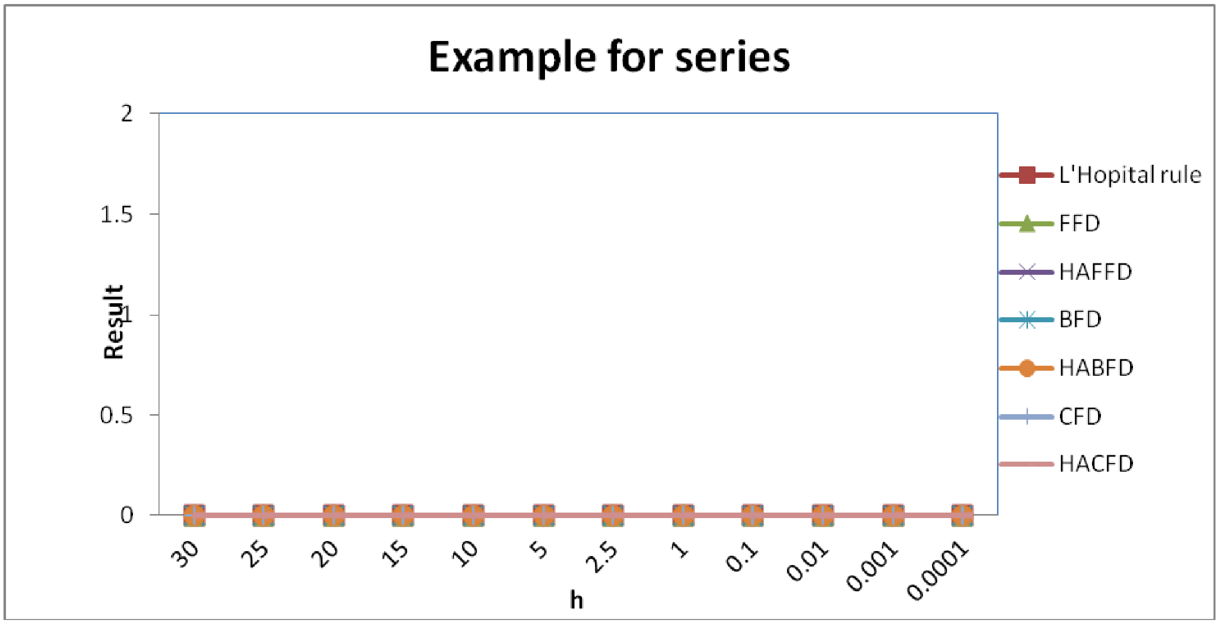

Figure 5. Comparison of all methods for example: $\lim_{x\to 0}\frac{cos(x)-sin\left(\frac{\pi}{2}-x\right)}{2x}$

To Figures 3, 4, 5, as h is to be decreased in all methods, accuracy leads to an increase. HACFD is the most accurate formulation even if the step size is not so reduced. Exact results can be obtained for all examples.

5. Conclusions

This paper presents alternative methods with use of CFD, BFD, FFD, HACFD, HABFD and HAFFD for various indeterminate limit forms including: 0/0, ∞/∞ , 1/0, -∞·0 in teaching mathematics. So, there is no need to use L' Hôpital rule with taking derivatives. Functions and series are selected according to the difficulty and not applicability of L' Hôpital rule. Numerical results of various indeterminate limit forms for some functions and series are given for proof of the purpose of this study. It is the first time in the literature that finite difference techniques are employed for indeterminate limits in mathematics courses. It is also shown that among the other finite difference techniques, HACFD can be applied to all forms: 0/0, ∞/∞ , 1/0, -∞·0 of both functions of single variable and series with the highest accuracy that L'Hôpital rule is impractical and complicated for evaluations of limits. Mathematics teachers can conveniently teach these methods as well as L' Hôpital rule for indeterminate limit chapter.

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