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The pitfalls of the unconventional process capability indices

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Abstract

For assessing capability of a normal process with upper specification limit (USL) conventionally C_{pu} index is estimated to facilitate better decision making in product and process management. But, in practice, many quality characteristics having USL only, e.g. count data, proportion defective etc. are discrete and follow Poisson or binomial distributions. Some unconventional indices (e.g. C_u, C_{fu}, C_{pcu} and C_{pyu}) are proposed in literature for assessing capability of Poisson or binomial processes. Due to legacy of usages of C_{pu} index and its interpretations, a user of an unconventional index often tends to interpret its values with reference to the values of C_{pu} for the bad, good or highly capable normal processes, and get a false impression about the capability of the concerned Poisson or binomial process. In this paper, the key features of those unconventional indices are highlighted and then some numerical analysis is carried out for assessing the interpretation issues associated with these unconventional indices. The results of these analyses reveal that although there is no interpretation issue for the unconventional index C_u, there are serious interpretation issues with all other unconventional indices. The mathematical relationships of estimates of other unconventional indices with the estimate of C_u index are established. It is recommended to convert the estimates of other unconventional indices into estimated C_u value using those relationships before any decision making. Otherwise, users of the other unconventional indices may inadvertently be led to erroneous decision making.

Keywords: Unconventional PCI, C_{pu} index, C_u index, C_{fu} index, C_{pcu} index, C_{pyu} index

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1. Introduction

The behavior of a process is often described by a probability distribution. In order to assess its adequacy, the hypothesized distribution has to be compared with the corresponding specifications. Process capability index (PCI) measures the extent of variation a process experiences relative to its specification limits. There is a large body of literature dealing with PCIs. Mention may be made of the books by Kotz and Johnson (1993), Kotz and Lovelace (1998), Pearn and Kotz (2006), Ryan (2011), Polhemus (2018) and Chakraborty and Chatterjee (2021). The indices help in the prevention of nonconforming products by establishing a benchmark capability. Being dimensionless, they facilitate communication between engineering and manufacturing departments and between manufacturers and suppliers. They aid in establishing the priority areas for process improvement and continuous improvement.

The most basic PCI for quality characteristic with both-sided specifications is C_p, which is defined as the quotient between the length of the acceptance level and six times the process standard deviation. Thus, if USL and LSL are the upper and lower specification limits respectively and σ is the population standard deviation, then

$$C_p = (USL - LSL)/6\sigma \quad (1)$$

When there is only USL or only LSL for a quality characteristic, then the process capability indices are defined as

$$C_{pu} = (USL - \mu)/3\sigma \quad (2)$$

$$C_{pl} = (\mu - LSL)/3\sigma \quad (3)$$

where μ is the population mean. Commonly used other indices are C_{pk}, C_{pm}, and C_{pmk}. Computation of all these indices requires assumption that the quality characteristic is a continuous variable and follows normal distribution. The details about these indices are available in Kane (1986), Kotz and Johnson (2002), Yum and Kim (2011), Chen et al. (2017) and Yum (2023). The generalization of these indices for continuous but non-normal variables is suggested by Clements (1989), Chen (2000), Kovarik and Sarga (2014), and Safder et al. (2019).

In practice, many quality characteristics in manufacturing and service set ups are attribute in nature (Gauri and Pal, 2020). For example, number of defects in 100 square metre cloths, number of customers served per hour, proportion of improperly sealed orange juice can, proportion of defective purchase orders etc. The attribute data are typically obtained by counting the number of occurrences of some condition (e.g. defect, error etc.) in an inspection unit or by counting number of defective units (r) within a given number of sample units(n), and so, these data are discrete in nature. It is well established that attribute data usually follow Poisson or binomial distribution (Montgomery, 2019). These attribute data generally have upper specification limit only. The standard formulas (that are developed for normal processes with one-sided specification) cannot be used for computation of capability indices of a process involving such characteristics.

In order to overcome the problem, some alternative indices are proposed in literature for measuring capability of a process involving attribute quality characteristic with USL. These are C_u index (Borges and Ho, 2001), C_{fu} index (Yeh and Bhattacharya, 1998), C_{pcu} index (Perakis and Xekalaki, 2002, 2005) and C_{pyu} index (Maiti et al., 2010). The procedures for computation of these indices are different from the standard practice for computation of process capability indices from a normal process. Thus these are referred to as unconventional indices. The indices C_{fu} , C_{pcu} and C_{pyu} are computed as the ratio of two probabilities, and thus these indices can be computed for both continuous as well as discrete quality characteristics, and no assumption is required on the distributions of these quality characteristics. On the other hand, C_u is computed by mapping the expected proportion of nonconformance above USL of a characteristic to the Z-score in the right side of standard normal distribution. That is, C_u responds to changes in the nonconforming region and not to changes in the distribution of the observed quality characteristic. Consequently, computations of all these indices are feasible in any process regardless of whether the quality characteristics are discrete or continuous and their probability distributions. Pal and Gauri (2020[a], 2020[b]) compared performance of these unconventional indices in evaluating capabilities of Poisson and binomial processes. Gauri and Pal (2020) showed that the same value of different unconventional (called generalized) indices computed from a Poisson or binomial process and the C_{pu} value computed from a normal process signifies different capabilities for these processes and thus there exists interpretation issues for the unconventional indices.

In fact, over the years we have been accustomed to evaluate the process capability indices from normal processes and interpret the same. For example, the capability of a process is considered good if $\hat{C}_{pu} \geq 1$ and the capability is considered very good if $\hat{C}_{pu} \geq 1.33$. Due to legacy of usages of conventional process capability indices and its interpretations, a user of an unconventional index may tend to interpret its values with reference to the values of the conventional process capability indices for the bad, good or very good capable normal processes, and thus he/she may unknowingly arrive at an erroneous decision based on the estimate of an unconventional index. Consequently, product and process management may become inefficient. This understanding motivates us to critically analyze the pros and cons of different unconventional indices in quantifying the capability of a process, and develop ways to overcome the problems of difficult to interpret unconventional indices. The article is organized as follows: The methods for computation of different unconventional indices along with their key features are described in Section 2. Some numerical analysis, which reveals usefulness and limitations of different unconventional indices, are presented in section 3. In section 4, mathematical relationships of some unconventional indices with C_u index are established. Section 5 concludes the paper.

2. Different Unconventional Process Capability Indices

The unconventional indices for process capability proposed by Borges and Ho (2001), Yeh and Bhattacharya (1998), Perakis and Xekalaki (2002, 2005) and Maiti et al. (2010) and their variants for unilateral specification are presented in the following subsections.

For computation of estimates of many of the unconventional indices from a Poisson process we need to compute first expected proportion of nonconforming units with respect to USL ($\widehat{PNU}_U$) and expected proportion of nonconforming units with respect to LSL ($\widehat{PNU}_L$). For convenience, let us consider that a single unit of product represents an inspection unit. Suppose number of occurrences of the events (e.g. defects or errors) is observed in each of the m units collected from in-control process. Let the random variable K denotes the number of occurrences of the event in a unit, and k_i is the number of events occurred in the i^{th} unit ($i = 1,2,3 \dots, m$). Then, the unknown parameter λ can be estimated as $\hat{\lambda} = \bar{k} = \sum_{i=1}^{m} k_i/m$, and the values of $\widehat{PNU}_U$ and $\widehat{PNU}_L$can be obtained as follows:

$$\widehat{PNU}_U = P\{k > k_U\} = 1 - P\{k \le k_U\} = 1 - \sum_{k=0}^{k_U} e^{-\bar{k}}(\bar{k})^k/k! \quad (4)$$

$$\widehat{PNU}_L = P\{k < k_L\} = \sum_{k=0}^{k_L-1} e^{-\bar{k}}(\bar{k})^k/k! \quad (5)$$

where k_U and k_L are the USL and LSL of number of defects in a unit product.

Similarly, for computation of estimates of many unconventional indices from a binomial process, we need to compute first expected proportion of nonconforming lots with respect to USL ($\widehat{PNL}_U$) and expected proportion of nonconforming lots with respect to LSL ($\widehat{PNL}_L$). Let a production process is operating in a stable manner, such that the probability that any unit will be nonconforming to specification is p and successive units produced are independent. Suppose a random sample of n units of product is selected from the process and the number of nonconforming products observed is d, i.e. sample fraction nonconforming is $f = d/n$. If the random variable D denotes the number of units of product that are nonconforming to the specification, then D has a binomial distribution with parameters n and p. The cumulative distribution function of sample fraction nonconforming, $f = d/n$ can be obtained by using the binomial distribution as

$$P\{f \le a\} = P\left\{\frac{d}{n} \le a\right\} = P\{d \le na\} = \sum_{d=0}^{[na]} \binom{n}{d} p^d (1-p)^{n-d} \quad (6)$$

where $[na]$ denotes the largest integer less than equal to na. It can be shown that $E(f) = p$ and $E(\sigma_f^2) = p(1-p)/n$ (Montgomery, 2009). If f is STB type, then it will have only USL (say, $USL = f_U$) and if f is LTB type, then it will have only LSL (say, $LSL = f_L$).

Suppose, m samples of size n_i (i = 1, 2, 3,..,m) are collected from a stable process and number of defectives observed in i^{th} sample is d_i. Then, fractions nonconforming in the i^{th} sample is $f_i = d_i/n$ ($i = 1,2,3, \dots, m$) and the unknown parameter p is estimated as $\hat{p} = \bar{f} = (\sum_{i=1}^{m} d_i)/mn$ and average sample size is estimated as $\bar{n} = (\sum_{i=1}^{m} n_i)/m$. So the values of $\widehat{PNL}_U$ and $\widehat{PNL}_L$ can be obtained as follows:

$$\widehat{PNL}_U = P\{f > f_U\} = 1 - P\{D \le \bar{n}f_U\} = 1 - \sum_{d=0}^{[\bar{n}f_U]} \binom{\bar{n}}{d} \bar{f}^d (1-\bar{f})^{\bar{n}-d} \quad (7)$$

$$\widehat{PNL}_L = P\{f < f_L\} = P\{D \le \bar{n}f_L\} = \sum_{d=0}^{[\bar{n}f_L]} \binom{\bar{n}}{d} \bar{f}^d (1-\bar{f})^{\bar{n}-d} \quad (8)$$

2.1 C index

Borges and Ho (2001) suggested a new measure of process capability, called C index, which has one-to-one correspondence (mapping) between the proportion of nonconformance (π) and Z-value of the standard normal distribution. The C index for a quality characteristic X is defined as follows:

$$C = \frac{1}{3} \times \Phi^{-1}\left(1 - \frac{\pi}{2}\right) \quad (9)$$

where, $\pi = 1 - P(LSL \le X \le USL)$ is the proportion of nonconformance in in-control process.

In case there is only USL, the expected proportion of nonconformance of a characteristic above USL is mapped to the Z-score in the right side of standard normal distribution, and 1/3rd of this Z-score is considered as the measure of the process capability with respect to USL and it is denoted as C_u. Thus, the estimates of C_u from a Poisson process and from a binomial process can be obtained using equations (10) and (11) respectively.

$$\hat{C}_u = (1/3) \times \Phi^{-1}\left(1 - \widehat{PNU}_U\right) \quad (10)$$

$$\hat{C}_u = (1/3) \times \Phi^{-1}\left(1 - \widehat{PNL}_U\right) \quad (11)$$

Similarly, the estimates of process capability with respect to LSL (denoted as C_l) can be obtained from Poisson and binomial processes.

Features of C_u index

- It may be noted from equations (10-11) that if $\widehat{PNU}_U$ or $\widehat{PNL}_L$ is greater than 0.5, then $\hat{C}_u$ will become negative which is not meaningful. Therefore, $\hat{C}_u$ is to be considered as zero if $\widehat{PNU}_U$ or $\widehat{PNL}_L$ is greater than equal to 0.5.
- The value of $\hat{C}_u$ will be reasonably high for a very highly capable process. For example, if the proportion of nonconformance is 0.00001, the value of $\hat{C}_u$ will be 1.42.
- The value of $\hat{C}_u$ will be one if the proportion of nonconformance in the process is 0.00135.
- The expected proportion of conformance (EPC) in the concerned Poisson or binomial process can be obtained based on the estimate of C_u using the following equation:

$$EPC = \Phi(3\hat{C}_u) \quad (12)$$

2.2 C_{pc} index

Perakis and Xekalaki (2002, 2005) proposed to measure C_{pc} index for assessing capability of a process. Saha et al. (2022) discussed estimation of C_{pc} index when the process follows exponentiated exponential distribution. The index C_{pc} is defined as follows:

$$C_{pc} = (1 - p_0)/(1 - p) \quad (13)$$

where, p_0 is the minimum allowable proportion of conformance and p is the actual proportion of conformance. If the characteristic has USL only, then the process capability index can be represented as C_{pcu}, and it can be defined as follows:

$$C_{pcu} = (1 - p_0^U)/(1 - p_U) \quad (14)$$

where, p_0^U is the desired proportion of conformance with respect to USL, and p_U is the actual proportion of conformance with respect to USL.

Perakis and Xekalaki (2002, 2005) recommend that 0.9973 is a good choice for the desired proportion of conformance (p_0) for both sided specifications and thus, when only USL is specified then a good choice for the minimum allowable proportion of conformance with respect to USL would be p_0^U = 0.99865, i.e. $1 - p_0^U = 0.00135$. On the other hand, the actual proportion of nonconformance with respect to USL for Poisson and binomial processes are obtained as $\widehat{PNU}_U$ and $\widehat{PNU}_L$ respectively. Thus, the estimates of C_{pcu} from the Poisson process and binomial processes can be obtained using equations (15) and (16) respectively.

$$\hat{C}_{pcu} = 0.00135/\widehat{PNU}_U \quad (15)$$

$$\hat{C}_{pcu} = 0.00135/\widehat{PNL}_U \quad (16)$$

Similarly, the estimates of process capability with respect to LSL (denoted as C_{pcl}) can be obtained from Poisson and binomial processes.

Features of C_{pcu} index

- In general, the value of $\widehat{PNU}_U$ or $\widehat{PNL}_U$ is expected to be very low and so the estimate of C_{pcu} is derived from the ratio of two very small numbers. Thus, the estimate is highly impacted due to a minor deviation in the value of actual proportion of nonconformance from the acceptable proportion of nonconformance.
- The value of $\hat{C}_{pcu}$ will be unreasonably very high for a highly capable process. For example, if the proportion of nonconformance is 0.00001, the value of $\hat{C}_{pcu}$ will be 135.
- Again, the value of $\hat{C}_{pcu}$ will be unreasonably small even for a moderately capable process. For example, if the proportion of nonconformance is 0.01, the value of $\hat{C}_{pcu}$ will be 0.135.
- The value of $\hat{C}_{pcu}$ will be one if proportion of nonconformance in the process is 0.00135.
- The expected proportion of conformance (EPC) in the concerned Poisson or binomial process can be obtained based on the estimate of C_{pcu} using the following equation:

  $$EPC = 1 - 0.00135/\hat{C}_{pcu}, \text{ where } \hat{C}_{pcu} \geq 0.00135 \quad (17)$$

- The value of $\widehat{PNU}_U$ or $\widehat{PNL}_U$ cannot be more than 1, and hence estimated $\hat{C}_{pcu}$ value will always be more than equal to 0.00135.

It may be mentioned here that Yeh and Bhattacharya (1998) defined C_f index for measuring the capability of a process as follows:

$$C_f = min(\alpha_0^L/\alpha_L, \alpha_0^U/\alpha_U) \quad (18)$$

where, α_0^U and α_0^L are the proportions of nonconformance the manufacturer can tolerate on the USL and LSL respectively, and α_U and α_L are the actual proportion of nonconformance with respect to USL and LSL respectively. If the characteristic has USL only, then the process capability index can be represented as C_{fu}, where

$$C_{fu} = \alpha_0^U/\alpha_U \quad (19)$$

Yeh and Bhattacharya (1998) recommended that a good choice for α_0^U should be 0.00135. On the other hand, the actual proportion of nonconformance with respect to USL for Poisson and binomial processes are obtained as $\widehat{PNU}_U$ and $\widehat{PNU}_L$ respectively. Thus, in case of unilateral specification, the $\hat{C}_{fu}$ index becomes essentially the same as $\hat{C}_{pcu}$ index. Similarly, the $\hat{C}_{fl}$ becomes essentially the same as $\hat{C}_{pcl}$ index. Therefore, only $\hat{C}_{pcu}$ index (not $\hat{C}_{fu}$ index) is taken into consideration for subsequent analysis/discussions.

2.3 C_{py} index

Maiti et al. (2010) proposed C_{py} index as a measure of process capability. Saha et al. (2019) addressed different methods of estimation of C_{py} index from both frequentist and Bayesian view points of generalized Lindley distribution. EL-Sagheer and Hasaallah (2020) discussed inference issues of C_{py} index in 3-Burr XII distribution. The C_{py} index is defined as follows:

$$C_{py} = \frac{F(U) - F(L)}{1 - \alpha_0^U - \alpha_0^L} \quad (20)$$

where, $F(U)$ and $F(L)$ are cumulative probability distribution function of the quality characteristic at USL and LSL respectively, and α_0^U and α_0^L are the maximum allowable proportion of nonconformance at upper tail and lower tail of the distribution of the quality characteristic. Here the numerator, $F(U) - F(L)$, gives the measure of the actual process yield (i.e. actual proportion of conformance) and the denominator, $(1 - \alpha_0^U - \alpha_0^L)$ gives the measure of the desired process yield (i.e. desired proportion of conformance).

Maiti et al. (2010) suggested that in case of unilateral specification, median of the distribution (μ_e) should be taken as the process target and the process centre should be located such that $F(\mu_e) = [F(U) + F(L)]/2 = 1/2 = 0.5$, and the value of α_0^U is conventionally taken as 0.00135. On the other hand, the cumulative probability up to USL (i.e. $F(U)$) for Poisson and binomial processes are obtained as $1 - \widehat{PNU}_U$ and $1 - \widehat{PNU}_L$ respectively. Therefore, the estimate of C_{pyu} can be obtained using equations (21) and (22) respectively.

$$\hat{C}_{pyu} = \frac{F(U) - F(\mu_e)}{1 - \alpha_0^U - F(\mu_e)} = \frac{1 - \widehat{PNU}_U - 0.5}{1 - 0.00135 - 0.5} = \frac{0.5 - \widehat{PNU}_U}{0.49865} \quad (21)$$

$$\hat{C}_{pyu} = \frac{F(U) - F(\mu_e)}{1 - \alpha_0^U - F(\mu_e)} = \frac{1 - \widehat{PNU}_U - 0.5}{1 - 0.00135 \quad .5} = \frac{0.5 - \widehat{PNL}_U}{0.49865} \quad (22)$$

Features of C_{pcu} index

- It may be noted from equations (21-22) that if $\widehat{PNU}_U$ or $\widehat{PNL}_L$ is greater than 0.5, then $\hat{C}_{pyu}$ will become negative which is not meaningful. Therefore, $\hat{C}_{pyu}$ is to be considered as zero if $\widehat{PNU}_U$ or $\widehat{PNL}_L$ is greater than equal to 0.5.
- The value of $\hat{C}_{pyu}$ will be unreasonably small even for a very highly capable process. For example, if the proportion of nonconformance is 0.00001, the value of $\hat{C}_{pyu}$ will be 1.0027.
- Again, the value of $\hat{C}_{pyu}$ will be unreasonably high even for a poorly capable process. For example, if the proportion of nonconformance is 0.30, the value of $\hat{C}_{pyu}$ will be 0.4011.
- The value of $\hat{C}_{pyu}$ will be one if the proportion of nonconformance in the process is 0.00135.
- The expected proportion of conformance (EPC) in the concerned Poisson or binomial process can be obtained based on the estimate of C_{pyu} using the following equation:

$$EPC = 0.5 + \hat{C}_{pyu} \times 0.49865 \quad (23)$$

3. Analysis and Discussions

Since introduction of the concept of process capability index, conventionally the indices C_{pu} and C_{pl} are estimated from normal processes for one-sided specification to facilitate better decision making in product and process management. Suppose in a normal process, the proportion of nonconformance of a characteristic X with respect to USL is PNC_U. Then,

$$PNC_U = P(X > USL) = 1 - P(X \le USL) = 1 - P\left(z \le 3 \times \frac{USL - \hat{\mu}}{3\hat{\sigma}}\right) = 1 - \Phi(3 \times \hat{C}_{pu})$$

$$\Rightarrow \hat{C}_{pu} = \frac{\Phi^{-1}(1 - PNC_U)}{3} \quad (24)$$

This implies that there is one to one correspondence between the proportion of nonconformance and the estimated C_{pu} value in case of a normal process. Similarly, it can be shown that there is one to one correspondence between the proportion of nonconformance and the estimated C_{pl} value in case of a normal process. Over the years, process managers, engineers and other decision makers have become accustomed to relate the estimates of process capability indices and the expected proportion of product conformance to specifications taking into account the normal processes. Accordingly, general thumb rule being followed among the users of the indices is that the capability of a process is good if $\hat{C}_{pu} \ge 1$ and the capability is very good if $\hat{C}_{pu} \ge 1.33$.

Therefore, it is desired that the values of the unconventional indices with respect to USL computed from Poisson or binomial processes should match as closely as possible with C_{pu} values corresponding to different values of proportion of nonconformance with respect to USL in a normal process. Otherwise, the users (process managers, design engineers, vendors, customers etc.) of the unconventional indices may unknowingly get a false impression about the capability of the concerned Poisson or binomial process, which may lead him/her to erroneous decision making. Consequently, product and process management may become inefficient.

In order to assess the usefulness of the unconventional indices, it is decided to compute the estimates of different unconventional indices corresponding to different values of proportion of nonconformance with respect to USL in the Poisson or binomial processes, and also to compute the estimate of C_{pu} values corresponding to the same proportion of nonconformance with respect to USL in a normal process, and then to compare the estimated unconventional and conventional indices with respect to the same one-sided proportion of nonconformance. To facilitate comparison, we categorize the process performance into following two categories based on the proportion of nonconformance produced in a process:

1) Capable process if $PNC_U \le 0.00135$
2) Incapable process if $PNC_U > 0.00135$

For a capable normal process, we arbitrarily consider some possible proportion of nonconformance with respect to USL (PNC_U) ranging from 0.00135 to 0.00001, and for each chosen value of proportion of nonconformance, the expected C_{pu} value is computed using equation (24). Similarly, for the same chosen value of proportion of nonconformance a capable Poisson (or binomial) process, the expected values of unconventional indices i.e. $\hat{C}_u$, $\hat{C}_{pcu}$, and $\hat{C}_{pyu}$ are computed using equations 10 (or 11), 15 (or 16) and 21 (or 22) respectively. The expected values of C_{pu} index in a capable normal process and expected values of unconventional indices, i.e. $\hat{C}_u$, $\hat{C}_{pcu}$, and $\hat{C}_{pyu}$ in a capable Poisson or binomial process corresponding to different PNC_U values are shown in Table 1.

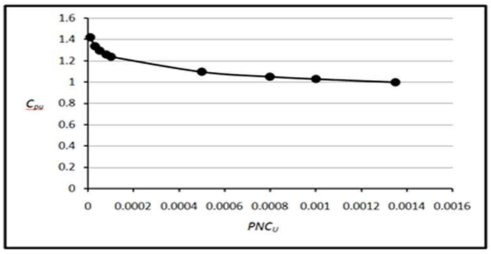
Table 1. Expected values of C_{pu} and unconventional indices (i.e. $\hat{C}_u$, $\hat{C}_{pcu}$, and $\hat{C}_{pyu}$) corresponding to the same PNC_U value in capable normal and Poisson/binomial processes

PNC_U	$\hat{C}_{pu}$	$\hat{C}_u$	$\hat{C}_{pcu}$	$\hat{C}_{pyu}$
0.00001	1.422	1.422	135.0000	1.00269
0.00003	1.338	1.338	45.0000	1.00265
0.00005	1.297	1.297	27.0000	1.00261
0.00008	1.258	1.258	16.8750	1.00255
0.0001	1.240	1.240	13.5000	1.00251
0.0005	1.097	1.097	2.700	1.0017
0.0008	1.052	1.052	1.688	1.0011
0.0010	1.030	1.030	1.350	1.0007
0.00135	1.000	1.000	1.000	1.0000

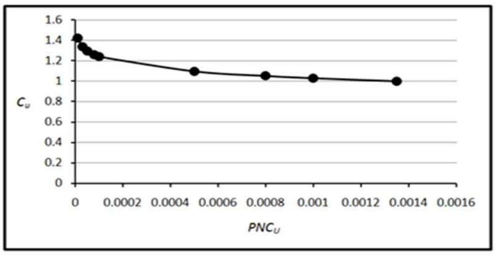
It can be noted from Table 1 that the expected values of C_{pu} and all the unconventional indices are the same (equal to one) when the proportion of nonconformance is 0.00135. As the proportions of nonconformance decrease, the values of $\hat{C}_u$ increase similarly to the values of $\hat{C}_{pu}$. This implies that the estimated $\hat{C}_u$ value from a capable Poisson or binomial process can be interpreted similarly to the interpretation of the estimated $\hat{C}_{pu}$ value from a capable normal process. However, as the proportion of nonconformance decreases the changes in the values of $\hat{C}_{pcu}$ and $\hat{C}_{pyu}$ become completely different from the changes in the values of $\hat{C}_{pu}$. Whereas the values of $\hat{C}_{pcu}$ rapidly increases to abnormally high values with the decrease in proportion of nonconformance (e.g. $\hat{C}_{pcu}$ changes from 1 to 27 when PNC_U value changes from 0.00135 to 0.00005), the values of $\hat{C}_{pyu}$ increases very marginally even when the proportion of nonconformance is very low (e.g. $\hat{C}_{pyu}$ value changes from 1 to 1.00261 when PNC_U value changes from 0.00135 to 0.00005). Therefore, it is very likely that the users of the C_{pcu} and C_{pyu} indices will arrive at a false impression about the capability of the concerned Poisson or binomial process by examining estimates of these indices. For example, if the proportion of nonconformance is 0.0001, the estimated $\hat{C}_{pcu}$ value will be 13.5 (whereas the estimated $\hat{C}_{pu}$ value in a normal process will be 1.24), which gives an impression that the concerned Poisson or binomial process is very highly capable. On the other hand, for the same proportion of nonconformance, i.e. 0.0001, the estimated $\hat{C}_{pyu}$ value will be 1.00251, which gives an impression that the concerned Poisson or binomial process is only little better than the just capable although in reality the process is highly capable.

To understand better the patterns in changes in expected values of C_{pu} and different unconventional indices over the changes in proportion of nonconformance (PNC_U) in capable processes, Figures 1a-1d are prepared. It can be noticed from these figures that only the pattern of changes in the unconventional index C_u over the changes in the proportion of nonconformance matches with the pattern of changes in C_{pu} index (compare Fig. 1a and Fig. 1b). The pattern of changes in the unconventional indices C_{pcu} and C_{pyu} over the changes in the proportion of nonconformance grossly deviates from the pattern of changes in C_{pu} (compare Fig. 1c with Fig. 1a and Fig. 1d with Fig. 1a). This implies that there is high chances of making wrong impressions about a Poisson or binomial process by examining the estimates of C_{pcu} and C_{pyu} indices.

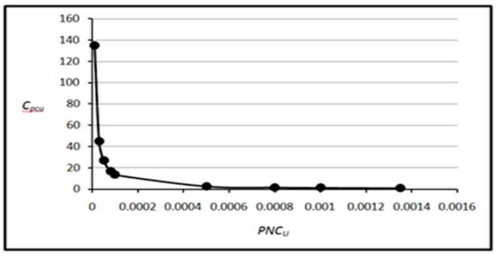
Again for an incapable normal process, we arbitrarily consider some possible proportion of nonconformance with respect to USL (PNC_U) ranging from 0.002 to 0.30, and for each chosen value of proportion of nonconformance, the expected C_{pu} value is computed using equation (24). Similarly, for the same chosen value of proportion of nonconformance in an incapable Poisson (or binomial) process, the expected values of unconventional indices, i.e. C_u, C_{pcu}, and C_{pyu} are computed using equations 10 (or 11), 15 (or 16) and 21 (or 22) respectively. The expected values of C_{pu} index in an incapable normal process and expected values of unconventional indices, i.e. $\hat{C}_u$, $\hat{C}_{pcu}$, and $\hat{C}_{pyu}$ in an incapable Poisson or binomial process corresponding to different PNC_U values are shown in Table 2.



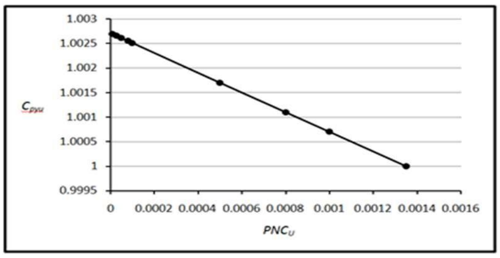
a) Scatter plot of PNC_U vs. C_{pu}



b) Scatter plot of PNC_U vs. C_u



c) Scatter plot of PNC_U vs. C_{pcu}



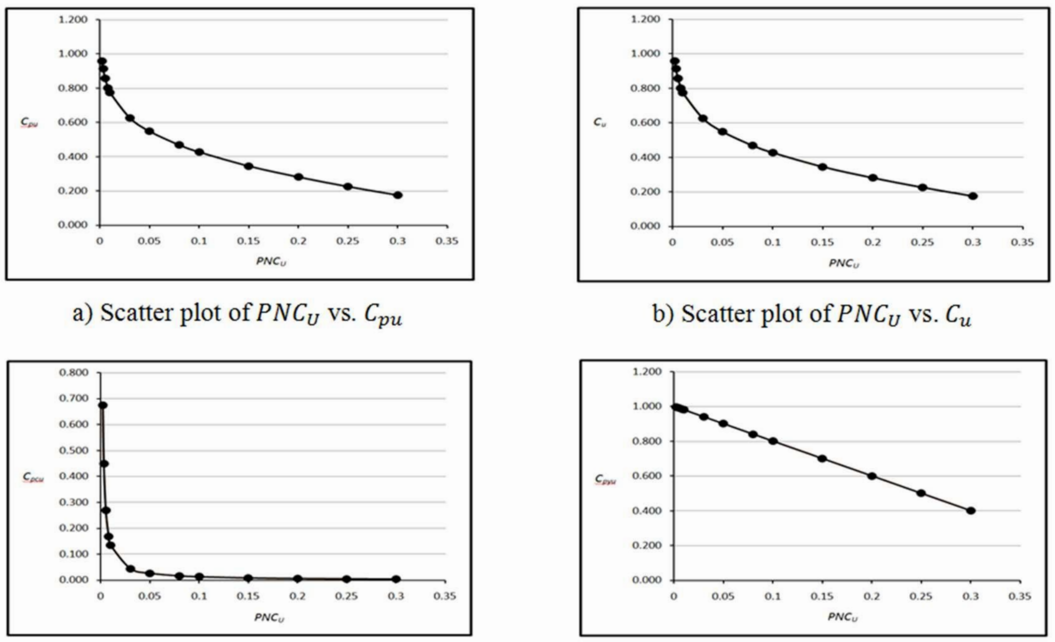
d) Scatter plot of PNC_U vs. C_{pyu}

Fig. 1: Patterns in changes of different indices over the changes in PNC_U values in capable processes

Figures 2a-2d shows the patterns in changes in C_{pu} and different unconventional indices over the changes in PNC_U in incapable processes. Comparison of Fig. 2b and Fig.2a reveals that both the values of C_u and C_{pu} indices decreases with the increase in the values of proportion of nonconformance, and the pattern of decreases in the values of C_u index is similar to the pattern of decreases in the values of C_{pu} index. Therefore, interpretation of the estimated $\hat{C}_u$ value in an incapable Poisson or binomial process will be similar to the estimated $\hat{C}_{pu}$ value for a incapable normal process. On the other hand, comparisons of Fig. 2c and Fig. 2d with Fig 2a reveal that the patterns of changes in the values of C_{pcu} and C_{pyu} indices in incapable Poisson or binomial processes are completely different from the pattern of changes in the values of the C_{pu} index in incapable normal process. Whereas the estimates of C_{pcu} index rapidly decrease to abnormally low values with the increase in proportions of nonconformance, the estimates of C_{pyu} index decrease very slowly in a straight line with the increase in proportions of nonconformance. Therefore, it is very likely that the users of the C_{pcu} and C_{pyu} indices will land into a false impression about the capability of the concerned Poisson or binomial process. For example, if the process nonconformance is 0.01, the estimate of C_{pcu} will be 0.1350 (whereas the estimate of C_{pu} will be 0.775), which gives an impression that the capability of process is very very poor. On the other hand, for the same proportion of nonconformance, i.e. 0.01, the estimate of C_{pyu} will be 0.98265, which gives an impression that the capability of the process is only little inferior to a just capable process although in reality the process is quite inferior to a just capable process.

Table 2. Expected values of C_{pu} and unconventional indices (i.e. $\hat{C}_u$, $\hat{C}_{pcu}$, and $\hat{C}_{pyu}$) corresponding to the same PNC_U value in incapable normal and Poisson/binomial processes

PNC_u	$\hat{C}_{pu}$	$\hat{C}_u$	$\hat{C}_{pcu}$	$\hat{C}_{pyu}$
0.002	0.959	0.959	0.6750	0.99900
0.003	0.916	0.916	0.4500	0.99700
0.005	0.859	0.859	0.2700	0.99270
0.008	0.803	0.803	0.1688	0.98666
0.01	0.775	0.775	0.1350	0.98265
0.03	0.627	0.627	0.0450	0.94254
0.05	0.548	0.548	0.0270	0.90244
0.08	0.468	0.468	0.0169	0.84227
0.10	0.427	0.427	0.0135	0.80217
0.15	0.345	0.345	0.0090	0.70190
0.20	0.281	0.281	0.0068	0.60162
0.25	0.225	0.225	0.0054	0.50135
0.30	0.175	0.175	0.0045	0.40108



a) Scatter plot of PNC_U vs. C_{pu}

b) Scatter plot of PNC_U vs. C_u

c) Scatter plot of PNC_U vs. C_{pcu}

d) Scatter plot of PNC_U vs. C_{pyu}

Fig. 2: Patterns in changes of different indices over the changes in PNC_U values in incapable processes

So it is observed that for both the capable and incapable processes the pattern of changes in the unconventional index C_u over the changes in the proportion of nonconformance matches with the pattern of changes in C_{pu}. This implies that the estimate of C_u index obtained fron a Poisson or binomial process can always be interpreted similarly to the interpretation of the estimate of C_{pu} index from a normal process. However, for both capable and incapable processes, the pattern of changes in the unconventional indices C_{pcu} and C_{pyu} over the changes in the proportion of process nonconformance grossly deviates from the pattern of changes in C_{pu}. This is indicative that there is always high chances of making wrong impressions about a Poisson or binomial process by examining the estimates of C_{pcu} and C_{pyu} indices. The problem with the C_{pcu} index is that it is estimated as ratio of two very small numbers, where numerator is 0.00135 (acceptable proportion of nonconformance) and denominator is actual proportion of nonconforming units/lots with respect to USL. Thus, the estimate is highly impacted due to a minor deviation in the value of actual proportion of nonconformance from the acceptable proportion of nonconformance. The C_{pyu} index suffers from another problem. The value of the ratio $[F(U) - 0.5]/(0.5 - \alpha_0^U)$ is considered as the estimate of C_{pyu}. Since the value α_0^U is usually taken as 0.00135, the denominator is always equal to 0.49865. On the other hand, the values of the numerator can be at most 0.5. Therefore, the maximum value of $\hat{C}_{pyu}$ (or $\hat{C}_{pyl}$) in a process can be 0.5/0.49865 = 1.0027. Thus the estimates of C_{pyu} index fail to make distinction between just capable process and highly capable process. Similarly, the estimates of C_{pyu} index fail to discriminate between just capable process and poorly capable process.

It is already noted that the estimate of C_u index obtained fron a Poisson or binomial process can always be interpreted similarly to the interpretation of the estimate of C_{pu} index from a normal process. Thus, it may be possible to resolve the interpretation issues of C_{pcu} and C_{pyu} indices, if a relationship between C_u and C_{pcu} indices, and C_u and C_{pyu} indices can be established.

4. Establishing Relationship Between estimates of C_u and C_{pcu} indices and C_u and C_{pyu} indices

The estimate of C_u from a Poisson process can be obtained using equation (10) as

$$\hat{C}_u = (1/3) \times \Phi^{-1}\left(1 - \widehat{PNU}_U\right)$$
$$\Rightarrow 1 - \widehat{PNU}_U = \Phi(3\hat{C}_u)$$
$$\Rightarrow \widehat{PNU}_U = 1 - \Phi(3\hat{C}_u) \tag{25}$$

Again the estimate of C_{pcu} from a Poisson process can be obtained using equation (15) as

$$\hat{C}_{pcu} = 0.00135/\widehat{PNU}_U$$

$$\Rightarrow \widehat{PNU}_U = 0.00135/\hat{C}_{pcu} \quad (26)$$

Comparing equations (25) and (26) we get,

$$1 - \Phi(3\hat{C}_u) = 0.00135/\hat{C}_{pcu}$$

$$\Rightarrow \Phi(3\hat{C}_u) = 1 - 0.00135/\hat{C}_{pcu}$$

$$\Rightarrow \hat{C}_u = (1/3) \times \Phi^{-1}(1 - 0.00135/\hat{C}_{pcu}) \quad (27)$$

On the other hand, the estimate of C_{pyu} from a Poisson process can be obtained using equation (21) as

$$\hat{C}_{pyu} = \frac{0.5 - \widehat{PNU}_U}{0.49865}$$

$$\Rightarrow 0.49865 \times \hat{C}_{pyu} = 0.5 - \widehat{PNU}_U$$

$$\Rightarrow \widehat{PNU}_U = 0.5 - 0.49865 \times \hat{C}_{pyu} \quad (28)$$

Comparing equations (25) and (28) we get,

$$1 - \Phi(3\hat{C}_u) = 0.5 - 0.49865 \times \hat{C}_{pyu}$$

$$\Rightarrow \Phi(3\hat{C}_u) = 0.5 + 0.49865 \times \hat{C}_{pyu}$$

$$\Rightarrow \hat{C}_u = (1/3) \times \Phi^{-1}(0.5 + 0.49865 \times \hat{C}_{pyu}) \quad (29)$$

Similarly, it can be shown that the relationship between estimates of C_u and C_{pcu} indices obtained from a binomial process will be the same as equation (27) and relationship between estimates of C_u and C_{pyu} indices obtained from a binomial process will be the same as equation (29).

The users of C_{pcu} and C_{pyu} indices should convert the estimated $\hat{C}_{pcu}$ and $\hat{C}_{pyu}$ values into $\hat{C}_u$ value using equations (27) and (29) respectively before decision making. Otherwise, they may inadvertently be led to erroneous decision making.

5. Conclusions

For assessing capability of a normal process with USL conventionally C_{pu} index is estimated to facilitate better decision making in product and process management. Due to legacy of usages of this index and its interpretations, a user of an unconventional index (used for assessing capability of a Poisson or binomial process) may tend to interpret its values with reference to the values of C_{pu} for the bad, good or highly capable normal processes, and get a false impression about the capability of the concerned process. In this paper, the patterns of changes in values of unconventional indices over the changes in the proportions of nonconformance in Poisson or binomial process is compared with the pattern in changes in the values of conventional C_{pu} index over the proportions of nonconformance in normal process. It is observed that the pattern of changes in the values of C_u index over the changes in the proportions of nonconformance matches with the pattern of changes in C_{pu}, which implies that C_u index can safely be used for assessing capability of a Poisson or binomial process. However, the patterns of changes of other unconventional indices grossly deviates from the pattern of changes in C_{pu}. Mathematical relationships of estimates of other unconventional indices with the estimate of C_u index are established. It is recommended to convert the estimates of other unconventional indices into estimated C_u value using those relationships before any decision making. Otherwise, users of the other unconventional indices may inadvertently be led to erroneous decision making.

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