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Numerical treatment of singularly perturbed delay reaction-diffusion equations

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Abstract

This paper presents a uniform convergent numerical method for solving singularly perturbed delay reaction-diffusion equations. The stability and convergence analysis are investigated. Numerical results are tabulated and the effect of the layer on the solution is examined. In a nutshell, the present method improves the findings of some existing numerical methods reported in the literature.

Keywords: Singularly perturbed, Time delay, Reaction-diffusion equation, Layer

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1. Introduction

Singularly perturbed delay differential equations are applicable in the mathematical modeling of various physical and biological phenomena. For example, micro-scale heat transfers, hydrodynamics of liquid helium, second-sound theory, thermo-elasticity, reaction-diffusion equations, stability, and a variety of models for physiological processes (File *et al*., 2017). However, the treatment of such problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions (Kadalbajoo and Reddy, 1989). The accuracy of the numerical scheme is increased by increasing the number of grid points (Kadalbajoo and Ramesh, (2007).

In recent years, various numerical methods for solving delay and other differential equations are presented by different authors. For example, Ramesh and Kadalbajoo, (2011); Swamy, (2014); Swamy *et al.*, (2015); Gadisa and File, (2019); Phaneendra and Lalu, (2019); Vaid and Arora, (2019); Melesse *et al.*, (2019); Sahu and Mohapatra, (2019); Chekole *et al.*, (2019) and etc, are presented different numerical schemes for solving singularly perturbed problems. However, to date, ε-uniformly convergent methods have not been sufficiently developed for a broad class of singularly perturbed delay differential equations (Pratima and Sharma, 2011).

In this paper, we develop the uniform convergence numerical method to solve singularly perturbed delay reaction-diffusion equations. The work can also help to introduce the technique of establishing and making analysis for the stability and convergence of the present method, which is the crucial part of the numerical analysis. Moreover, the present method gives more accurate results than some currently existing numerical methods reported in the literature. Therefore, this paper is essential for science (such as mathematics, physics, and engineering) researchers who are working in this area.

2. Mathematical Formulation

Consider singularly perturbed delay reaction-diffusion equation (SPDRDE) of the form:

$$£\, y''(x) + a(x)\,y(x-¢) + b(x)\,y(x) = f(x),\ 0 < x < 1 \qquad (1)$$

with the interval and boundary conditions,

$$y(x) = ¤(x),\ -¢ \le x \le 0 \text{ and } y(1) = \tag{2}$$

where £ is small parameter, $0 < £ << 1$ and ¢ is also small delay parameter, $0 < ¢ << 1$; $a(x),\ b(x),\ f(x)$ and $¤(x)$ are bounded smooth functions in $(0,1)$ and is a given constant. The condition of the layer or oscillatory behavior is described in File *et al*., (2017).

By using Taylor series expansion in the neighborhood of the point x, we have:

$$y(x-¢) \approx y(x) - ¢\, y'(x) + o(¢^2) \tag{3}$$

Substituting Eq. (3) into Eq. (1), we obtain an asymptotically equivalent singularly perturbed two-point boundary value problem of the form:

$$Ly(x) \equiv y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{4}$$

under the boundary conditions,

$$y(0) = ¤_0 \text{ and } y(1) = \quad . \tag{5}$$

where, $p(x) = \dfrac{-¢\, a(x)}{£}$, $q(x) = \dfrac{a(x)+b(x)}{£}$ and $r(x) = \dfrac{f(x)}{£}$.

Using the uniform mesh discretization $x_i = x_0 + ih,\ i = 0(1)N$ and making use of Taylor's series expansions of y_{i+1} and y_{i-1} up to $O(h^5)$, we get the finite difference approximations for y_i' & y_i'':

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + T_1 \tag{6}$$

where, $T_1 = -\dfrac{h^4}{120} y^{(5)}(¶_1)$, for $¶_1 \in [x_{i-1},\ x_i]$.

and $$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + T_2 \tag{7}$$

where, $T_2 = -\dfrac{h^4}{360} y^{(6)}(¶_2)$, for $¶_2 \in [x_{i-1},\ x_i]$.

Substituting Eqs. (6) and (7) into Eq. (4), we obtain:

$$\frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + \frac{p_i}{2h}\left(y_{i+1} - y_{i-1}\right) - \frac{h^2}{6} p_i y_i''' + q_i y_i = r_i + T \tag{8}$$

where, $T = \dfrac{h^2}{12} y^{(4)}(¶_2) - p_i T_1 - T_2$ is the local truncation error and $p(x_i) = p_i$, $q(x_i) = q_i$, $r(x_i) = r_i$, $y(x_i) = y_i$.

Differentiating both sides of Eq. (4) concerning x and evaluating at x_i, we get:

$$y_i''' = r_i' - p_i y_i'' - \left(p_i' + q_i\right) y_i' - q_i' y_i \tag{9}$$

Substituting Eq. (9) into Eq. (8) for y_i''' and using central difference approximation for y_i'' and y_i', we obtain:

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \text{ for } i = 1, 2, \ldots, N-1 \tag{10}$$

where,

$$E_i = \frac{1}{h^2} - \frac{p_i}{2h} + \frac{p_i^2}{6} - \frac{h}{12} p_i \left(p_i' + q_i \right), \qquad F_i = \frac{2}{h^2} + \frac{p_i^2}{3} - q_i - \frac{h^2}{6} p_i q_i'$$

$$G_i = \frac{1}{h^2} + \frac{p_i}{2h} + \frac{p_i^2}{6} + \frac{h}{12} p_i \left(p_i' + q_i \right) \quad \text{and} \quad H_i = r_i + \frac{h^2}{6} p_i r_i'.$$

3. Stability and Convergence Analysis

Case 1: Layer Behavior $\left(a(x) + b(x) = q(x) < 0, \text{ for } x \in (0,1)\right)$.

First, we present the stability of the discrete problem in Eq. (10) for the case of layer behavior.

Lemma 1. The finite difference operator L^N in Eq. (10) has the discrete minimum principle, if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \leq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in (0,1)$.

Proof. Suppose that there exists a positive integer k such that $w_k < 0$ and $w_k = \min_{0 \le i \le N} w_i$.

Then, from Eq. (10), we have:

$$\begin{aligned} L^N w_k &\equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1} \\ &= \left(\frac{1}{h^2} + \frac{p_k^2}{6} \right)\left(w_{k-1} - w_k \right) + \left(\frac{1}{h^2} + \frac{p_k^2}{6} \right)\left(w_{k+1} - w_k \right) + \left(\frac{p_k}{2h} + \frac{h}{12} p_k \left(p_k' + q_k \right) \right)\left(w_{k+1} - w_{k-1} \right) \\ &\quad + \left(q_k + \frac{h^2}{6} p_k q_k' \right) w_k \end{aligned}$$

For sufficiently small h (*i.e.*, as $h \to 0$) and for suitable value of p_k, we obtain:

$L^N w_k > 0$. Since, $w_k < 0$ (by the assumption) and $\left(q_k + \frac{h^2}{6} p_k q_k' \right) \to q_k < 0$. But, this is a contradiction.

Hence, $w_i \geq 0$ for all $x_i \in (0,1)$.

Theorem 1. The finite difference operator L^N in Eq. (10) is stable for $a(x) + b(x) < 0$, if w_i is any mesh function, then $\left| w_i \right| \leq C \max \left\{ \left| w_0 \right|, \max_{x_i \in (0,1)} \left| L w_i \right| \right\}$, for some constant $C \geq 1$.

Proof. We define two functions, $\cdot_i^{\pm} \equiv C \max \left\{ \left| w_0 \right|, \max_{x_i \in (0,1)} \left| L w_i \right| \right\} \pm w_i$. Then, we get:

$\cdot_0^{\pm} \geq 0$ and

$L \cdot_i^{\pm} \equiv C q_i \max \left\{ \left| w_0 \right|, \max_{x_i \in (0,1)} \left| L w_i \right| \right\} \pm L w_i \leq 0$, since $a_i + b_i < 0 \Rightarrow q_i < 0$ and $C \geq 1$.

Therefore by Lemma 1, we get:

$$\cdot_i^{\pm} \geq 0, \text{ for all } x_i \in (0,1). \ \Rightarrow \cdot_i^{\pm} \equiv C \max \left\{ \left| w_o \right|, \max_{x_i \in (0,1)} \left| L w_i \right| \right\} \pm w_i \geq 0.$$

Thus, $\left| w_i \right| \leq C \max \left\{ \left| w_o \right|, \max_{x_i \in (0,1)} \left| L w_i \right| \right\}$.

This proves the stability of the scheme for the case of layer behavior.

Case 2: Oscillatory Behavior $\left(a(x)+b(x)=q(x)>0, \text{ for } x\in(0,1)\right)$.

Lemma 2. The finite difference operator L^N in Eq. (10) has the discrete maximum principle, if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \geq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in (0,1)$.

Proof. Suppose that there exists a positive integer k such that $w_k < 0$ and $w_k = \max_{0\le i\le N} w_i$.

Then, from Eq. (10), we have:

$$L^N w_k \equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1}$$
$$=\left(\frac{1}{h^2}+\frac{p_k^2}{6}\right)(w_{k-1}-w_k)+\left(\frac{1}{h^2}+\frac{p_k^2}{6}\right)(w_{k+1}-w_k)+\left(\frac{p_k}{2h}+\frac{h}{12}p_k\left(p_k'+q_k\right)\right)(w_{k+1}-w_{k-1})$$
$$+\left(q_k+\frac{h^2}{6}p_k q_k'\right)w_k$$

For sufficiently small h and for suitable value of p_k, we obtain:

$L^N w_k < 0$. Since, $w_k < 0$ (by the assumption) and $\left(q_k+\frac{h^2}{6}p_k q_k'\right) \to q_k > 0$. But, this is a contradiction.

Hence, $w_i \geq 0$ for all $x_i \in (0,1)$.

Theorem 2. The finite difference operator L^N in Eq. (10) is stable for $a(x)+b(x)>0$, $\left(i.e.\ q(x)>0\right)$, if w_i is any mesh function, then $|w_i| \leq K \max\left\{|w_0|, \max_{x_i\in(0,1)}|Lw_i|\right\}$, for some constant $K \geq 1$.

Proof. The proof is similar to Theorem 1.

This proves the stability of the scheme for the case of oscillatory behavior.

Definition 1 (Uniform Convergence): Let y be a solution of Eqs. (1) and (2). Consider a difference scheme for solving Eqs. (1) and (2). If the scheme has a numerical solution y^N that satisfies

$$\left\|y-y^N\right\| \leq C h^p,$$

where $C>0$ and $p>0$ are independent of £ and of the mesh size h, then we say the scheme uniformly converges to y concerning the norm $\|.\|$, (O'Riordan and Stynes, 1991).

Theorem 3. Let $y(x)$ be the analytical solution of the problem in Eqs. (4) and (5) and $y^N(x)$ be the numerical solution of the discretized problem of Eq. (10). Then, $\left\|y-y^N\right\| \leq C h^2$ for sufficiently small h and C is positive constant.

Proof. Multiplying both sides of Eq. (10) by $-h^2$ and simplifying, we get:

$$\left(-1+u_i\right)y_{i-1}+\left(2+v_i\right)y_i+\left(-1+w_i\right)y_{i+1}+g_i+T_i=0 \tag{11}$$

where, $T_i(h)=\frac{h^4}{12}y^{(4)}(\P_2)+O\left(h^6\right)$ is a local truncation error, for $i=1,2,\ldots,N-1$,

$$u_i=\frac{h}{2}p_i-\frac{h^2}{6}p_i^2+\frac{h^3}{12}p_i\left(p_i'+q_i\right), \qquad v_i=\frac{h^2}{3}p_i^2-h^2q_i-\frac{h^4}{6}p_iq_i'$$
$$w_i=-\frac{h}{2}p_i-\frac{h^2}{6}p_i^2-\frac{h^3}{12}p_i\left(p_i'+q_i\right), \qquad g_i=h^2\left(r_i+\frac{h^2}{6}p_ir_i'\right)$$

Incorporating the boundary conditions $y_0 = ¤(x_0) = ¤_0$, $y_N = y(1) =$ in Eq. (11), we get the systems of equations of the form:

$$\left(D+P\right)y + M + T(h) = 0 \tag{12}$$

where,

$$D = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & -1 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} v_1 & w_1 & 0 & \cdots & 0 \\ u_2 & v_2 & w_2 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & u_{N-1} & v_{N-1} \end{bmatrix}$$ are tri-diagonal matrices of order $N-1$, and

$M = \left[\left(g_1 + (-1+u_1)¤(0)\right), g_2, g_3, \cdots, \left(g_{N-1} + (-1+w_{N-1}) \quad \right)\right]^T$, $T(h) = O\left(h^4\right)$ and

$y = \left[y_1, y_2, \cdots, y_{N-1}\right]^T$, $T(h) = \left[T_1, T_2, \cdots, T_{N-1}\right]^T$, $\overline{0} = \left[0, 0, \cdots, 0\right]^T$ are the associated vectors of Eq. (12).

Let $y^N = \left[y_1^N, y_2^N, \cdots, y_{N-1}^N\right]^T \cong y$ be the solution which satisfies the Eq. (12), we have:

$$\left(D+P\right)y^N + M = 0 \tag{13}$$

Let $e_i = y_i - y_i^N$, for $i = 1, 2, \cdots, N-1$ be the discretization error, then,

$$y - y^N = \left[e_1, e_2, \cdots, e_{N-1}\right]^T.$$

Subtracting Eq. (12) from Eq. (13), we get:

$$\left(D+P\right)\left(y^N - y\right) = T(h) \tag{14}$$

Let $\left|p_i\right| \le C_1$, $\left|p_i'\right| \le C_2$, $\left|q_i\right| \le K_1$, $\left|q_i'\right| \le K_2$

Let t_{ij} be the $(i, j)^{th}$ element of the matrix P, then:

$$\left|t_{i,i+1}\right| = \left|w_i\right| \le h\left\{\frac{C_1}{2} + \frac{h}{6}C_1^2 + \frac{h^2}{12}C_1\left(C_2 + K_1\right)\right\}, \quad i = 1, 2, \cdots, N-2$$

$$\left|t_{i,i-1}\right| = \left|u_i\right| \le h\left\{\frac{C_1}{2} + \frac{h}{6}C_1^2 + \frac{h^2}{12}C_1\left(C_2 + K_1\right)\right\}, \quad i = 2, 3, \cdots, N-1.$$

Thus, for sufficiently small h, we have:

$$-1 + \left|t_{i,i+1}\right| < 0, \; i = 1, 2, \cdots, N-2 \quad \text{and} \quad -1 + \left|t_{i,i-1}\right| < 0, \; i = 2, 3, \cdots, N-1.$$

Hence, the matrix $\left(D+P\right)$ is irreducible (Varga, 1962).

Let S_i be the sum of the elements of the i^{th} row of the matrix $\left(D+P\right)$, then:

$$S_i = 1 + v_i + w_i = 1 + h\left(-\frac{p_i}{2}\right) + h^2\left(\frac{p_i^2}{6} - q_i\right) + h^3\left(-\frac{p_i}{12}\left(p_i' + q_i\right)\right) + h^4\left(-\frac{p_i q_i'}{6}\right), \quad \text{for } i = 1$$

$$S_i = u_i + v_i + w_i = h^2\left(-q_i\right) + h^4\left(-\frac{p_i q_i'}{6}\right), \quad \text{for } i = 2, 3, \cdots, N-2$$

$$S_i = 1 + u_i + v_i = 1 + h\left(\frac{p_i}{2}\right) + h^2\left(\frac{p_i^2}{6} - q_i\right) + h^3\left(\frac{p_i}{12}\left(p_i' + q_i\right)\right) + h^4\left(-\frac{p_i q_i'}{6}\right), \text{ for } i = N-1$$

For sufficiently small h, $(D+P)$ is monotone (Varga, 1962).

Hence, $(D+P)^{-1}$ exists and $(D+P)^{-1} \geq 0$.

From the error Eq. (14), we have:

$$\left\| y - y^N \right\| \leq \left\| (D+P)^{-1} \right\| \left\| T(h) \right\| \tag{15}$$

For sufficiently small h, we have:

$$S_i > h^2 K_{1*}, \text{ for } i = 1, 2, \cdots, N-1, \text{ where } K_{1*} = \min_{1 \leq i \leq N-1} |q_i|.$$

Let $(D+P)_{i,k}^{-1}$ be the $(i,k)^{th}$ element of $(D+P)^{-1}$ and we define,

$$\left\| (D+P)^{-1} \right\| = \max_{1 \leq i \leq N-1} \sum_{k=1}^{N-1} (D+P)_{i,k}^{-1} \text{ and } \left\| T(h) \right\| = \max_{1 \leq i \leq N-1} |T_i| \tag{16}$$

Since $(D+P)_{i,k}^{-1} \geq 0$, then from the theory of matrices, we have:

$$\sum_{k=1}^{N-1} (D+P)_{i,k}^{-1} . S_k = 1, \; i = 1, 2, \cdots, N-1.$$

Hence,

$$\sum_{k=1}^{N-1} (D+P)_{i,k}^{-1} \leq \frac{1}{\min\limits_{1 \leq k \leq N-1} S_k} < \frac{1}{h^2 Q}, \text{ since } 0 < £ << 1 \tag{17}$$

where, $Q = \min\limits_{1 \leq i \leq N-1} |a_i + b_i|$.

Now, from Eqs. (15) - (17), we get:

$$\left\| y - y^N \right\| \leq \left(\frac{y^{(4)}(¶_2)}{12Q} \right) h^2 = Ch^2 \tag{18}$$

where $C = \dfrac{y^{(4)}(¶_2)}{12Q}$. Thus, the present scheme is £ -uniform convergent.

4. Illustrative Examples and Results

The presented scheme is validated by taking four numerical examples, two with twin boundary layers and two with oscillatory behavior. Since those examples have no exact solution, so the numerical solutions are computed using a double mesh principle (File *et al*., 2017).

Example 1. Consider the SPDRDE with layer behavior,

$$£ y''(x) + 0.25 y(x - ¢) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \; -¢ \leq x \leq 0 \text{ and } y(1) = 0.$$

Table 1. The maximum absolute errors of Example 1, for different values of ¢ with $£ = 0.1$.

¢ ↓	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Present Method					
0.03	2.4645e-05	6.1616e-06	2.7385e-06	1.5404e-06	9.8587e-07
0.05	2.4393e-05	6.0986e-06	2.7105e-06	1.5247e-06	9.7581e-07
0.09	2.3947e-05	5.9872e-06	2.6611e-06	1.4969e-06	9.5799e-07
Results in Swamy *et al*., (2015)					
0.03	2.1999e-03	1.1041e-03	7.3705e-04	5.5315e-04	4.4269e-04
0.05	2.2012e-03	1.1049e-03	7.3749e-04	5.5345e-04	4.4293e-04
0.09	2.1999e-03	1.1038e-03	7.3676e-04	5.5289e-04	4.4247e-04

Table 2. The maximum absolute errors of Example 1, for different values of £ with $¢ = 0.5£$.

£ ↓	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$
Present Method					
2^{-4}	1.5070e-03	3.7828e-04	9.4715e-05	2.3685e-05	5.9215e-06
2^{-5}	2.6509e-03	6.6781e-04	1.6749e-04	4.1894e-05	1.0475e-05
2^{-6}	4.8151e-03	1.2413e-03	3.1158e-04	7.8047e-05	1.9517e-05
2^{-7}	9.2994e-03	2.4334e-03	6.1499e-04	1.5434e-04	3.8604e-05
2^{-8}	1.8030e-02	4.8019e-03	1.2303e-03	3.0956e-04	7.7486e-05
2^{-9}	3.3607e-02	9.3674e-03	2.4542e-03	6.1966e-04	1.5557e-04
2^{-10}	5.2477e-02	1.8177e-02	4.8372e-03	1.2385e-03	3.1168e-04
Results in Swamy *et al*., (2015)					
2^{-4}	1.8632e-02	9.6189e-03	4.8865e-03	2.4643e-03	1.2376e-03
2^{-5}	2.8161e-02	1.4818e-02	7.6255e-03	3.8713e-03	1.9509e-03
2^{-6}	3.7958e-02	2.0967e-02	1.0977e-02	5.6273e-03	2.8498e-03
2^{-7}	5.0640e-02	2.8316e-02	1.5267e-02	7.9105e-03	4.0287e-03
2^{-8}	6.3580e-02	3.7706e-02	2.0984e-02	1.1012e-02	5.6555e-03
2^{-9}	8.3843e-02	5.0477e-02	2.8297e-02	1.5261e-02	7.9111e-03
2^{-10}	9.9137e-02	6.3529e-02	3.7660e-02	2.0974e-02	1.1011e-02

Example 2. Consider the SPDRDE with layer behavior,

$$£ y''(x) - 2y(x - ¢) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \ -¢ \le x \le 0 \ \text{ and } \ y(1) = 0 \ .$$

Table 3. The maximum absolute errors of Example 2, for different values of ¢ with $£ = 0.1$.

¢ ↓	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Present Method					
0.03	5.5262e-05	1.3819e-05	6.1422e-06	3.4551e-06	2.2112e-06
0.05	6.1292e-05	1.5325e-05	6.8113e-06	3.8314e-06	2.4521e-06
0.09	7.5050e-05	1.8764e-05	8.3405e-06	4.6916e-06	3.0026e-06
Results in Swamy *et al*., (2015)					
0.03	3.1674e-03	1.6058e-03	1.0754e-03	8.0837e-04	6.4760e-04
0.05	3.1437e-03	1.5949e-03	1.0685e-03	8.0338e-04	6.4367e-04
0.09	3.0784e-03	1.5660e-03	1.0502e-03	7.9000e-04	6.3310e-04

Table 4. The maximum absolute errors of Example 2, for different values of £ with ¢ $= 0.5$£ .

£ $\downarrow$	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$
Present Method					
2^{-4}	3.5264e-03	8.9037e-04	2.2369e-04	5.5986e-05	1.4001e-05
2^{-5}	6.2964e-03	1.6598e-03	4.1737e-04	1.0450e-04	2.6149e-05
2^{-6}	1.1914e-02	3.1276e-03	7.9216e-04	1.9981e-04	4.9993e-05
2^{-7}	2.1388e-02	5.8351e-03	1.5338e-03	3.8613e-04	9.6851e-05
2^{-8}	3.2782e-02	1.1174e-02	2.9520e-03	7.5112e-04	1.8935e-04
2^{-9}	4.1139e-02	2.0396e-02	5.6170e-03	1.4743e-03	3.7135e-04
2^{-10}	4.1585e-02	3.1521e-02	1.0818e-02	2.8673e-03	7.3159e-04
Results in Swamy *et al.*, (2015)					
2^{-4}	2.1118e-02	1.1692e-02	6.1941e-03	3.1887e-03	1.6178e-03
2^{-5}	2.7872e-02	1.6023e-02	8.6367e-03	4.4957e-03	2.2948e-03
2^{-6}	3.5711e-02	2.1293e-02	1.1869e-02	6.2731e-03	3.2240e-03
2^{-7}	4.6679e-02	2.8350e-02	1.6107e-02	8.6728e-03	4.5120e-03
2^{-8}	5.4895e-02	3.6018e-02	2.1373e-02	1.1929e-02	6.2847e-03
2^{-9}	5.7371e-02	4.7254e-02	2.8581e-02	1.6140e-02	8.6961e-03
2^{-10}	5.7878e-02	5.5695e-02	3.6153e-02	2.1406e-02	1.1956e-02

Example 3. Consider the SPDRDE with oscillatory behavior,

$$£\, y''(x) + 0.25 y(x - ¢) + y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1,\ -¢ \le x \le 0 \text{ and } y(1) = 0 .$$

Table 5. The maximum absolute errors of Example 3, for different values of ¢ with £ $= 0.1$.

¢ $\downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Present Method					
0.03	5.2227e-04	1.3061e-04	5.8052e-05	3.2655e-05	2.0899e-05
0.05	5.1649e-04	1.2916e-04	5.7409e-05	3.2293e-05	2.0668e-05
0.09	5.0518e-04	1.2634e-04	5.6156e-05	3.1588e-05	2.0217e-05
Results in Swamy *et al.*, (2015)					
0.03	2.5991e-03	1.2872e-03	8.5528e-04	6.4039e-04	5.1179e-04
0.05	2.6270e-03	1.3013e-03	8.6474e-04	6.4750e-04	5.1749e-04
0.09	2.6813e-03	1.3289e-03	8.8320e-04	6.6139e-04	5.2863e-04

Example 4. Consider the SPDRDE with oscillatory behavior,

$$£\, y''(x) + y(x - ¢) + 2y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1,\ -¢ \le x \le 0 \text{ and } y(1) = 0 .$$

Table 6. The maximum absolute errors of Example 4, for different values of ¢ with £ $= 0.1$.

¢ $\downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Present Method					
0.03	8.3415e-04	2.0833e-04	9.2578e-05	5.2071e-05	3.3324e-05
0.05	8.8299e-04	2.2050e-04	9.7980e-05	5.5110e-05	3.5269e-05
0.09	9.7538e-04	2.4370e-04	1.0828e-04	6.0909e-05	3.8980e-05
Results in Swamy *et al.*, (2015)					
0.03	1.5929e-02	7.4850e-03	4.8816e-03	3.6202e-03	2.8764e-03
0.05	1.5470e-02	7.2782e-03	4.7473e-03	3.5209e-03	2.7975e-03
0.09	2.1396e-02	1.0097e-02	6.5922e-03	4.8916e-03	3.8879e-03

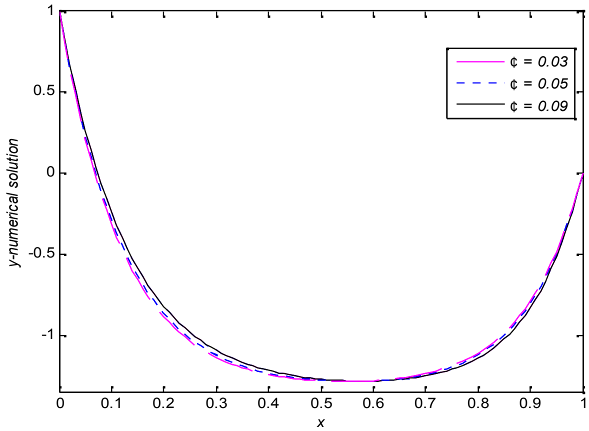

Fig. 1. The numerical solution of Example 1 with £ $= 0.01$.

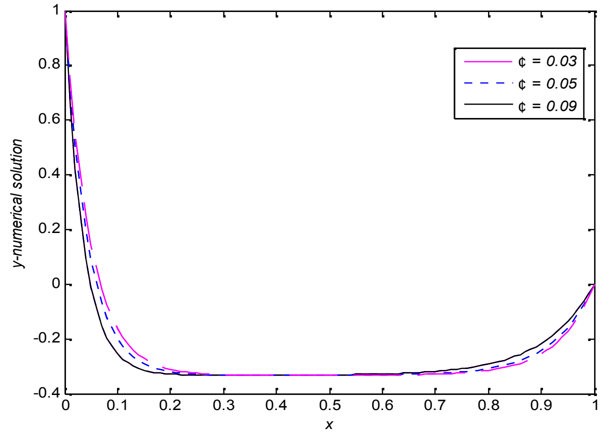

Fig. 2. The numerical solution of Example 2 with £ $= 0.01$.

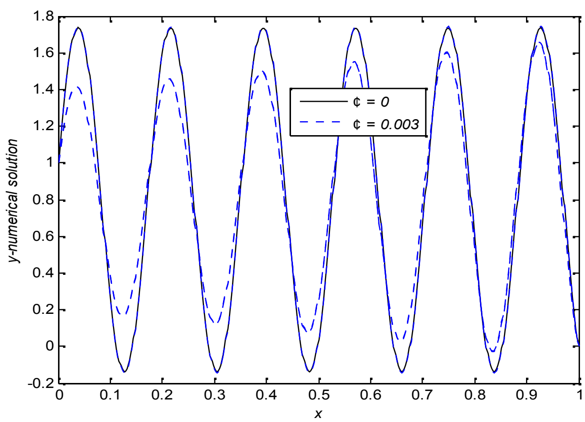

Fig. 3. The numerical solution of Example 3 with £ $= 0.001$.

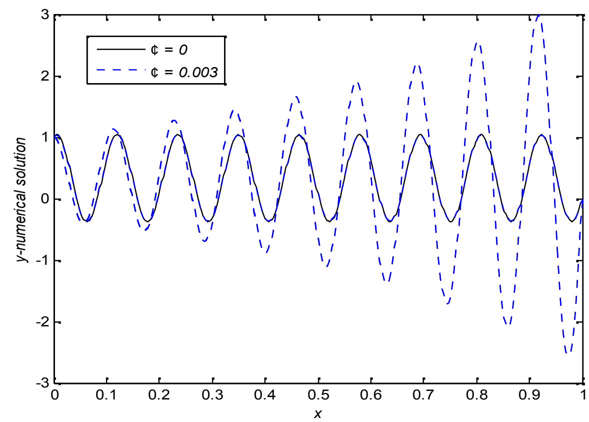

Fig. 4. The numerical solution of Example 4 with £ $= 0.001$.

5. Conclusion

The parameter uniform numerical method for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behavior has been presented. The stability and ε-uniform convergence of the scheme are investigated and established well. The numerical solutions are tabulated in terms of maximum absolute errors and observed that the present method improves the findings of Swamy *et al*., (2015). Furthermore, the effect of layer behavior on the solution is investigated. Concisely, the present method gives more accurate solution and is uniformly convergent for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behavior.

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