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Bifurcations and feedback control of a stage-structure exploited prey-predator system

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Abstract

The present paper describes a bioeconomic modelling of a stage-structure prey-predator system with differential algebraic equations. The criterion for coexistence of the equilibrium points and their stability nature are investigated. Singularity induced bifurcation are studied for zero economic profit and in this perspective, feedback control is designed to preserve the persistence property of the system. In contrast to zero profit, an interior equilibrium point remains stable for positive economic profit. The reasons behind the different nature of the interior equilibriums for zero and positive profit are discussed in conclusion section. Some numerical simulations are given to verify the analytical results. How the maximum profit hampers the system is provided through saddle-node bifurcation in the last subsection of numerical simulation.

Keywords: Prey-predator, stage-structure, singularity induced bifurcation, feedback control.

1. Introduction and model description

A major current focus related to the interacting prey-predator bioeconomic systems in presence of harvesting effort is to investigate the dynamical behaviour of the ecosystem towards the positive economic interest. Biological resource of prey-predator system is recently harvested unscientifically and exported with the aim of positive economic profit which gradually shortages the resources and the ecosystem is collapsed eventually. Nowadays, few research articles have proposed some harvesting strategies and management policies for long run biological resources. Idels and Wang (2008) have investigated the consequences of various harvesting strategies in single species fish population. Kar and Pahari (2007), Xiao *et al.* (2006) and Kumar *et al.* (2002) have studied the prey-predator model with harvesting and observed various complexity of the system namely, Bogdanov-Takens bifurcation, Hopf bifurcation, limit cycle, heteroclinic bifurcation and so on. Mazoudi *et al.* (2008) have considered age-structure fishery model and a Liapunav function is adapted to study the stability and stabilization of the system around the non-trivial steady states.

We shall now discuss some research articles which studied the dynamical behaviour of the bioeconomic model systems governed by some first order ordinary differential equations together with few algebraic equations; such systems are called differential algebraic equations (DAEs) systems. Zhang *et al.* (2009) have taken a differential algebraic prey-predator system with time delay where predator population is harvested continuously. They analyzed the transcritical bifurcation at a boundary point, singularity induced bifurcation at the unique singular point with respect to economic profit and well known Hopf bifurcation regarding the time delay parameter. Kar and Chakraborty (2010) have discussed with the same bioeconomic model harvesting the prey populations and removed completely the singularity induced bifurcation as well as the instability behaviour towards the positive economic profit by means of feedback control theory (Dai L. R., 1989). Zhang and Zhang (2009) modeled a differential algebraic equation system with a single harvesting population equation and a single algebraic equation, and the optimal control strategy is applied to eliminate the singularity induced bifurcation and minimize the cost energy on zero economic profit case. Liu *et al.* (2008, 2009) have constructed a harvested differential algebraic prey predator system and they demonstrated that the system is unstable for any positive economic interest (profit not very closed to zero) due to singularity induced bifurcation theory. But our recent model does not agree with the same. In such situations, the singularity induced bifurcation theory is not well fitted to describe the stabilization of the equilibrium points of the system.

Here, we have considered a stage structure prey-predator model with stage structure for predator which is organized as follows:

$$\begin{aligned}\frac{dN}{d\tau} &= r_1 N(1-\frac{N}{K}) - \alpha N N_2,\\ \frac{dN_1}{d\tau} &= \beta N_2 - r_2 N_1,\\ \frac{dN_2}{d\tau} &= r_3 N_2 + m\alpha N N_2 + \gamma N_1 - \delta {N_2}^2,\end{aligned} \tag{1}$$

where r_1 is the intrinsic growth rate of prey population N, K is the environmental carrying capacity for the prey, α is the predation rate of the mature predator N_2 over the prey N, β is the transition rate from mature predator population N_2 to immature predator population N_1, r_2 and r_3 are the natural death rates of the immature and mature predator population respectively, m measures how many portion of biomass is added to the mature predator population after predation, γ is the conversion rate from immature to mature predator, δ is the coefficient of intraspecific competition of the mature predator and τ is the current time. All the biological meaningful parameters are positive.

We take the transformation for the state and time variable as follows: $N=Kr_2x/r_1$, $N_1=\beta y/(m\alpha)$, $N_2=r_2z/(m\alpha)$ and $\tau=t/r_2$, then the system (1) is converted to

$$\begin{aligned}\dot{x} &= ax - x^2 - bxz,\\ \dot{y} &= z - y,\\ \dot{z} &= -c_1 z + dxz + ey - sz^2,\end{aligned} \tag{2}$$

where $a=r_1/r_2$, $b=1/m$, $c_1=r_3/r_2$, $d=m\alpha K/r_1$, $e=\gamma\beta/r_2^2$, $s=\delta/(m\alpha)$ and $\dot{x}$, $\dot{y}$, $\dot{z}$ represent the derivatives of x, y, z with respect to t respectively.

Using the phase catch-per-unit-effort (CPUE) hypothesis (Clark, 1990) to describe an assumption that catch per unit effort is proportional to stock level we take the harvested term on mature predator as $H = qEz$, where E is the harvesting effort and q is the catchability co-efficient of the mature predator. Simultaneously an algebraic equation is also developed by considering the economic interest of harvesting according to Gordon's economic theory of a common property resource (Gordon, 1954). He established the economic interest of the yield of harvest effort as:

$$\text{Net Economic Revenue } (NER) = \text{Total Revenue } (TR) - \text{Total Cost } (TC). \tag{3}$$

In our present problem we suppose $TR = pqEz$ and $TC = cE$, where p and c respectively stand for the constant price per unit harvested biomass and constant harvesting cost per unit effort. Let us assume that v is the NER, then the algebraic equation looks like

$$(pqz - c)E = v. \tag{4}$$

Finally, the differential algebraic model system with harvesting predator takes the form

$$\begin{aligned}\dot{x} &= ax - x^2 - bxz,\\ \dot{y} &= z - y,\\ \dot{z} &= -c_1 z + dxz + ey - sz^2 - qEz,\\ 0 &= \left(pqz - c\right)E - v.\end{aligned} \tag{5}$$

We now set $f := \begin{pmatrix} f_1(x, y, z, \mu) \\ f_2(x, y, z, \mu) \\ f_3(x, y, z, \mu) \end{pmatrix} = \begin{pmatrix} ax - x^2 - bxz \\ z - y \\ -c_1 z + dxz + ey - sz^2 - qEz \end{pmatrix},$

$g(x, y, z, \mu) = (pqz - c)E - v$ and $X = [x, y, z]$ is a three dimensional column vector.

The matrix representation of equation (5) is ultimately expressed as $A\begin{pmatrix} \dot{X} \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$,

$$\text{where } A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is a singular matrix.} \tag{6}$$

2. Equilibria and their stability analysis in zero economic profit

When economic profit is zero then the model system (5) reduces to

$$\begin{aligned} \dot{x} &= ax - x^2 - bxz, \\ \dot{y} &= z - y, \\ \dot{z} &= -c_1 z + dxz + ey - sz^2 - qEz, \\ 0 &= (pqz - c)E. \end{aligned} \tag{7}$$

The above system always possesses two equilibrium points $P_0(0, 0, 0, 0)$ and $P_1(a, 0, 0, 0)$. The boundary equilibrium point $P_2(x_2, y_2, z_2, E_2)$ and the interior equilibrium point $P_*(x_*, y_*, z_*, E_*)$ exist under some considerations, where $x_2 = (as + b(c_1 - e))/(bd + s)$, $y_2 = (ad + e - c_1)/(bd + s)$, $z_2 = (ad + e - c_1)/(bd + s)$, $E_2 = 0$; and $x_* = (apq - bc)/pq$, $y_* = c/pq$, $z_* = c/pq$, $E_* = (pq(ad + e - c_1) - c(bd + s))/pq^2$. If $apq > bc$ and $pq(ad + e - c_1) > c(bd + s)$, then P_* exists. The existence of P_* ensures the existence of P_2 together with the condition $(as + bc_1) > be$. But, if P_2 does not exit yet $(as + bc_1) > be$, then P_* never exists. Thus both the equilibrium exist when $(as + bc_1) > be$, $apq > bc$ and $pq(ad + e - c_1) > c(bd + s)$. Out of these P_* is the only interior singular equilibrium (definition would be lunched shortly) point. The interior nature of P_* prevents P_2 not to be singular which follows from $E_* > 0$ *i.e.* $x_2 > x_*$. To describe the stability of different equilibrium points we evaluate the Jacobian matrix J_3 of the system (7) at an arbitrary point is defined in a special form as

$$J_3 = D_X f - D_E f (D_E g)^{-1} D_X g$$

$$= \begin{pmatrix} -x & 0 & -bx \\ 0 & -1 & 1 \\ zd & e & -c_1 + xd - 2sz - qE + \dfrac{pq^2 Ez}{(pqz - c)} \end{pmatrix}. \tag{8}$$

First of all we shall discuss the stability analysis of the three equilibrium points *viz.*, P_0, P_1 and P_2 by calculating the eigen values of the Jacobian matrix J_3 at the corresponding points.

At P_0, one eigen value of the community matrix is a. Consequently P_0 is an unstable node or unstable focus. Also $-a$ is an eigen value at P_1 of the linearised system (7), hence the stability nature can be completely determined by the solutions of the equation

$$\lambda^2 + (1 + c_1 - ad)\lambda + (c_1 - ad - e) = 0.$$

Theorem 2.1 The equilibrium point P_2 is stable for all biological parameters.

Proof. The characteristic equation of the jacobian matrix at P_2 is

$$\tau^3 + w_1\tau^2 + w_2\tau + w_3 = 0, \text{ where}$$

$w_1 = e_1 y_2 / z_2 + s z_2 + x_2 + 1$, $\quad w_2 = (x_2+1)(e y_2 / z_2 + s z_2) + b d x_2 z_2 + x_2 - e \quad$ and $\quad w_3 = x_2[(e y_2 / z_2 + s z_2) + b d z_2 - e]$.

It is easy to follow that $w_1 > 1$, $w_3 > 0$ and $w_2 - w_3 = (1+e)x_2 + s z_2$. Therefore, $w_1 w_2 - w_3 > 0$ for any sets of biological parameters. Hence by Routh-Hurwitz criterion, P_2 is a stable equilibrium point.■

Theorem 2.2 When the economic profit is negative and closed to zero, the system (7) is stable at P_* for any set of meaningful biological parameters.

Proof. Let us assume that m_{11}, m_{22}, and m_{33} are the principal diagonal minors of the community matrix at P_* for arbitrary small $v < 0$. Then $m_{11} = -x_* < 0$, $m_{22} = x_* > 0$ and

$$m_{33} = x_* z_* \left(\frac{pq^2 E_*}{pq z_* - c} - (bd + s) \right) < 0.$$

So we can demonstrate that all the eigen values corresponding to the system (7) lie in C^- (left half complex plane). Hence P_* is stable for very small negative economic profit.■

2.1 Singular induced bifurcation in a differential algebraic equations (DAEs) system

The DAEs system can be put in the form

$$\dot{x} = f(x, y, \mu), \quad f : \Re^{n+m+r} \to \Re^n, \tag{9a}$$

$$0 = g(x, y, \mu), \quad g : \Re^{n+m+r} \to \Re^m, \tag{9b}$$

where $x \in \Theta \subset \Re^n$, $y \in \Omega \subset \Re^m$, $\mu \in \Lambda \subset \Re^r$ with n, m and r are all positive integers. In this particular section, x is the dynamic state vector whose time evaluation is directly connected by the equation (9a) and y is the instantaneous state vector which satisfies the constraint equation (9b) and the parameter set μ defines a specific system configuration and operating condition.

We define the set of all equilibria of the DAEs system (9a)-(9b) to be EQ and the set of all stable equilibria OP as

$$EQ = \{(x, y, \mu) \in \Theta \times \Omega \times \Lambda \ : \ f(x, y, \mu) = 0, \ g(x, y, \mu) = 0\} \text{ and}$$

$$OP = \{(x, y, \mu) \in EQ \ : \ Det(D_y g) \neq 0, \ \mathrm{Re}(\lambda(J_n)) < 0\},$$

where $\lambda(J_n)$ is the set of all eigen values corresponding to the Jacobian matrix $J_n = D_x - D_y f (D_y g)^{-1} D_x g$ of the system (9a)-(9b). We also define the singular surface

$$S = \{(x, y, \mu) \in \Theta \times \Omega \times \Lambda \ : \ g(x, y, \mu) = 0, \ \Delta(x, y, \mu) := Det(D_y g) = 0\}$$

and corresponding point on S is known as singular point which plays an important role in differential algebraic system. In a DAEs system the singularity induced bifurcation (SIB) occurs if equilibrium crosses the singular surface S at bifurcation point. Trajectories cross the singularity in a finite time with an infinite speed and the system changes its stability due to an eigen value diverging to infinity. This type of bifurcation can be analyzed with the help of the following theorem.

Theorem 2.3 (Singularity induced bifurcation theorem)

Suppose the system (9a)-(9b) satisfies the following conditions at the singular equilibrium point (x_0, y_0, μ_0):

SIB1: $D_y g$ has a simple zero eigen value and $Trace\left(D_y f (D_y g)^{-1} D_x g\right)$ is nonzero.

SIB2: $\begin{pmatrix} D_x f & D_y f \\ D_x g & D_y g \end{pmatrix}$ is nonsingular.

SIB3: $\begin{pmatrix} D_x f & D_y f & D_\mu f \\ D_x g & D_y g & D_\mu g \\ D_x \Delta & D_y \Delta & D_\mu \Delta \end{pmatrix}$ is also nonsingular.

Then according to Venkatasubramanian (1992) & Venkatasubramanian *et al.* (1995), there exist a smooth curve of the equilibrium in $\Re^{n+m+r}$ which passes through (x_0, y_0, μ_0) and is transversal to the singular surface at (x_0, y_0, μ_0). When μ increases through μ_0 one eigen value of the Jacobian matrix J_n moves from C^- to C^+ if $M/N > 0$ (respectively from C^+ to C^- if $M/N < 0$) along the real axis by diverging through infinity. The rest $(n-1)$ eigen values remain bounded and stay away from the origin. The constants M and N can be computed by evaluating

$$M = -Trace\left(D_y f \left(adj\left(D_y g\right)\right) D_x g\right) \text{ and}$$

$$N = D_\mu \Delta - \left(D_x \Delta \quad D_y \Delta\right) \begin{pmatrix} D_x f & D_y f \\ D_x g & D_y g \end{pmatrix}^{-1} \begin{pmatrix} D_\mu f \\ D_\mu g \end{pmatrix}. \blacksquare$$

2.2 Singularity induced bifurcation at P_*

Theorem 2.4 Considering the positive meaningful biological parameters when the economic parameter v increases through 0, the system (7) undergoes singularity induced bifurcation at the equilibrium P_* and the stability of the equilibrium point P_* changes from stable to unstable.

Proof. SIB1: At P_*, $\Delta := D_E g = pqz - c$ has a simple zero eigen value and

$$Trace(D_E f (adj(D_E g)) D_X g)|_{P_*} = -pq^2 E_* z_* \neq 0,$$

SIB2: $$\begin{vmatrix} D_X f & D_E f \\ D_X g & D_E g \end{vmatrix}_{P_*} = \begin{vmatrix} -x & 0 & -bx & 0 \\ 0 & -1 & 1 & 0 \\ -zd & e & -e-sy & -qz \\ 0 & 0 & pqE & pqz-c \end{vmatrix}_{P_*} = pqE_* z_* \neq 0 \text{ and}$$

SIB3: $$\begin{vmatrix} D_X f & D_E f & D_\mu f \\ D_X g & D_E g & D_\mu g \\ D_X \Delta & D_E \Delta & D_\mu \Delta \end{vmatrix}_{P_*} = \begin{vmatrix} -x & 0 & -bx & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -zd & e & -e-sy & -qz & 0 \\ 0 & 0 & pqE & pqz-c & -1 \\ 0 & 0 & pq & 0 & 0 \end{vmatrix}_{P_*} = -pq^2 E_* x_* \neq 0.$$

Also $M = -Trace(D_E f(adj(D_E g)) D_X g)|_{P_*} = pq^2 E_* z_* > 0$ and

$$N = D_\mu \Delta\Big|_{P_*} - (\, D_X\Delta \quad D_E\Delta\,)\begin{pmatrix} D_X f & D_E f \\ D_X g & D_E g \end{pmatrix}^{-1}\begin{pmatrix} D_\mu f \\ D_\mu g \end{pmatrix}\Bigg|_{P_*} = \frac{1}{E_*} > 0.$$

Finally, $\dfrac{M}{N} = pq^2 E_*{}^2 z_* > 0.$

According to the theorem 2.3, the system (7) undergoes SIB at the equilibrium P_* when the bifurcation parameter $\mu := v = 0$. When v increases through 0, one eigen value of the system Jacobian J_3 at P_* moves from C^- to C^+ along the real axis and ultimately diverges to infinity as v is highly closed to 0. This eigen value approximately is equal to $(pq^2 E_*{}^2 z_* / v - e - sz_*)$ at the neighbourhood of P_* (or equivalently at the neighbourhood of $v = 0$). It brings an impulse in the said population system and the ecosystem collapsed very rapidly. The other two eigen values remain bounded and stay in C^-, away from the origin. Consequently the stability of model system (7) changes from stable to unstable at the equilibrium point P_* when the economic profit increases through zero. Hence the proof is complete.■

3. Design of the feedback control

In the case of zero economic profit the system (7) is unstable around the equilibrium point P_*. To eliminate the singularity induced bifurcation and stabilize the system (7), a state feedback control is designed under certain condition when economic profit $v=0$. In the next subsection, we also fit the same to stabilize an unstable equilibrium point $P^*(x^*, y^*, z^*, E^*)$ regarding to a suitable positive economic profit $v = v^*$.

The Jacobian matrix of (7) at an interior point can also be put in the form as follows:

$$J_4 = \begin{pmatrix} -x & 0 & -bx & 0 \\ 0 & -1 & 1 & 0 \\ -zd & e & -e-sy & -qz \\ 0 & 0 & pqE & pqz-c \end{pmatrix}. \tag{10}$$

Now rank $(J_4, AJ_4, A^2J_4, A^3J_4)_{P_*} = 4$, where A is described earlier. According to the Theorem 2-2.1 in Dai (1989), the model system (7) is locally controllable at P_*. In the status of the Theorem 3-1.2 in Dai (1989), a state feedback controller $u = k(E - E_*)$ can be applied to stabilized the differential algebraic system, where $E_* = (pq(ad + e - c_1) - c(bd + s))/pq^2$ is the E component of P_* and k is called the feedback gain. Hence the system (7) reduces to

$$\begin{aligned} \dot{x} &= ax - x^2 - bxz, \\ \dot{y} &= z - y, \\ \dot{z} &= -c_1 z + dxz + ey - sz^2 - qEz, \\ 0 &= (pqz - c)E + k(E - E_*). \end{aligned} \tag{11}$$

Remark 3.1 At P_*, J_3 and J_4 both are the Jacobian matrices of the same model system (7), but of different orders and elements. Also $Det(J_3) \neq Det(J_4)$. The characteristic polynomial of J_3 follows the expansion of $Det(I_3\lambda - J_3)$, but in the case of J_4 it is of $Det(A\lambda - J_4)$, which is expected as the system (7) consists of three differential equations of first order and first degree. These two characteristic polynomials are different with respect to the coefficients of λ, but of same degree. The first one is a monic polynomial of third degree, where as the second one is not monic. Both the polynomials agree at their zeros. We use these according to our necessity.□

Theorem 3.1 If the feedback gain k satisfies either

$$k > Max\{\, pq^2 E_* z_* /(sz_* + x_* + e + 1), \;\; pq^2 E_* /(\, bd + s)\} \;\text{ or}$$

$$k > Max\{\ pq^2E_*z_*/(sz_*+x_*+e+1),\ \ pq^2E_*/(\ bd+s),\ k_*\},$$

where k_* is the largest positive root of the equation

$$k^2+\frac{B_*}{C_*}k+\frac{A_*}{C_*}=0,$$

then the system (11) is asymptotically stable at P_*.

Here $A_*=(x_*+1)(\ pq^2E_*z_*)^2$, $B_*=-pq^2E_*z_*[(1+x_*^2)^2+2x_*(e+sz_*)+bdx_*z_*+2sz_*+e]$ and

$$C_*=[((1+x_*+e+sz_*)(1+x_*+x_*^2)+(1+x_*+e+sz_*)(bdx_*z_*+x_*-e)+(1+x_*)(e+sz_*)^2+ex_*-bdx_*z_*]$$
$$>e+bdx_*z_*-e(1+x_*+e+sz_*)+(e+sz_*)^2+ex_*-bdx_*z_*$$
$$=(e+sz_*)^2-e(e+sz_*)>0.$$

Proof. The Jacobian matrix of the system (11) at P_* is

$$\begin{pmatrix} -x & 0 & -bx \\ 0 & -1 & 1 \\ zd & e & -e-sz+\dfrac{pq^2Ez}{k} \end{pmatrix}_{P_*}. \tag{12}$$

Its characteristic equation is $\lambda^3+\sigma_1\lambda^2+\sigma_2\lambda+\sigma_3=0$, where $\sigma_1=1+x_*+e+sz_*-pq^2E_*z_*/k$, $\sigma_2=(x_*+1)(e+sz_*-pq^2E_*z_*/k)+bdx_*z_*+x_*-e$ and $\sigma_3=x_*(sz_*+bdz_*-pq^2E_*z_*/k)$.

Now $\sigma_1>0 \Rightarrow k>pq^2E_*z_*/(1+x_*+e+sz_*)$ and $\sigma_3>0 \Rightarrow k>pq^2E_*/(bd+s)$. Also $\sigma_1\sigma_2-\sigma_3=A_*/k^2+B_*/k+C_*$. For any set of biological parameters $A_*>0$ always holds. Now, if $B_*^2-4A_*C_*<0$, then the expression $\sigma_1\sigma_2-\sigma_3$ is always positive for any real value of k. Hence by Routh-Hurwitz criteria (Kot, 2001), the DAEs system (11) at P_* is stable and the corresponding feedback gain k satisfies

$$k > Max\{\ pq^2E_*z_*/(sz_*+x_*+e+1),\ \ pq^2E_*/(\ bd+s)\}.$$

Otherwise, if $B_*^2-4A_*C_*<0$ does not hold, then the expression $\sigma_1\sigma_2-\sigma_3=(C_*/k^2)[k^2+(B_*/C_*)k+(A_*/C_*)]$ is greater than zero for any $k>k_*$. Consequently, the system is stable for

$$k > Max\{\ pq^2E_*z_*/(sz_*+x_*+e+1),\ \ pq^2E_*/(\ bd+s),\ k_*\}.$$

Hence the proof.■

From a bioeconomic system, a society or the government of a country always expects some positive profit. In a real fishery management, the fishery agencies are interested towards the positive economic rent from the fishery. But, we can not expect a high economic profit in real life situation, as it hampers the persistent property of the ecosystem. Thus the economic profit runs over a suitable interval $(0,\ v_e)$. Suppose v^* is our targeted positive profit over $(0,\ v_e)$, then it ensures at least one interior positive equilibrium point $P^*(x^*,\ y^*,\ z^*,\ E^*)$ of the model system and for $v^*\geq v_e$ the system has no positive equilibrium point at all. Liu *et al.* in (2008, 2009) have demonstrated that any positive equilibrium point corresponding to a positive profit in their bioecononic DAEs system is always unstable according to the singularity induced bifurcation theorem. But, no singular point exists for positive economic profit as $Det(D_Eg)=v/E>0$ and singularity induced bifurcation theory can not be permitted to apply for analyzing the stability nature of the equilibrium points. So, there may exist positive equilibrium point associated with a positive profit for which the model system (5) is stable in its own rights. Assume $P^*(x^*,\ y^*,\ z^*,\ E^*)$ is an unstable equilibrium point related to $v=v^*$ of the system (5), where $x^*=a-bz^*$, $y^*=z^*$, $E^*=v^*/(pqz^*-c)$ and z^* is a typical solution of the equation

$$pq(bd+s)z^2-[pq(ad+e-c_1)+c(bd+s)]z+[c(ad+e-c_1)+qv^*]=0. \tag{13}$$

Then we can, once again, design a feedback controller $u(t)=k(E-E^*)$, applying the same assumptions discussed in the front of this section. Adding the feedback controller into the model system (5) regarding to $v=v^*$, the reduced final model is

$$\begin{aligned}
\dot{x} &= ax - x^2 - bxz, \\
\dot{y} &= z - y, \\
\dot{z} &= -c_1 z + dxz + ey - sz^2 - qEz, \\
0 &= (pqz - c)E - v^* + k(E - E^*)
\end{aligned} \tag{14}$$

and the associated Jacobian matrix is

$$\begin{pmatrix} -x & 0 & -bx \\ 0 & -1 & 1 \\ zd & e & -e - sz + \dfrac{pq^2Ez}{(pqz-c)+k} \end{pmatrix}_{P^*}. \tag{15}$$

We now calculate k by means of numerical approach.

4. Numerical simulation

For verification of our previously discussed analytical results, we, here, would like to present some numerical simulations with the help of MATHEMATICA 5.2 and MATLAB 7.0 software packages.

(I) In this subsection, numerical support is provided for the singularity induced bifurcation with the hypothetical data as $a=5$, $b=0.5$, $c_1=15$, $d=2$, $e=10$, $s=0.05$, $p=20$, $q=0.2$, $c=15$ and v is the bifurcation parameter. For zero economic profit of the model system (7), there exists unique singular equilibrium point $P_*(3.125,\ 3.75,\ 3.75,\ 5.3125)$. Now the arbitrary variation of v (very closed to zero) changes the local stability of P_*. The variation of v and the corresponding eigen values are shown in the following Table 1 and together with the Figure 1 & Figure 2 below.

Table 1. The variation of v and the corresponding eigen values

Eigen values $\mapsto$ $v \downarrow$	Real(λ_1)	Real(λ_2)	Real(λ_3)
-0.001	-0.99999	-3.1252	-84684.97
-0.0001	-0.99999	-3.1250	-846696.68
0.001	-1.00012	-3.1248	84650.97
0.0001	-1.00001	-3.1249	846662.69

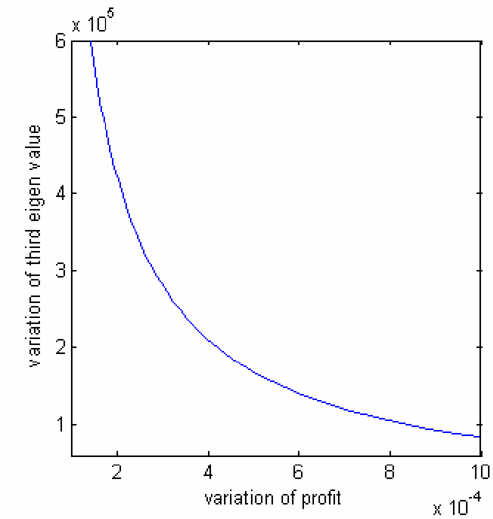

Figure 1. Variation of the third eigen value for positive v (very closed to zero).

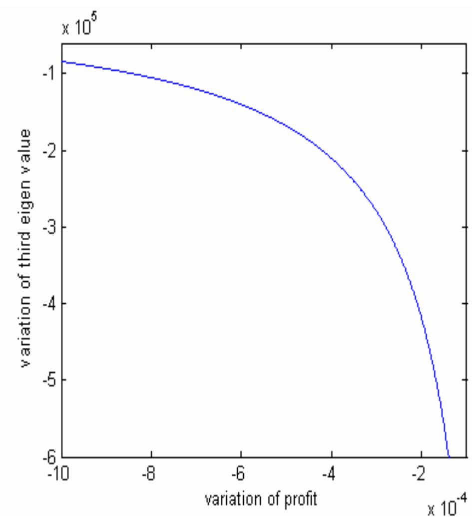

Figure 2. Variation of the third eigen value for negative v (very closed to zero).

(II) To eliminate the singularity induced bifurcation at $P_*(3.125,\ 3.75,\ 3.75,\ 5.3125)$, we take the feedback gain $k=20$, which is greater than $Max\{1.114, 4.048\}$. Using this condition the system (11) possesses the eigen values as -8.556, -4.716 and -0.243, consequently the system is stable at P_*.

(III) Numerically we can calculate that the maximum positive bioeconomic profit $v_e \approx$ 5.37574. If we want to investigate the positive profit $v^* = 3$, then there exists a typical equilibrium point $\hat{P}(2.7038, 4.5923, 4.5923, 0.8904)$, which is stable without any feedback control and no singularity induced bifurcation occurs for positive profit (positive profit not very closed to zero). In contrast to the above fact, if we take $v^* = 2.5$, then a typical equilibrium $P^*(3.0570, 3.8859, 3.8859, 4.5990)$ exists which is unstable in nature. Applying feedback control, we have $k > Max\{0.4044, 3.8538\}$ and the eigen values of the system (14) are -7.5757, -6.3977 and -0.1869 corresponding to $k=15$.

(IV) In previous section, we have demonstrated that there exists no equilibrium point in the system when economic profit exceeds its maximum value. For example, if we consider the same set of biological parameters as in (*I*), then the maximum profit $v_e \approx$ 5.37574. In particular, there exist a stable equilibrium $P^s(2.8051, 4.3897, 4.3897, 1.9540)$ and unstable equilibrium $P^u(2.9389, 4.1222, 4.1222, 3.3585)$ corresponding to $v = 5$. These two equilibrium approach to each other as v increases and disappear when v crosses v_e. Thus the maximum profit is a bifurcation parameter which compel the populations to be extinct forever. This phenomenon can be proved numerically through saddle-node bifurcation as follows:

When profit is positive all the equilibrium are nonsingular points. Therefore, applying the literature proposed by Venkatasubramanian *et al.* (1995), the differential algebraic system can be reduced to ordinary differential system as $\dot{X} = f_R(X, \mu)$ locally near any equilibrium point by a suitable (unique) function f_R. Now the new constructed system satisfies the following conditions near $P_e(x_e, y_e, z_e, E_e) = (2.8720, 4.2559, 4.2559, 2.6562)$ corresponding to v_e.

SND-1: $$\left(D_X f - D_E f\left(D_E g\right)^{-1} D_X g\right)_{P_e} = \begin{pmatrix} -2.8720 & 0 & -1.4360 \\ 0 & -1 & 1 \\ 8.5119 & 10 & -5.7441 \end{pmatrix}$$

has a simple zero eigen value with right vector

$$\vec{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} 0.3333 \\ -0.6667 \\ -0.6667 \end{pmatrix}$$

and left eigen vector $\vec{\Psi} = (\Psi_1 \quad \Psi_2 \quad \Psi_3) = (0.2827 \quad 0.9544 \quad 0.0954)$,

SND-2: $(\vec{\Psi} D_\mu f_R)_{P_e} = \left(\vec{\Psi}(D_\mu f - D_E f (D_E g)^{-1} D_\mu g)\right)_{P_e} = -\dfrac{q z_e \Psi_3}{pq z_e - c} = -0.04 \neq 0$ and

SND-3: $\left(\vec{\Psi}(D_X^2 f_R(\vec{\Phi},\vec{\Phi}))\right)_{P_e} = \left(\vec{\Psi}\sum_{i=1}^{3}(e_i \vec{\Phi}'(D_X(D_X f_i)')\vec{\Phi})\right)_{P_e}$

$$= -2\left[\Phi_1\Psi_1\{\Phi_1 + b\Phi_3\} + \Phi_3\Psi_3\left\{\left(s + \frac{pq^2 E_e c}{(pq z_e - c)^2}\right)\Phi_3 - d\Phi_1\right\}\right] = -0.705 \neq 0,$$

where $\vec{\Phi}'$ stands for transpose of $\vec{\Phi}$ and so on.

According to the literature (Guckenheimer & Holems, 1983), the system undergoes saddle node bifurcation at P_e.

5. Conclusions

The present paper deals with a stage structure prey-predator model with harvesting and it is proposed by means of a system of differential algebraic equations. Dynamical behaviour of the model is investigated due to the variation of the economic interest of harvesting. Singularity induced bifurcation and feedback control technique are studied. To stabilize the system at an interior point, the positive profit and the harvesting effort must satisfy the relation $E^2 / v < (bd + s)/ pq^2$ (from Theorem 2.2). For zero profit case the condition does not hold at all. In the last section of numerical simulation, P^s satisfies the above relation, but P^u does not. When v increases, the E component of P^s increases and for the case of P^u, it decreases continuously. At a certain value of v, the relation reduces to $E^2 / v \approx (bd + s)/ pq^2$ and community matrix possesses a simple zero eigen value which causes the saddle-node bifurcation in the system. Therefore, in fishery management, fishery agency always should take care of it.

The model and its dynamical behaviour are studied mainly on the deterministic framework. It will be more realistic to consider the model in a stochastic environment due to either ecological or economic fluctuations. This may be considered in future work.

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