A Quasi Fourth Order Root-Finding Numerical Method

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Abstract: *Newton's iteration formula* $x_{n+1} = x_n - \dfrac{f(x_n)}{f^{/}(x_n)}, \quad n = 0,1,2,\ldots$ *is a powerful numerical method for solving the root-finding problem* $f(x) = 0$. *Its simplicity and quadratic rate of convergence have significantly contributed to its popularity with numerical practitioners over its linearly convergent rival methods (bisection, secant and the regula-falsi). Masenge [1973: 51-53] derived a quasi third order convergent method* $x_{n+1} = x_n - \dfrac{2f(x_n)f^{/}(x_n)}{2\left[f^{/}(x_n)\right]^2 - f(x_n)f^{//}(x_n)}$ *, which involves both the first and second derivatives of* f. *In this article we present a quasi fourth order numerical method for solving the root-finding problem. The proposed method is based on Taylor's polynomial and represents significant improvement over Newton's method and Masenge's quasi third order method. The new method exploits the higher rate of convergence gained in Masenge's hybrid method to achieve an even faster rate of convergence.*

Numerical experiments carried out on a number of prototype test functions demonstrate unequivocally the gain both in accuracy and in speed of convergence achieved by the proposed method over previously published numerical methods. Our experience working with the method has shown that, in only two iterations, results with an accuracy of up to 10^{-10} *are achievable provided the starting value* x_0 *for the iteration is sufficiently close to the required root.*

Key words: Root-finding problem $f(x) = 0$; Single-step iteration methods; Taylor polynomial; order of convergence.

INTRODUCTION

Single step iterative methods for approximating an isolated root of a function $f(x)$ are derived using either the fixed point theorem or Taylor series expansion of the function about a point x_n assumed to be sufficiently close to the root ° being looked for. The fixed point theorem approach expresses the function $f(x)$ in the form $f(x) = x - g(x)$ where g is a function that can be formed in many different ways, but must satisfy certain conditions to guarantee the existence and uniqueness of a fixed point ° , which turns out to be a root of f. Numerical methods based on

Taylor series expansion use truncated versions of the series expansion of f at the approximate root x_n to obtain a better approximation x_{n+1} of the root $\circ$.

If the derivatives f', f'', f''' are not difficult to obtain and evaluate at a point x_n, then numerical methods based on Taylor series expansion are superior to those based on the fixed point theorem because of the relative ease of determining or estimating the corresponding rate of convergence.
Let $f \in C^3[a,b]$ and $\circ$ be an isolated root of f in (a,b). If $x_n \in (a,b)$ is sufficiently close to but not equal to $\circ$, then $f(x_n) \neq 0$. We seek to determine the adjustment h needed in x_n to obtain the exact root $\circ = x_n + h$. Since $\circ$ is a root of f it follows that

$$0 = f(\circ) = f(x_n + h) = f(x_n) + hf(x_n) + \frac{h^2}{2} f''(x_n) + \frac{h^3}{6} f'''(x_n) + \cdots \tag{1}$$

The classical Newton-Raphson method is obtained from (1) by considering only the first two terms of the series. This gives the result

$$0 = f(x_n) + hf'(x_n). \tag{2}$$

Due to the associated truncation error, the quantity $x_n + h$ will not be equal the root $\circ$ but will hopefully be a better approximation to $\circ$ than x_n. We therefore denote $x_n + h$ by x_{n+1} and write $x_{n+1} = x_n + h$. Solving equation (2) for h and observing that $h = x_{n+1} - x_n$, we arrive at the Newton-Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

If we truncate the right-hand side of equation (1) after the term containing the second derivative f'' we get the approximate equation

$$0 = f(x_n) + hf'(x_n) + \frac{1}{2}h^2 f''(x_n) \;=\; f(x_n) + h\left[f'(x_n) + \frac{1}{2}hf''(x_n) \right]$$

$$= f(x_n) + (x_{n+1} - x_n)\left[f'(x_n) + \frac{1}{2}(x_{n+1} - x_n) f''(x_n) \right] \tag{4}$$

The quantity $(x_{n+1} - x_n)$ appearing inside the square brackets in (4) is replaced by the quantity $-\dfrac{f(x_n)}{f'(x_n)}$ obtained from (3) to give the result

$0 = f(x_n) + (x_{n+1} - x_n)\left[f'(x_n) - \dfrac{f(x_n) f''(x_n)}{2 f'(x_n)} \right]$, which upon solving for x_{n+1} yields Masenge's hybrid formula

$$x_{n+1} = x_n - \frac{2f(x_n)f^{/}(x_n)}{2\left[f^{/}(x_n)\right]^2 - f(x_n)f^{//}(x)}$$
(5)

DERIVATION OF THE PROPOSED METHOD

The new method we are proposing uses terms of the Taylor series expansion (1) up to and including the third derivative term $f^{///}$:

$$0 = f(x_n) + hf^{/}(x_n) + \frac{1}{2}h^2 f^{//}(x_n) + \frac{1}{6}h^3 f^{///}(x_n)$$
$$= f(x_n) + h\left[f^{/}(x_n) + \frac{1}{2}hf^{//}(x_n) + \frac{1}{6}h^2 f^{///}(x_n)\right]$$
$$= f(x_n) + (x_{n+1} - x_n)\left[f^{/}(x_n) + \frac{1}{2}(x_{n+1} - x_n)f^{//}(x_n) + \frac{1}{6}(x_{n+1} - x_n)^2 f^{///}(x_n)\right]. \quad (6)$$

To derive Masenge's hybrid method (5) we used the Newton-Raphson method (3). Likewise, we shall use the hybrid formula (5) to derive the proposed new method. From (5) we define the quantity

$$A = x_{n+1} - x_n = -\left[\frac{2f(x_n)f^{/}(x_n)}{2\left(f^{/}(x_n)\right)^2 - f(x_n)f^{//}(x_n)}\right] \quad (7)$$

Substituting A for the two expressions $(x_{n+1} - x_n)$ that appear inside the square brackets in (6) one gets

$$0 = f(x_n) + (x_{n+1} - x_n)\left[f^{/}(x_n) + \frac{1}{2}Af^{//}(x_n) + \frac{1}{6}A^2 f^{///}(x_n)\right]$$
$$= f(x_n) + (x_{n+1} - x_n)B \quad (8)$$

where B represents the quantity enclosed inside the square brackets in (8):

$$B = f^{/}(x_n) + \frac{1}{2}Af^{//}(x_n) + \frac{1}{6}A^2 f^{///}(x_n) = f^{/}(x_n) + \frac{A}{6}\left[3f^{//}(x_n) + Af^{///}(x_n)\right]. \quad (9)$$

Solving equation (8) for x_{n+1} one gets $x_{n+1} = x_n - \dfrac{f(x_n)}{B}$ (10)

Equation (10) together with the expressions defining the quantities A and B as given by (7) and (9), respectively, constitute the proposed new root-finding numerical method.

ALGORITHM

An algorithm for implementing the method is outlined below:

If x_n approximates an isolated root ° of a function f, then an improved approximation x_{n+1} is obtained by calculating the following quantities:

Step 1 $f(x_n)$, $f^{/}(x_n)$, $f^{//}(x_n)$ and $f^{///}(x_n)$

Step 2 $$A = -\frac{2f(x_n)f^{/}(x)}{2\left[f^{/}(x_n)\right]^2 - f(x_n)f^{//}(x_n)},$$

$$B = f^{/}(x_n) + \frac{A}{6}\left[3f^{//}(x_n) + Af^{///}(x_n)\right]$$

Step 3 $$x_{n+1} = x_n - \frac{f(x_n)}{B}$$

The improved approximation x_{n+1} given by (10) after the first iteration may now be used to obtain the next improved value x_{n+2} in an obvious iterative manner.

ORDER OF CONVERGENCE

The order of convergence of a sequence of numbers $\{x_0, x_1, x_2, \ldots\}$ generated by a single step iteration formula of the general form $x_{n+1} = g(x_n)$ is a positive number p defined by the equation $£_{n+1} = K£_n^p$, where $£_i = {}^\circ - x_i$, $i = 0,1,2,\ldots$

Remark: The order of convergence of a numerical method derived by truncating the formula

$$0 = f(x_n) + hf(x_n) + \frac{h^2}{2}f^{//}(x_n) + \frac{h^3}{6}f^{///}(x_n) + \cdots \qquad (11)$$

of equation (1) is simply the power of h in the first neglected term. Consequently, the order of convergence of the Newton-Raphson method (3) in approximating an isolated root is $p = 2$ (second order or quadratic rate of convergence) because it is obtained by truncating the series of terms on the right hand side of (11) after the linear term in h. The power of h in the first neglected term is 2. Because we derived the new method (10) by truncated the series after the term involving h^3 it is tempting to think that the method is of order 4. However, this is not the case because subsequently we did not calculate the exact value of the adjustment $h = x_{n+1} - x_n$ but resorted to using an approximation previously derived in an earlier publication (Masenge [1973: 51-53]). Thus, the proposed method is almost (quasi-) fourth order, implying that $3 < p < 4$.

As can be inferred from the above remark, determination of the order of convergence of a hybrid method is not a straightforward matter. Instead, one relies on comparing results of numerical experiments obtained using methods with known rates of convergence. On this basis,

NUMERICAL EXPERIMENTS

We demonstrate the power of the proposed method by applying it on three prototype functions $f(x)$ and compare the results with those obtained using the Newton-Raphson method and Masenge's hybrid methods.

Example 1: $f(x) = x - \cos x, \quad x_0 = 0.5$

The exact root of the function correct to 9 decimal places is ° $= 0.739085133$.

The first three derivatives are:

$$f^{/}(x) = 1 + \sin x, \quad f^{//}(x) = \cos x, \quad f^{///}(x) = -\sin x.$$

	Newton-Raphson Method	Hybrid Method	New Method
Error! Objects cannot be created from editing field codes.	0.5	0.5	0.5
x_1	0.755222417	0.737262173	0.739122193
Error! Objects cannot be created from editing field codes.	0.739141654	0.739085132	0.739085133
x_3	0.739085133		

Example 2: $f(x) = x^4 + x^2 - 4, \quad x_0 = 1.5$

The exact root of the function correct to 9 decimal places is ° $= 1.249621068$.

The first three derivatives are

$$f^{/}(x) = 4x^3 + 2x, \quad f^{//}(x) = 12x^2 + 2, \quad f^{///}(x) = 24x.$$

	Newton-Raphson Method	Hybrid Method	New Method
Error! Objects cannot be created from editing field codes.	1.5	1.5	1.5
x_1	1.299242424	1.256236934	1.251350367
Error! Objects cannot be created from editing field codes.	1.251975432	1.249621215	1.249621068
x_3	1.249626632	1.249621362	
x_4	1.249621068		

Example 3: $f(x) = \ln(1 + x^2) - \cos x, \quad x_0 = 1.0$

The exact root of the function correct to 9 decimal places is ° $= 0.915857659$.

The first three derivatives are

$$f^{/}(x) = \frac{2x}{1+x^2} + \sin x, \quad f^{//}(x) = \frac{2}{1+x^2} - \frac{4x^2}{(1+x^2)^2} + \cos x,$$

$$f^{///}(x) = -\left[\frac{12x}{(1+x^2)^2} - \frac{16x^3}{(1+x^2)^3} + \sin x\right].$$

	Newton-Raphson Method	Hybrid Method	New Method
Error! Objects cannot be created from editing field codes.	1.0	1.0	1.0
x_1	0.916998489	0.915975350	0.915862341
Error! Objects cannot be created from editing field codes.	0.915857915	0.915857659	0.915857659
x_3	0.915857659		

CONCLUSION

A quasi fourth order convergent root finding iterative numerical method for solving the nonlinear equation $f(x) = 0$ has been derived. The method is based on Taylor series expansion of the function f at an approximate root x_n and requires evaluation of the first three derivatives of the function at the current approximate root x_n. Numerical experiments carried out on a number of prototype examples have shown that the rate of convergence is almost fourth order, giving almost four additional correct significant digits per iteration. We know that the order of convergence of the method p lies in the interval $3 < p < 4$ but the exact value of p is yet to be computed. This task is yet to be carried out and is a challenge for future research.

References

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