# Comparative Study of Mathematical Models for Ghana's Gold Production*

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## Abstract

Sigmoid functions were used to approximate the cumulative gold production in Ghana. The functions examined were the logistic, Gompertz, Gaussian, Probit and the Hill, which were then used to predict into the future. Although all the five approximated models gave a good estimation of the reality, the Gompertz function was identified to give the best approximation of the observed trend of gold production in Ghana. The model was selected based on its high proportion of variance explained ($R^2$ = 0.9402) as well as having the least value in terms of error (RMSE) and information loss (AIC). The model suggested that gold production could escalate to a level of 7040813 $O_z$ in the future. It also showed that, industrial scale production of gold should be possible even after the year 2050 provided the current upward trend of gold production in Ghana continues.

**Keywords:** Sigmoid, Logistic, Gompertz, Gaussian, Probit, Hill, Gold

## 1 Introduction

Gold is a major export commodity for Ghana. The precious metal bringing in nearly 48 percent of the country's revenue. After South Africa, Ghana is the largest gold producer in Africa and currently has the honour of having one of the largest gold reserves in the world. The sector's contribution to wealth creation, employment and the economy make it one of the nation's most important livelihood activities, directly employing an estimated one million people and supporting approximately 4.5 million more. However, the sector for the past decades has faced numerous challenges which necessitated several rigorous changes in the gold mining industry (Gbireh *et al.*, 2007). This is characterised by the provisions in the Minerals and Mining law together with improved exploration, mining and processing techniques which revolutionized Ghana's gold mining industry. As a result, Ghana experienced a fourth gold boom from 1987 to 1999 (Amankwah and Suglo, 2003).

However, the sector recorded 6%, 3% and 6% drops in production from the previous years in 2000, 2001 and 2002 respectively due to the fall in gold prices on the world market which led to the closure of some mines (Gbireh *et al.*, 2007). Similar fluctuations have been observed in the early 2000's and lately 2015.

According to Müller and Dirner (2010), the production figures of some mineral commodities like gold in certain countries can be seen as an empirical picture of the technical, geological, economic and political situation of that country. This is certainly true in the case of Ghana, formerly known as Gold Coast due to its abundance in minerals.

With this invaluable importance of the sector to Ghana's economy, having a fore knowledge or approximation of future gold production will prove very useful in decision making. Thus, the obvious question is, how gold production in future would look like? In the field of mathematical approximations sigmoid functions offer a wide range of applications, such as modelling biological or economic growth processes (Müller and Dirner, 2010; Appiah *et al.*, 2016).

In this paper, sigmoid functions are used to approximate the cumulative gold production in Ghana. The comparative study of the functions: logistic, Gompertz, Gaussian, Probit and the Hill functions are examined.

## 2 Resources and Methods Used

### 2.1 Logistic Function

The logistic growth function (referred to as the Verhulst model) was first proposed as a model for population growth by Verhulst (1838). The function has the form as shown in Equation (1).

$$P(t_i) = \frac{K}{1 + \text{Exp}\left[-A(t_i - r)\right]} \qquad (1)$$

where $P(t_i)$ is the production at time $(t_i)$, $K$ is the maximum cumulative production (carrying capacity), $A$ and $r$, the growth rate and inflection point respectively. The unknown parameters $K$, $A$ and $r$ are approximated to fit the observed

data points. The inflection point of this function is exactly at 50% of $K$ (*i.e.* $0.5 * K$).

## 2.2 Gompertz Function

The Gompertz function was originally derived to estimate human mortality by Gompertz (1825). The function is shown in Equation (2).

$$P(t_i) = K * \text{Exp}\left[-\text{Exp}\left(-A(t_i - r)\right)\right] \qquad (2)$$

where $P(t_i)$ is the production at time $(t_i)$, $K$ is the maximum cumulative production (carrying capacity), $A$ and $r$, the growth rate and inflection point respectively. The unknown parameters $K$, $A$ and $r$ are approximated to fit the observed data points. The inflection point is at $K * \text{Exp}(-1)$ value (*i.e.* at approximately 36.8% of $K$).

## 2.3 Gaussian Function

The Gaussian normal distribution is widely used in probability calculations (Hagen *et al.*, 2007; Guo, 2011). The density function is given by Equation (3)

$$P(t_i) = K * \text{Exp}\left[-\left[0.5\left(\frac{(t_i - r)}{A}\right)^2\right]\right] \qquad (3)$$

where $P(t_i)$ is the production at time $(t_i)$, $K$, the peak value, $A$ and $r$ parameters (*i.e.* the growth rate and inflection point respectively). The unknown parameters $K$, $A$ and $r$ are approximated to fit the observed data points.

## 2.4 Probit (4P)

The Probit (4P) distribution is widely used in probability calculations (Bliss, 1934). The density function is given by Equation (4).

$$P(t_i) = L + (K - L) * Norm\left(\frac{(t_i - r)}{A}\right) \qquad (4)$$

where $P(t_i)$ is the production at time $(t_i)$, $L$, $K$, the minimum and maximum cumulative production (*i.e.* lower and upper asymptote), $A$ and $r$ parameters (*i.e.* the growth rate and inflection point respectively). The unknown parameters $L$, $K$, $A$ and $r$ are approximated to fit the observed data points.

## 2.5 Hill Function (Logistic 4P)

The Hill function is widely used in modelling (Weiss, 1997; Goutelle *et al.*, 2008). The function has the form as shown in Equation (5).

$$P(t_i) = L + \frac{K - L}{1 + 10^{[A(r - t_i)]}} \qquad (5)$$

where $P(t_i)$ is the production at time $(t_i)$, $L$, $K$, the minimum and maximum cumulative production (*i.e.* lower and upper asymptote), $A$ and $r$ parameters (*i.e.* the growth rate and inflection point respectively). The unknown parameters $L$, $K$, $A$ and $r$ are approximated to fit the observed data points.

## 3 Results and Discussion

### 3.1 Logistic Approximation

Table 1 shows the parameter estimates as well as the summary of the logistic model for gold production in Ghana.

**Table 1 Summary of Approximation**

| Parameter | Estimate | Standard Error |
|---|---|---|
| $A$ | 0.12 | 0.02 |
| $R$ | 2004.23 | 2.96 |
| $K$ | 5191950.80 | 749002.64 |

AICc=1055.8605, BIC=1061.0542, RSME=350363.2700, $R^2$=0.9349

Using the parameter estimates from Table 1, the logistic model for gold production in Ghana is approximated using Equation (6).

$$P(t_i) = \frac{5191950.80}{1 + \text{Exp}\left[-0.12(t_i - 2004.23)\right]} \qquad (6)$$

Fig. 1 shows the observed and the approximated logistic curve for gold production in Ghana for the first half of the 21st century. The model estimates the cumulative production of gold to the tune of 5191950.80 $O_z$ (area under the ash curve). The inflection point is in the year 2004, with a production value of 2595975.40 $O_z$. According to the logistic model, gold production on an industrial scale should be possible even in the latter half of the 21st century.
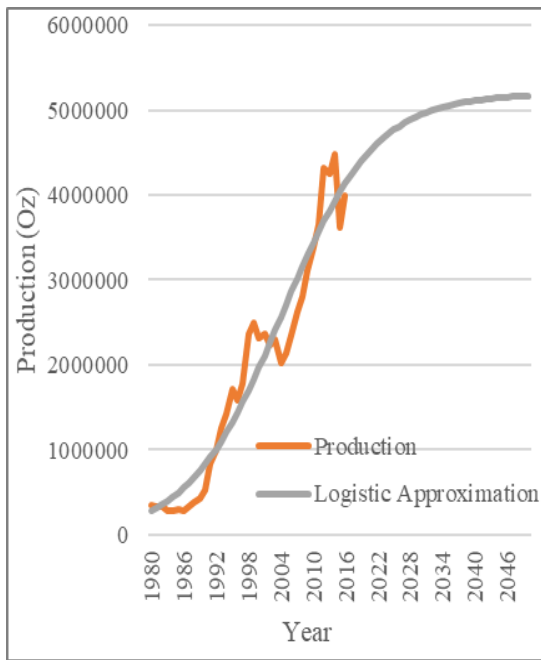
**Fig. 1 Observe Gold Production with Approximated Logistic Function**

## 3.2 Gompertz Approximation

Table 2 shows the parameter estimates as well as the summary of the Gompertz model for gold production in Ghana.

**Table 2 Summary of Approximation**

| Parameter | Estimate | Standard Error |
|-----------|----------|----------------|
| $A$ | 0.05 | 0.01 |
| $r$ | 2004.09 | 5.25 |
| $K$ | 7040813.40 | 1837862.80 |

AICc=1052.7336,        BIC=1057.9273,
RMSE=335866.9400,     $R^2$=0.9402

Using the parameter estimates from
Table **2**, the Gompertz model for gold production in Ghana is approximated using Equation (7).

$$P(t_i) = 7040813.40 * \mathrm{Exp}\left[-\mathrm{Exp}\left(-0.05(t_i - 2004.9)\right)\right] \quad (7)$$

Fig. 2 shows the observed and the approximated Gompertz curve for gold production in Ghana for the first half of the 21st century. The model shows that the cumulative production gold in Ghana is 7040813.40 $O_z$. The inflection point is in the year 2004, with a production value of 2590170.50 $O_z$. Analogous to the logistic model, the Gompertz model shows that industrial scale gold production should be possible in the latter half of the 21st century.
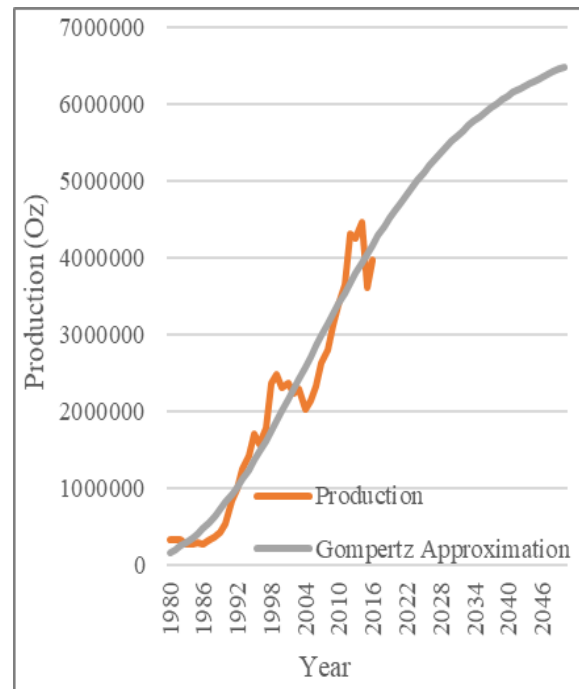


**Fig. 2 Observe Gold Production with Approximated Gompertz Function**

## 3.3 Gaussian Approximation

Table 3 shows the parameter estimates as well as the summary of the Gaussian model for gold production in Ghana.

**Table 3 Summary of Approximation**

| Parameter | Estimate | Standard Error |
|-----------|----------|----------------|
| $A$ | 18.25 | 2.66 |
| $r$ | 2023.63 | 5.62 |
| $K$ | 4542263.70 | 658681.62 |

AICc=1054.7695,        BIC=1059.9631,
RSME=345235.3300,     $R^2$=0.9368

Using the parameter estimates from Table 3, the Gaussian model for gold production in Ghana is approximated using Equation (8).

$$P(t_i) = 4542263.70 * \mathrm{Exp}\left[-\left[0.5\left(\frac{(t_i - 2023.63)}{18.25}\right)^2\right]\right] \quad (8)$$

Fig. 3 shows the observed and the approximated Gaussian curve for gold production in Ghana for the first half of the 21st century. The model shows that the cumulative production of gold in Ghana is 4542263.70 $O_z$. The inflection point is in the year 2024, with a production value of 4541323.31 $O_z$. According to the Gaussian model, gold production on an industrial scale will start to decline in the year 2024**.**
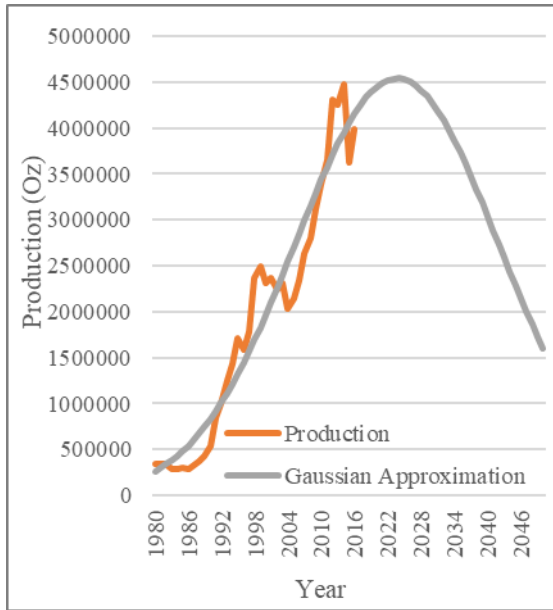
**Fig. 3 Observe Gold Production with Approximated Gaussian Function**

## 3.4 Probit Approximation

Table 4 shows the parameter estimates as well as the summary of the Probit model for gold production in Ghana.

**Table 4 Summary of Approximation**

| Parameter | Estimate | Standard Error |
|-----------|----------|----------------|
| $A$ | 22.51 | 18.79 |
| $r$ | 2009.77 | 18.18 |
| $L$ | -677130.90 | 1293417.00 |
| $K$ | 7392909.40 | 6235108.30 |

AICc=1055.4561,  BIC=1061.5752,
RSME=341088.1400,  $R^2$=0.9401

Using the parameter estimates from Table 4, the Probit model for gold production in Ghana is approximated using Equation (9).

$$P(t_i) = -677130.90 + \left(7392909.40 + 677130.90\right)$$
$$* Norm\left(\frac{(t_i - 2009.77)}{22.51}\right) \quad (9)$$

Fig. 4 shows the observed and the approximated Probit curve for gold production in Ghana for the first half of the 21st century. The model shows that the cumulative production of gold in Ghana is 7392909.40 $O_z$. The inflection point is in the year 2010, with a production value of 3390794.31 $O_z$. Similar to the logistic and Gompertz models, the

Probit model shows that industrial scale gold production should be possible in the latter half of the 21st century.
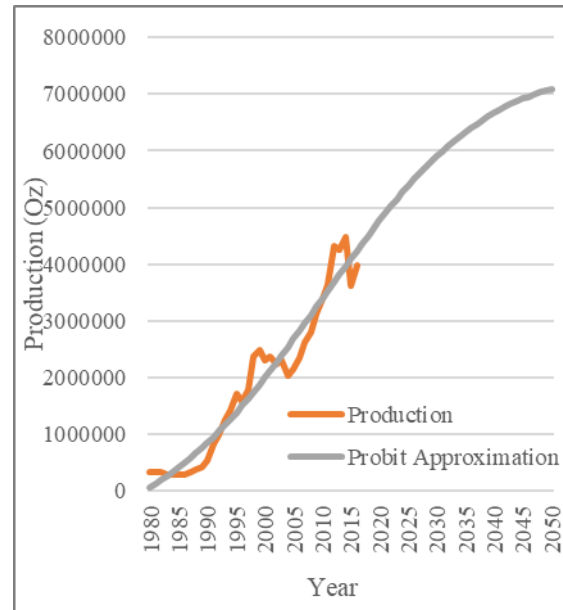


**Fig. 4 Observe Gold Production with Approximated Probit Function**

## 3.5 Hill Approximation

Table 5 shows the parameter estimates as well as the summary of the Hill model for gold production in Ghana.

**Table 5 Summary of Approximation**

| Parameter | Estimate | Standard Error |
|-----------|----------|----------------|
| $A$ | 0.03 | 0.02 |
| $r$ | 2010.77 | 20.92 |
| $L$ | -1117420.00 | 1803637.00 |
| $K$ | 8113213.30 | 7596951.00 |

AICc=1055.5797,  BIC=1061.6989,
RSME=341658.6700,  $R^2$=0.9399

Using the parameter estimates from

Table 5, the Hill model for gold production in Ghana is approximated using Equation (10).

$$P(t_i) = -1117420 + \frac{8113213.30 + 1117420}{1 + 10^{[0.03(2010.77 - t_i)]}} \quad (10)$$

Fig. 5 shows the observed and the approximated Hill curve for gold production in Ghana for the first half of the 21st century. The model shows that the cumulative production of gold in Ghana is 8113213.30 $O_z$. The inflection point is in the year 2010, with a production value of 3387541.26 $O_z$. Similar to the logistic, Gompertz and Probit

models, the Hill's model shows that industrial scale gold production should be possible even in the latter half of the 21st century.
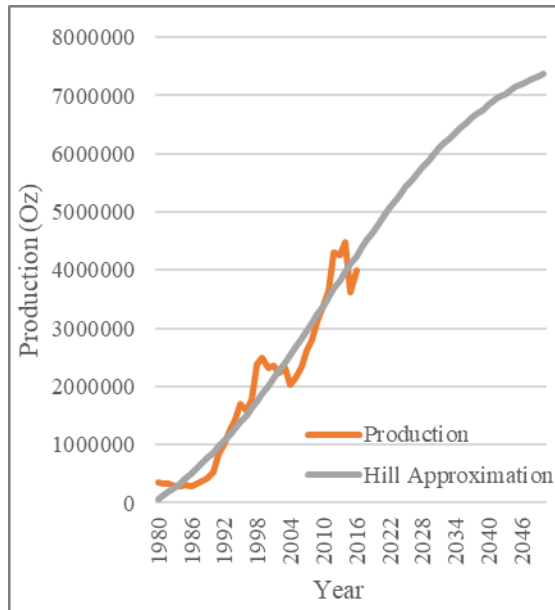


**Fig. 5 Observe Gold Production with Approximation by Hill function**

Five different production models have been approximated based on historical gold production data. Table 6 shows the summary of the approximated production models. It can be seen from Table 6 that, all the models approximated yielded a coefficient of determination ($R^2$) above 0.9300. This implies that all the approximated models are deemed sufficient in term of proportion of variance being explained. Moreover, all the models suggest that, the amount of gold produced in Ghana is yet to reach its maximum, since all the models estimated their respective cumulative production, greater than 4,472,379 $O_z$ which is the highest amount of observed gold production in Ghana.

**Table 6 Summary of the Approximated Gold Production Models**

| Model | $K$ | AICc | BIC | RMSE | $R^2$ |
|---|---|---|---|---|---|
| Logistic | 5191950.80 | 1055.86 | 1061.05 | 350363.27 | 0.9349 |
| Gompertz | 7040813.40 | 1052.73 | 1057.93 | 335866.94 | 0.9402 |
| Gaussian | 4542263.70 | 1054.77 | 1059.96 | 345235.33 | 0.9368 |
| Probit | 7392909.40 | 1055.46 | 1061.58 | 341088.14 | 0.9401 |
| Hill | 8113213.30 | 1055.58 | 1061.70 | 341658.67 | 0.9399 |

Among the approximated models, it can be observed from Table 6 that, the Gompertz model had the least values (in terms of AICc, BIC and RMSE). This suggest that, the approximated Gompertz model yielded a better prediction/approximation among the five models. This subsequently led to it achieving the least error

value (RMSE). Also, the Gompertz model explains the highest proportion of variance (as indicated by $R^2$ of 0.9402), suggesting that the model approximates closely to the observed trend of gold production in Ghana. Hence, the approximated Gompertz model is selected as the suitable model to represent the dynamics of annual gold production in Ghana.

With the exception of the Gaussian model, all the remaining approximated models suggest that, gold production in Ghana is bound to increase gradually until it reaches its peak. This could be attributed to the fact that Ghana's gold reserve has been constant (8.74 tonnes) throughout the years, despite the continuous mining of the mineral. If the current trend of gold production should continue, it is expected that industrial scale production of gold in Ghana should be possible even after the year 2050.

## 4 Conclusion

This paper sought to approximate the cumulative gold production in Ghana using five sigmoid functions. The logistic, Gompertz, Gaussian, Probit and the Hill functions were examined, and used to predict into the future the cumulative gold production in Ghana. Although all the five approximated models gave a good estimation of the reality, the Gompertz function was identified to give the best approximation of the observed trend of gold production in Ghana. The best model, Gompertz model suggested that gold production could rise up in the future to a level of 7040813 Oz. It also showed that, industrial scale production of gold should be possible even after the year 2050 provided the current upward trend of gold production in Ghana continues.

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