THE CORONA PROBLEM IN CARLEMAN ALGEBRAS ON NON-STEIN DOMAINS IN $\mathbb{C}^n$.

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Abstract

New estimates are obtained for the $\bar{\partial}-$operator on non-stein domains in $\mathbb{C}^n$ and the results are applied to the Corona problem in Carleman algebras on those domains.

Introduction

Let Ω be an open subset of a complex manifold X, and let p be a non-negative function on Ω. Denote by $A_p(\Omega)$ the (Carleman) algebras of all holomorphic functions f in Ω such that for some positive constants c_1 and c_2

$$|f(z)| \leq c_1 \exp(c_2 p(z)), z \in \Omega. \quad (1)$$

In [3] where $X = C^n$ and Ω is pseudoconvex, and in Darko (2005), where X is a complex manifold and Ω is a relatively compact Stein open subset, a condition is given on p such that a given finite set $f_1, ..., f_N \in A_p(\Omega)$ generates $A_p(\Omega)$ if and only if

$$|f_1(z)| + |f_2(z)| + ... + |f_N(z)| \geq c_1 \exp(-c_2 p(z)), z \in \Omega \quad (2)$$

for some constants $c_1 > 0, c_2 > 0$. Both in Darko [2005] and Hormander (1967), Ω was Stein. As is always the case, it is natural to ask whether the condition of Steinness can be dropped. We show here that it can, if Ω is a domain in $\mathbb{C}^n$ and we modify the condition in Darko (2005) and Hormander (1967) to the following Condition (H) :

(a) p is a non-negative upper semicontinuous function on Ω.

(b) all polynomial belongs to $A_p(\Omega)$.

(c) there exist positive constants $K_1, ..., K_4$ such that $z \in \Omega$ and $|z - \xi| \leq \exp(-K_{1p}(z) - K_2) \Rightarrow \xi \in \Omega$ and $p(\xi) \leq K_{3p}(z) + K_4$. The only difference between the condition in Hormander(1967) and the Condition (H) here is the replacement of "plurisubharmonic" with "upper semicontinuous".

Note that if Ω is an arbitray domain in $\mathbb{C}^n$, and $d(z)$ denotes the distance from $z \in \Omega$ to the complement of Ω in $\mathbb{C}^n$, $p(z) = \log 1/d(z)$ satisfies Condition (H) on Ω.

If Ω is a domain in $\mathbb{C}^n$ and p satisfies Condition (H)on Ω, then we have the following two lemmas as in Hormander (1967).

Lemma 1.1
If $f \in A_p(\Omega)$ it follows that $\frac{\partial f}{\partial z_j} \in A_p(\Omega), 1 \leq j \leq n$.

Lemma 1.2 If f is holomorphic in Ω, then $f \in A_p(\Omega)$ if and only if for some $K > 0$

$$\int_\Omega |f|^2 \exp(-2K_p(z)d\lambda < \infty,$$

where $d\lambda$ denotes Lebesgue measure.

Our main Theorem is therefore the following
Theorem Let Ω be a domain in $\mathbb{C}^n$ and p a fuction on Ω satisfying Condition (H). Then a finite set of functions in $A_p(\Omega), f_1, ..., f_N$ generates $A_p(\Omega)$ if and only if (2) is valid.

To prove this theorem we follow the homological argument given in Hormander(1967), almost word for word, using Lemma 1.1 and 1.2 and $\mathcal{L}^p-$Carleman estimates for the $\bar{\partial}-$operator on Ω,which we establish in the next section.

$\mathcal{L}^p-$Carleman Estimates For The $\bar{\partial}-$Operetor

For $1 \leq p \leq \infty$, let $\mathcal{L}^p_{(r,q)}(U)$ denote the space of forms of type (r,q) with coefficients in $\mathcal{L}^p(U)$,

$$f = \sum_{|I|=r}{}' \sum_{|J|=q}{}' f_{I,J} dz^I \wedge d\bar{z}^J \qquad (3)$$

where $\sum'$means that the summation is performed only over strictly increasing multi-indices,

$$I = (i_1, ..., i_r), J = (j_i, ..., j_q),$$

$$dz^I = dz_{i_1} \wedge ... \wedge dz_{i_r}, d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q},$$

and U is open in C^n. The norm of the $(r,q)-$form in (3) is defined by

$$||f||_{\mathcal{L}^p_{(r,q)}(U)} = \left\{ \sum_I{}' \sum_J{}' ||f_{I,J}||^p_{\mathcal{L}^p(U)} \right\}^{\frac{1}{p}},$$

$1 \leq p < \infty$. Let $B_q(\xi, z)$ be the Bochner-Martinelli-Koppelman kernel of degree $(0,q)$ in z and degree $(n, n-q-1)$ in ξ, so that, with $\beta = |\xi - z|^2$,

$$B_q(\xi, z) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} \beta^{-n} \partial_\xi \beta \wedge \left(\bar{\partial}_\xi \partial_\xi \beta\right)^{n-q-1} \wedge \left(\bar{\partial}_z \partial_\xi \beta\right)^q \qquad (4)$$

for $0 \leq q \leq n$.

An upper semicontinuous function φ is said to be admissible in an open set U in $\mathbb{C}^n$, if for every coefficient $b_q(\xi, z)$ of $B_q(\xi, z), 0 \leq q \leq n$,

$$\int_U |b_q(\xi, z)| e^{-\varphi(z)} d\lambda(z) \leq C, \int_U |b_q(\xi, z)| e^{-\varphi} d\lambda(\xi) \leq C \qquad (5)$$

where $C > 0$ is a constant and λ is Lebesgue measure.

For an upper semicontinuous φ we define $\mathcal{L}^p(U, \varphi)$ where U is open in C^n by

$$\mathcal{L}^p(U,\varphi) := \left\{ g \text{ is measurable on } U : \int_U |g|^p e^{-\varphi} d\lambda < \infty \right\}, \quad (6)$$

$1 \leq p < \infty$, and

$$||g||_{\mathcal{L}^p(U,\varphi)} = \left\{ \int_U |g|^p e^{-\varphi} d\lambda \right\}^{1/p}.$$

$\mathcal{L}^p_{(r,q)}(U,\varphi)$ is the space of $(r,q)-$forms with coefficients in $\mathcal{L}^p(U,\varphi)$, and if f is as in (3),

$$||g||_{\mathcal{L}^p(U,\varphi)} = \left\{ \sum_I{}' \sum_J{}' ||f_{I,J}||^p_{\mathcal{L}^p(U,\varphi)} \right\}^{1/p}$$

$1 \leq p < \infty$.

Our second main result is

Theorem 2.1
Let Ω be a domain in $\mathbb{C}^n$ and let $f \in \mathcal{L}^p_{(0,q+1)}(\Omega\varphi)$ be $\bar{\partial}-$closed, ,$1 < p < \infty$ and φ an upper semi continuous function admissible in Ω. Then there is $u \in \mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ such that $\bar{\partial}u = f$ and

$$||u||_{\mathcal{L}^p_{(0,q)}(\Omega,\varphi)} \leq \delta ||f||_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)},$$

where δ is independent of f.

To prove Theorem we need a lemma about Sobolev Space estimates for the $\bar{\partial}-$operator on bounded domains in $\mathbb{C}^n$ with boundaries of Lebesgue measure zero. Accordingly, let $W^{1,1}(U)$ be the space of functions which together with their distributional derivatives of order one are in $\mathcal{L}^1(U)$, with the usual norm, and $W^{1,1}_{(r,q)}(U)$ is the space of $(r,q)-$forms with coefficients in $W^{1,1}(U)$. We then have

Lemma 2.2
Let Ω be a bounded domain in $\mathbb{C}^n$ with boundary of Lebesgue measure zero. Let $f \in W^{1,1}_{(0,q+1)}(\Omega)$ be $\bar{\partial}-$closed, then there is a $u \in W^{1,1}_{(0,q)}(\Omega)$ such that $\bar{\partial}u = f$.

To prove Lemma 2.2 we need the Bochner-Martinelli-Koppelman formula:

Theorem 2.3
Let Ω be any bounded domain in $\mathbb{C}^n$ with C^1 boundary. for $f \in C^1_{(0,q)}(\Omega), 0 \leq q \leq n$, we have

$$f(z) = \int_{\partial\Omega} B_q(\cdot,z) \wedge f + \int_{\Omega} B_q(\cdot,z) \wedge \bar{\partial}_\xi f$$
$$+ \bar{\partial}_z \int_\Omega B_{q-1}(\cdot,z) \wedge f, z \in \Omega \quad (7)$$

where $B_q(\xi,z)$ is in (4).

Proof of Lemma 2.2. With Ω and f as in Lemma 2.2, if

$$u(z) = \int_\Omega B_q(\cdot,z) \wedge f, z \in \Omega, \quad (8)$$

then $\bar{\partial}u = f$:

Let $f = \sum_J' f_J d\bar{z}^J$ be defined as zero outside Ω and regularize f coefficient-wise: $f_m = \sum_J (f_J)_m d\bar{z}^J$,

where

$$(f_J)^{(z)}_m = \int_{C^n} f_J(z - \xi/m)\varphi(\xi) d\lambda(\xi)$$

$$m^{2n} \int_{C^n} f_J(\xi)\varphi(m(z-\xi)) d\lambda(\xi)$$

and $\psi \in C^\infty_0(\mathbb{C}^n), \int \psi d\lambda = 1, \psi \geq 0$,supp $\psi = \{z \in \mathbb{C}^n : |z| \leq$

$1\}$, and λ is Lebesgue measure. Then $||f_m||_{\mathcal{L}^p_{(0,q+1)}(\mathbb{C}^n)}, f_m \to f$ in $\mathcal{L}^1_{(0,q+1)}(\Omega)$ as $m \to \infty$ and f_m is $\bar{\partial}-$closed in $\mathbb{C}^n$.

Now let

$$u_m(z) = \int_{\mathbb{C}^n} B_q(\cdot, z) \wedge f_m. \qquad (9)$$

Then from theorem 2.3, we have $\bar{\partial} u_m = f_m$, and since $f_m \to f$ in $\mathcal{L}^1_{(0,q+1)}(\Omega)$, we have $u_m \to u$ in $\mathcal{L}^1_{(0,q)}(\Omega)$, and $\bar{\partial} u = f$. Proof of Theorem 2.1. We first assume that Ω is bounded. It is clear that there is a sequence $\Omega_1 \subset\subset \Omega_2 \subset\subset ...$ of bounded domains, each with boundary of Lebesgue measure zero, such that $U^{\infty}_{\nu=1}\Omega_v = \Omega$. We construct a sequence of $(0,q)-$forms $\{u_v\}^{\infty}_{v=1}$ with $u_v \in \mathcal{L}^p_{(0,q)}(\Omega,\varphi), \bar{\partial} u_v = f$ in Ω_v and

$$||u_v||_{\mathcal{L}^p_{(0,q)}(\Omega_v,\varphi)} \leq K||f||_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)},$$

where K is the same for all $v, 1 < p < \infty$.Let us regularize f as above. For v fixed, if m is sufficiently large, $f_m \in W^{1,1}_{(0,q+1)}(\Omega_v)$ and $\bar{\partial} f_m = 0$ in Ω_v. For such an m(sufficiently large) define

$$g_m = \begin{cases} f_m & \text{in } \Omega_\nu \\ 0 & \text{outside } \Omega_\nu \end{cases}$$

Then from Lemma 2.2, if

$$u_{v,m} = \int_{\Omega v} B_q(\cdot, z) \wedge g_m,$$

$$\bar{\partial} u_{v,m} = g_m \text{ in } \Omega_v$$

and since φ is admissible on Ω_v

$$||u_{v,m}||_{\mathcal{L}^p_{(0,q)}(\Omega_v,\varphi)} \leq K||f||_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)}.$$

Now it is clear that as $m \to \infty, g_m \to f$ in $\mathcal{L}^1_{(0,q+1)}(\Omega_v)$ and $u_{v,m} \to$some u_v in $\mathcal{L}^1_{(0,q)}(\Omega_v), \bar{\partial} u_v = f$ and

$$||u_v||_{\mathcal{L}^P_{(0,q)}(\Omega_v,\varphi)} \leq K||f||_{\mathcal{L}^P_{(0,q+1)}(\Omega,\varphi)}. \qquad (10)$$

Define u_v as zero outside Ω_v, then since $\mathcal{L}^P_{(0,q)}(\Omega,\varphi)$ is reflexive, for $1 < p < \infty$, by the Banach-Alaoglu Theorem, there is u in $\mathcal{L}^P_{(0,q)}(\Omega,\varphi)$ with

$$||u||_{\mathcal{L}^P_{(0,q)}(\Omega_v,\varphi)} \leq ||f||_{\mathcal{L}^P_{(0,q+1)}(\Omega,\varphi)}, \qquad (11)$$

$(1 < p < \infty)$, and a subsequence $\{u_{v\lambda}\}$ of $\{u_v\}$ such that $u_{v\lambda} \to u$ weekly in $\mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ as $\lambda \to \infty$. In particular, $u_{v\lambda} \to u$ in the sense of distributions, as $\lambda \to \infty$. Therefore $\bar{\partial} u = f$. If Ω is not bounded, we can find a sequence of bounded domains $\Omega_1 \subset\subset \Omega_2 \subset\subset ...$ exhausting Ω and a sequences of $(0,q)-$forms $\{u_v\}^{\infty}_{v=1}$ as above, such that $\bar{\partial} u_v = f$ on Ω_v and

$$||u_v||_{\mathcal{L}^P_{(0,q)}(\Omega_v,\varphi)} \leq K||f||_{\mathcal{L}^P_{(0,q+1)}(\Omega,\varphi)}. \qquad (12)$$

and K is the same for all v.

Treating the sequence in (12) as the sequence in (10) was treated, we get an $(0,q)-$form $u \in \mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ with $\bar{\partial} u = f$ and

$$||u_v||_{\mathcal{L}^P_{(0,q)}(\Omega_v,\varphi)} \leq K||f||_{\mathcal{L}^P_{(0,q+1)}(\Omega,\varphi)}.$$

The format of the proof is the same as that in (2) : Because of (1) and (2), where $|f|^2 = |f_1|^2 + ... + |f_N|^2$, for each

$$V_j = \frac{\bar{f}_j}{|f|^2}$$

there is $K > 0$ such that

$$\int_\Omega |V_j|^2 \exp(-2Kp) d\lambda < \infty \qquad (13)$$

and it is clear that

$$\sum_{j=1}^{N} V_j f_j = 1. \qquad (14)$$

For non-negative integers s amd r let L_r^s denote the set of all differential forms h of type $(0, r)$ with values in $\Lambda^s C^N$, such that for some $K > 0$

$$\int_\Omega |h|^2 \exp(-2Kp) d\lambda < \infty. \qquad (15)$$

This means that for each multi-index $I = (i_1, ..., i_s)$ of lenght $[I] = s$ with indices between 1 and N inclusively, h has component h_I which is a differential form of type $(0, r)$ such that h_I is skew symmetric in I and

$$\int_\Omega |h_1|^2 \exp(-2Kp) d\lambda < \infty. \qquad (16)$$

As in [3], $\bar\partial$ is an unbounded operator from L_r^s to L_{r+1}^s and the interior product P_f by $(f_1, ..., f_N)$ maps L_r^{s+1} into L_r^s.

$$(P_I(h))_I = \sum_{j=1}^N h_{I_j} f_j, \quad |I| = s \quad (17)$$

If we define $P_f L_r^0 = 0$, then clearly $P_f^2 = 0$ and P_f commutes with $\bar\partial$, so we have a double complex. We now have (as in Hormander [1967]) the following

Theorem 3.1
For every $g \in L_r^s$ with $\bar\partial g = Pfg = 0$ one can find $h \in L_r^{s+1}$ so that $\bar\partial h = 0$ and $P_f h = g$.

Now from (14) $P_f \bar\partial V = \bar\partial P_f V = \bar\partial(1) = 0$, where $V = (V_1, ..., V_N)$, therefore by Theorem 3.1 there exist $w \in L_1^2$ with $P_f w = \bar\partial V$ and $\bar\partial w = 0$. let $k \in L_0^2$ solve $\bar\partial k = w$ and set

$$h = V - P_f k \in L_0^1 \qquad (18)$$

Then $\bar\partial h = \bar\partial V - P_f w = 0$ and

$$P_f(f) = P_f V = 1 \qquad (19)$$

i.e. there exist $h_1, ..., h_N \in A_p(\Omega)$ such that

$$\sum_{j=1}^N h_j f_j = 1. \qquad (20)$$

Chen S.C & Shaw M.C.(2001) *Partial Differential Equations in Several Complex Variables.* Studies in Advanced Mathematics, No. 19 AMS-International Press.

Darko P.W.(2005) The Corova Problem in Carleman Algebras on Stein Manifolds. Result. Math **48** 203-205.

Hormander L. (1967) Generators for Some Rings of Analytic Functions. Bull. AMS **73**, 943-949.