Ghana J. Sci. **50** (2010), 29-33

THE ELLIPTIC MONGE-AMPÈRE EQUATION-EXACT SOLUTION

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Abstract
Using fundamental solutions, exact solutions are constructed for the real Monge-Ampère equation. Continuous, L^p, Soblev, and Hölder estimates are also obtained for the solutions. Finally, for the Dirichlet problem for the equation, a solution is obtained.

Introduction

Starting from the results in [1] we constructed what must be called a fundamental solution of the complex Monge-Ampère operator in [2], and we obtained continuous and L^p estimates for that operator. We consider the real Monge-Ampère operator on arbitrary bounded domains in $\mathbb{R}^n$, unlike the classical results which were obtained on convex domains. We obtain solutions in distributions, viscosity solutions, L^p-estimates, Sobolev estimates and Holder estimates. For the free boundary problem we obtain exact solutions using integral representation constructed from the fundamental solution, from which we get the estimates. We finish by obtaining a solution for the Dirichlet problem for the same equation. References for the recent works on the Monge-Ampère equation are in the references in [3].

We consider the Monge-Ampère equation in the form

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \text{ on } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where at least $f \geq 0$ in Ω, $f^{\frac{1}{n}} \in L^1(\Omega)$, Ω being bounded and open in $\mathbb{R}^n$.

For the distributional solution, the viscosity solution, the L^p-estimates and the Dirichlet problem the boundary of Ω need not be smooth. For the sobolev estimates we require the boundary of Ω to have Lesbesgue measure zero, and for the Hölder estimates we require the boundary of Ω to be smooth.

Preliminaries

Our estimates come from the explicit integral representation which we use and the estimates are almost immediate. Here we recall the definition of Sobolev Spaces and Hölder Spaces.

Let Ω be an open set in ^{n}, $1 \le s \le \infty$, $1 \le p \le \infty$, the Sobolev space W (Ω) := the space of all distributions u defined in Ω, such that

(a) $D^\alpha \in L^p(\Omega)$, for $|\alpha| \le m$, when $s = m$ is a nonnegative integer,

(b) $u \in W^m_p(\Omega)$ and $\int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{n+\sigma p}} dxdy < \infty$, for $|\alpha| = m$, when $s = m + \sigma$ is nonnegative and is not an linteger.

With norm

$$||u||_{m,p,\Omega} = \sum_{|\alpha| \le m} \int_\Omega |D^\alpha u|^p \quad \frac{1}{p} \qquad (2.1)$$

in case (a) and

$$||u||_{s,p,\Omega} = \quad ||u||^p_{m,p,\Omega} + \sum_{|\alpha| = m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{n+\sigma p}} dxdy \quad \frac{1}{p} \qquad (2.2)$$

in case (b).

For Ω still an open set in $0 < \alpha < 1$, $k \ge 0$ an integer, the Hölder Space $^{k,\alpha}(\Omega)$:= the space of functions on Ω such that

$$|f|_{C^{k,\alpha}}(\Omega) := \sup_\Omega |f| + \sum_{0<|\omega|<\kappa} \sup_{x \ne y\, x,y \in \Omega} \frac{|D^w f(x) - D^w f(y)|}{|x-y|^\alpha} < \infty \quad (2.3)$$

where $D^w \Omega$ is a derivative of order $|w|$, $w = (w_1, ..., w_n)$, $w_j \ge 0$.

Results

Throughout the section is a bounder domain in ^{n}.

Theorem 3.1. Let $f \ge 0$, $f \in L^1(\Omega)$ then there is u such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \text{ in distributions} \qquad (3.1)$$

*Theorem 3.2***.** Let $f \ge 0$, f continuous and $f \in L^p(\Omega)$. Then there is a continuous u, such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \qquad (3.2)$$

Therefore u is a viscosity soultion.

Theorem 3.3. Let $f \geq 0$ and $f \in L^p(\Omega)$, $1 \leq p \leq \infty$. Then there is $u \in L^p(\Omega)$, such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f, \qquad (3.3)$$

and there is a K independent of f such that

$$\|u\|_{L^p((\Omega)} < K \| f \|_{L^p(\Omega)} \qquad (3.4)$$

Theorem 3.4. Let $f > 0$, $f \in W \;\; (\Omega)$, $1 \leq p \leq \infty$, $1 \leq s \leq \infty$, and let the boundary of Ω be of Lebesque measure zero, Then there is $u \in W^{\;\;+2}(\Omega)$, such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \qquad (3.5)$$

and $\| u\|_{s+2,p,\Omega} \leq K \| f \|_{s,p,\Omega}$ for some constant K independent of f.

Theorem 3.5. Let $f \geq 0$, $f \in {}^{k,\alpha}(\Omega)$, $0 < \alpha < 1$, $k \geq 0$ an integer, and let the boundary of Ω be smooth. Then there is a $u \in {}^{k+2,\alpha}(\Omega)$ such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \qquad (3.6)$$

and $$|u|_{k+2,\alpha}(\Omega) \leq K \, | f |_{k,\alpha} \, \mathbb{C}_{k,\alpha}(\Omega) \qquad (3.7)$$

for some constant K independent of f.

Theorem 3.6. Let $f \geq 0$, $f \in {}^{0}(\;)$ and $g \in {}^{0}(\;)$. Then there is a u defined on $u \in {}^{0}(\;)$, such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \qquad (3.8)$$

and $$u = g \text{ on } \partial\Omega \qquad (3.9)$$

Solutions and Estimates

In this section we prove those parts of the theorems that need to be proved.

For Theorem 1, let *e* be a fundamental solution for $\frac{\partial^2}{\partial x^2}$ in , that is $\frac{\partial^2 e}{\partial x^2} = \delta$, the Dirac delta in .

Define the distribution *Ej* in ^{n} by

$$\text{Ej}(\varphi) = e(\varphi(0, 0, ..., j, ..., 0, 0)) \tag{4.1}$$

the action of *e* being in the *j*th coordinate; $\varphi \in$ D(Rn) – a test function.

Let *f* be zero outside Ω and define *u* by

$$v = (E_1 + E_2 + ... + E_n) * (f \quad) \tag{4.2}$$

where * is convolution. It is then clear that *u*, the restriction of *v* to Ω satisfies

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \text{ in distributions.} \tag{4.3}$$

To prove Theorem 2, let Ω_1, Ω_2 Ω_3 ..., with U $\Omega = \Omega$, be an exhaustion of Ω. Let $\{\varphi_v\} \infty 1$ be a sequence of functions with $\varphi_v \in$ (Ω_{v+1}), φ_v 1 on Ω_v, $0 \leq \varphi_v \leq 1$.

Define $Uv \in {}^0(\ {}^n)$ by

$$\text{Uv} = (E_1 + E_2 + ... + E_n) * (\varphi_v \, f^{\,i}_{\,n}), \tag{4.4}$$

where again * is convolution.

Now, it is clear that det $\frac{\partial^2 v_v}{\partial x_i \, \partial x_j}$ $= f$ in Ω_v and $\{\nu_v\}$ tends locally uniformly to a continuous u such that

$$\det \frac{\partial^2 u}{\partial x_i \, \partial x_j} = f \text{ on } \Omega \tag{4.5}$$

Therefore, *u* is a viscosity soultion of the equation.

To prove Theorem 3, let E_j $(1 \leq j \leq n)$ be as above, let *f* be zero outside Ω and Define.

$$v = (E_i + E_2 + ... + E_n) * (f \quad), \tag{4.6}$$

where * is again convolution. Then u, the restriction of v to Ω satisfies the conclution of Theorem 3.

Also if f and Ω satisfy the hypothesis of Theorem 4 and f is defined to be zero outside Ω as above. Then the u constructed above satisfies the conclusion of Theorem 4.

Now, let f and Ω satisfy the hypothesis of Theorem 5, Extend f to be in $C^{k,\alpha}(\ ^{n})$ and let $\psi \in\ _0^{\infty}(\ ^{n})$ be identically equal to one in a compact neighborhood of . Define

$$v = (E_1 + E_2 + ... + E_n) * (f^{\frac{i}{n}}) \qquad (4.7)$$

Then u, the restriction of v to Ω, satisfies the conclusion of Theorem 5.

Finally to prove Theorem 6, let $\{\varphi_v\}$ be the sequence in $_0^{\infty}(\Omega)$ constructed in part two of this section and define

$$u_v := \{ (E_1 + E_2 + ... + E_n) * (\varphi_v f \quad)\}\ \varphi_v + (1 - \varphi_v)g \qquad (4.8)$$

Then $\det \dfrac{\partial^2 v_v}{\partial x_i\, \partial x_j} = f$ in Ω_v and $\{u_v\}$ converges locally uniformly in Ω to a function u in $C^0(\Omega)$ such that

$$\det \frac{\partial^2 u}{\partial x_i\, \partial x_j} = f \ \text{ in } \ \Omega \qquad (4.9)$$
$$u = g \quad on \quad \partial\Omega.$$

References

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Received 23 May 09; revised 2 Sept 09.