SOBOLEV AND L^P-CARLEMAN ESTIMATES FOR THE $\bar{\partial}-$OPERATOR ON BOUNDED DOMAINS IN $\mathbb{C}^n$

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Abstract

Sobolev and Carleman estimates are obtained for the $\bar{\partial}-$operator on all bounded domains in $\mathbb{C}^n$ with boundaries of Lebesgue measure zero

Introduction

The work in this paper stems from the confluence of three ground breaking results. The first is Hormander's L^2-estimates for the $\partial-$operator on pseudoconvex domains Hormander 1965.The second is L^p-estimates for the ∂-operator on strongly pseudoconvex domains obtained by Kerzman,(1971) and Ovrelid (1971) . The last is the work by Beals, Greiner, and Stanton (1987) on domains which satisfy the so-called Condition $z(q)$. Three different methods are used in the above mentioned works. Hormander uses Hilbert space methods, Kerzman and Ovrelid use integral representation methods of Ramirez-Henkin type and Beals, Greiner and Stanton use psuedodifferential operations. In extending their results to bounded domains in $\mathbb{C}^n$ with boundaries with Lebesgue measure zero, we use integral representation methods of Martinelli, Bochner and Koppelman type. These are not as sophisticated as the Ramirez-Henkin types, but they are still very powerful. We first obtain L^p-Sobolev estimates for the $\bar{\partial}$-operator on all bounded domains in $\mathbb{C}^n$ with boundries with Lebesgue measure zero and then use these estimates to obtain L^p-Carleman estimates for the ∂-operator on all bounded in C^n (regardless of boundaries). For $1 \leq p \leq \infty$, let $L^p_{(r,q)}$ denote the space of forms of type (r, q) with coefficients in $L^p(U), i.e$

$$f = \sum_{|I|=r}{}' \sum_{|J|=q}{}' f_{I,J} dz^I \wedge d\bar{z}^J, \quad (1)$$

where $\sum'$means that the summation is performed only over strictly increasing multi-indicies,

$$I = (i_1, ..., i_r),\ J = (j_1, ..., j_q),$$

$dz^I = dz_{i_1} \wedge ... \wedge dz_{i_r}, d\bar{z}^J = d\bar{z}_{j1} \wedge ... \wedge d\bar{z}_{j_q}, U$ is open in $\mathbb{C}^n$. The norm of the $(r, q)-$form in (1) is defined by

$$||f||_{L^p_{(r,q)}}(U) = \left\{ \sum_I{}' \sum_J{}' ||f_{IJ}||^p_{L^p(U)} \right\}^{\frac{1}{p}},$$

$1 \leq p < \infty$, and

$$||f||_{L^\infty_{(r,q)}(U)} = \max_{I,J} \ ||f_{I,J}|| L^\infty(U).$$

Let $W^{k,p}(U), 1 \leq p \leq \infty, k = 1,2,3...$ be the space of functions which together with their distributional derivatives of order through k are in $L^p(U)$, with the actual norm, and $W^{k,p}_{(r,q)}(U)$ the space of $(r,q)-$forms with coefficients in $W^{k,p}(U)$, with norm defined by

$$||f||_{W^{k,p}_{(r,q)}(U)} := \left\{ \sum_I {}' \sum_J {}' ||f_{I,J}||_{W^{k,p}(U)} \right\}^{\frac{1}{p}}, 1 \leq p < \infty,$$

and

$$||f||_{W^{k,\infty}_{(r,q)}(U)} := \max_{I,J} ||f_{I,J}||_{W^{k,\infty}(U)}.$$

Let $B_q(\xi, z)$ be the Bochner-Martinelli-Koppelman kernel of degree $(0,q)$ in z and degree $(n, n-q-1)$ in ξ, so that, with $\beta = |\xi - z|^2$,

$$B_q(\xi, z) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} \beta^{-n} \partial_\xi \beta \wedge (\bar{\partial}_\xi \partial_\xi \beta)^{n-q-1} \wedge (\bar{\partial}_z \partial_\xi \beta)^q \qquad (2)$$

for $0 \leq q \leq n$.

A plurisubharmonic function φ is said to be admissible on a bounded open set U in $\mathbb{C}^n$, if for every coefficient $b_q(\xi, z)$ of $B_q(\xi, z) 0 \leq q \leq n$

$$\int_U |b_q(\xi, z)| e^{-\varphi(z)} d\lambda(z) \leq C, \int_U |b_q(\xi, z)| e^{-\varphi(\xi)} d\lambda(\xi) \leq C \qquad (3)$$

where $C > 0$ is a constant and λ is Lebesgue measure.

For a plurisubharmonic φ we define $L^p(U, \varphi)$ where U is open in $\mathbb{C}^n$ by

$$L^p(U, \varphi) := \left\{ g \text{ is measurable on } U : \int_U |g|^p e^{-\varphi} d\lambda < \infty \right\}, 1 \leq p < \infty, \qquad (4)$$

and

$$||g|| L^p(U, \varphi) = \left\{ \int_U |g|^p e^{-\varphi} d\lambda \right\}^{\frac{1}{p}}.$$

$L^p_{(r,q)}(U, \varphi)$ is the space of $(r,q)-$forms with coefficient in $L^p(U, \varphi)$, and if f is as in (1) our results are

$$||f|| L^p_{(r,q)} = \left\{ \sum_I{}' \sum_J{}' ||fI, J||^p_{L^p(U,\varphi)} \right\}^{\frac{1}{p}}.$$

Theorem 1

Let Ω be a bounded domain in $\mathbb{C}^n$ with boundary of Lebesgue measure zero. Let for $k \geq 1 f \in W^{k,p}_{(0,q+1)}(\Omega)$ be a $\bar{\partial}-$closed, then there is a $u \in W^{k,p}_{(0,q)}(\Omega)$ such that $\bar{\partial} u = f$ and

$$||u||_{W^{k,p}_{(0,q)}(\Omega)} \leq \delta ||f||_{W^{k,p}_{(0,q+1)}}(\Omega)$$

where δ is independent of $f (1 \leq p \leq \infty)$.

Theorem 2

Let Ω be any bounded domain in C^n and let $f \in L^p_{(0,q+1)}(\Omega, \varphi)$be $\bar{\partial}-$closed, $1 < p < \infty$, and φ plurisubharmonic and admissible in Ω. Then there is $u \in L^p_{(0,q)}(\Omega, \varphi)$ such that $\bar{\partial} u = f$ and

$$||u||_{L^p_{(0,q)}(\Omega,\varphi) \leq \partial ||f||_{L^p_{(0,q+1)}}(\Omega,\varphi)},$$

where δ is independent of f.

Bochner-Martinelli-Koppelman Formular and $\bar{\partial} u = f$

Theorem 3

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. For $f \in C^1_{(0,q)}(\Omega)$, $0 \leq q \leq n$, we have

$$f(z) = \int_{\partial\Omega} B_q(.,z) \wedge f + \int_{\Omega} B_q(.,z) \wedge \bar{\partial}_\xi f$$

$$+ \bar{\partial}_z \int_{\Omega} B_q - 1(.,z) \wedge f, z \in \Omega \qquad (5)$$

where $B_q(\xi, z)$ is as in (2).

Lemma 4

With Ω and f as in Theorem 1.1, if

$$u(z) = \int_{\Omega} B_q(.,z) \wedge f, z \in \Omega \qquad (6)$$

then $\bar{\partial} u = f$.

Proof.

Let $f = \sum'_J f_J d\bar{z}^J$ defined as zero outside Ω and regularize f coefficientwise: $f_m = \sum'(f_J)_m d\bar{z}^J$, where

$$(f_J) = \int_{C_n} f_J(z - \xi/m)\psi(\xi) d\lambda(\xi)$$

$$= m^{2n} \int_{\mathbb{C}^n} f_J(\xi)\psi(m(z-\xi)) d\lambda(\xi) \qquad (7)$$

and $\psi \in C^\infty_0(\mathbb{C}^n), \int \psi d\lambda = 1, \psi \geq 0$, supp $\psi = \{z \in \mathbb{C}^n : |z| \leq 1\}$ and λ is Lebesgue measure. Then $||fm||_{L_{(0,q+1)}(C^n)} \leq ||f||_{L^p_{(0,q+1)}(\mathbb{C}^n)}$ for$1 \leq p \leq \infty, f_m \to f$ in$L^1_{(0,q+1)}(\Omega)$ as $m \to \infty$ and f_m is $\bar{\partial}-$closed in $\mathbb{C}^n$.

$$u_m(z) = \int_{\mathbb{C}^n} B_q(\cdot, z) \wedge f_m. \qquad (8)$$

Then, from Theorem 3, we have

$$\bar{\partial} u_m = f_m,$$

and since $f_m \to f$ in $L^1_{(0,q+1)}(\Omega)$, we have $u_m \to u$ in $L^1_{(0,q)}$, and $\bar{\partial} u = f$.

L^p-Sobolev etimates

In this section we indicate how the estimates in Theorem 1 should be arrived at. Now, from (8)

$$\partial^\alpha u_m(z) = \int_{\mathbb{C}^n} B_q(\cdot, z) \partial^\alpha f_m, \qquad (9)$$

where

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial_{y_1}^{\alpha_2} ... \partial x_n^{\alpha_{2n-1}} \partial y_n^{\alpha_{2n},}}$$

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_{2n-1}, \alpha_{2n})$$

$$z = (x_1 + iy_1, ..., x_n + iy_n) \; i = \sqrt{-1}$$

and the derivatives are taken coefficientwise. Therefore, the desired estimate follows upon letting $m \to \infty$ in (9) and estimating.

L^p-Carleman etimates

1. In this section we prove Theorem 2. Let Ω, φ and f be as in Theorem 2. We then have the following

   Lemma 5
   There is a sequence $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots$ of bounded domains,each with boundary of Lebesgue measure zero, such that $\bigcup_{\nu=1}^{\infty} \Omega_\nu = \Omega$, and sequence of $(0,q)-$forms $\{u_\nu\}_{\nu=1}^{\infty}$ with $u_\nu \in L^p_{(0,q)}(\Omega_{\nu,\varphi}), \bar{\partial} u_\nu = f$ in Ω_ν and

   $$||u_\nu||_{L^p_{(0,q)}(\Omega,\varphi)} \leq K||_{L^p_{(0,q+1)}(\Omega,\varphi)},$$

   where K is the same for all $\nu, 1 < p < \infty$.

   Proof.
   The first part is clear. Let us regularize f as in the proof of theorem 1.

   For ν fixed, if m is sufficiently large $f_m \in W^{1.}_{(0,q+1)}(\Omega_\nu)$ and $\bar{\partial} f_m = 0$ in Ω_ν. For such an m(sufficiently large) define

   $$g_m = \begin{cases} f_m & \text{in } \Omega_\nu \\ 0 & \text{outside } \Omega_\nu \end{cases}$$

   Then from Lemma 4, if

   $$u_{\nu,m}(z) = \int_{\Omega_\nu} B_q(\cdot, z) \wedge g_m$$

   $$\bar{\partial} u_{\nu.m} = g_m \text{ in } \Omega_\nu$$

   and since φ is admissible on Ω_ν

   $$||u_{\nu,m}||_{L^p_{(0,q)}(\Omega_\nu,\varphi)} \leq K||f||_{L^p_{(0,q+1)}(\Omega,\varphi)},$$

   Now it is clear that as $m \to \infty, g_m \to f$ in $L^1_{(0,q+1)}(\Omega_\nu)$, and $u_{\nu,m} \to$ some u_ν in $L^1_{(0,q)}(\Omega_\nu), \bar{\partial} u_\nu = f$ in Ω_ν and

   $$||u_\nu||_{L^p_{(0,q)}}(\Omega_{\nu,q}) \leq K||f||_{L^p_{(0,q+1)}(\Omega,\varphi)}. \quad (10)$$

2. Now define u_ν as zero outside Ω_ν, then since $L^p_{(0,q)}(\Omega,\varphi)$ is reflexive for $1 < p < \infty$, by the Banach-Alaoglu Theorem, there is u in $L^p_{(0,q)}(\Omega,\varphi)$ with

   $$||u||_{L^p_{(0,q)}}(\Omega,\varphi) \leq K||f||_{L^p_{(0,q+1)}(\Omega,\varphi)} \quad (11)$$

   $(1 < p < \infty)$, and a subsequence $\{u_{\nu N}\}$ of $\{u_\nu\}$ such that $u_{\nu N} \to u$ weekly in $L^p_{(0,q)}(\Omega,\varphi)$ as $N \to \infty$. In particular, $u_\nu \to u$ in the sense of distributions, as $N \to \infty$. Therefore, $\bar{\partial} u = f$ and we are done.

Conclution

Using the techniques of Darko(2000,2002) we can get L^p-Sobolev and L^p-Carleman regularity for the $\bar{\partial}-$operator on relatively compact Stein domains of complex manifolds.

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