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SOLUTION WITH THE EQUATION $\bar{\partial} u = f$ WITH BOUNDS ON L^∞ Z-DERIVATION ON MANIFOLDS WITH PIECEWISE STRICTLY PSEUDOCONVEX BOUNDARIES

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Abstract

What is called the Andreotti-Stoll technique is used to solve the equation $\bar{\partial}$u = *f* with bounds. Locally, the Cauchy kernel isused to solve the equation. Globally, Sheaf cohomology is used to patch the local solutions together.

Introduction

Of the L^p estimates ($\propto$) for the $\bar{\partial}$-operator on strictly pseudoconvex domains (cf. [4], L^2 and L^∞ estimates are the most useful. The L^∞ estimates were extended to domains with priecewise smooth strictly pseudoconvex case were also established for the piecewise smooth strictly pseudocovex case.

In [2] and [3] the L^2 estimates were extended to the case of domains with piecewise strictly pseudoconvex boundaries where there was no smoothness assumption. It, therefore, seemed natural to expect some of the approximation theorems to hold in the case of piecewise strictly pseudoconvex domains as defined in [2]. To prove these approximation theorems in the absence of uniform estimates for $\bar{\partial}$ on piecewise strictly pseudoconvex domains (without any smoothness assumption), we propose here to use a technique used in [1] to estimate the solutions of $\bar{\partial} u = f$ in a norm that reduces to the uniform norm in the case of holomorphic functions.

We use the Cauchy kernel to estimate the solutions of $\bar{\partial} u = f$ in the new norm locally, and then globalize by means of the technique used in [2] and [5]. We then state an approximation theorem which our estimates can be used to prove (by now) routinely.

Preliminaries

(1) Let *U* be an open set in an n-dimensional complex manifold *X*. Define *Z*(*U*) by

$$Z(U) := \left\{ f \text{ measureable } U : \| f \|\, z\,(u) := \|f\|l\, L^\infty(U) + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial \bar{z}_j} \right\|_{W_1{}^\infty(U)} < \infty \right.$$

If $f =$ $f_j \, d$ $_j$ is an (0, 1)-form on U we define

$$\| f \| \, z\,(U) := \quad \| \, f_j \| \, _{Z\,(U)},$$

where $W^{1,\infty}(U)$ is the Sobolev space of functions whose first order derivatives are in $L^{\infty}(U)$. and similarly for $(0, \gamma)$-forms.

(2) Now let Ω be a relatively compact subdomain of a complex n-dimensional manifold X, with Ω having a piecewise strictly pseudoconvex boundary as defined in [2] that is the boundary ∂Ω of Ω is covered by finitely many open subsets U_j $(1 \leq j \leq k)$ of X and there are C^2 strictly plurisubharmonic functions ϱ_j on U_j $(1 < j \leq k)$ such that $\Omega_{\cap}\ (U^{k}_{j=1}\ U_j)$ is the set of all $x \in U \quad U_j$ which, for every $1 \leq j \leq k$, satisfy $x \notin U_j$ or $\varrho(x) < 0$. And let there be a C^2 strictly plurisubharmonic function defined on Ω. Our main result is the following.

Theorem 2.1. If f is a $\bar{\partial}$-closed (0, 1)-form on Ω with $\|f\|_{Z\,(\Omega)} < \infty$, then there is a $u \in Z\,(\Omega)$ such that

$$u = f \text{ in distributions and}$$
$$\|u\|_{z\,(\Omega)} \leq C \| \, f \, \|_{z\,(\Omega)},$$

where C depends on Ω.

(3) For our last theorem, let Ω be as above, and $K = \Omega$ the closure of Ω in X. Let $C\,(K)$ denote the Banach Space of continuous complex-valued functions on K with the uniform norm, and let $H\,(K)$ denote the closure in $C(K)$ of the space of functions which are holomorphic in some neighbourhood of K. Then we have:

Theorem 2.2. If $f \in C\,(K)$ and if for each $x \in K$ there is a neighbourhood U_x of x such that $f \in H\,(K) \cap U_x$), then $f \in H\,(K)$.

Given Theorem 2.1, Theorem 2.2 folows routinely.

Local Estimates

Let now $\Omega \subset\subset \quad ^n$ by a polycylinder with $\Omega = \Omega_1 \quad \ldots \times \Omega_n$ and each Ω_j having a boundary of plane measure zero. We then have the following.

Lemma 3.1. If $f =$ $f \, dz_j$ is a -closed (0, 1)- form and $\|f\| \, z\,(\Omega) < \infty$, then there is $u \in Z\,(\Omega)$ such that

$$u = f \text{ is distributions} \tag{3.1}$$

and

$$\| \, u \, \|_{Z\,(\Omega)} \leq C \, \| \, f \, \|_{Z(\Omega)} \tag{3.2}$$

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where C depends on Ω.

Proof. Extend f to all of ^{n} by zero outside Ω and call it again in f. Then $\bar{\partial} f = 0$ in the distribution sense in ^{n}. Perhaps we should recall what this means: In Ω that $\partial f = 0$ in the sense of distributions means that $\frac{\partial f_j}{\partial \ _k} \ \frac{\partial f_k}{\partial \ _j}$ in the sense of distributions ($1 \leq j, k \leq n$), and since the $\frac{\partial f_j}{\partial \ _k}$ ($1 \leq j, k \leq n$) are L^∞ (Ω) function, $\frac{\partial f_j}{\partial \ _k} = \frac{\partial f_k}{\partial \ _j}$ almost everywhere in Ω. Therefore, if we extend the f_j ($1 < j < n$) by zero outside Ω, then the $\frac{\partial f_j}{\partial \ _k}$ are in L^∞ (^{n}), since the boundary of Ω is of Lebesgue measure zero, and $\frac{\partial f_j}{\partial \ _k} = \frac{\partial f_k}{\partial \ _j}$ ($1 \leq j, k \leq in$) almost everywhere in ^{n}. This means, in particular, tha $f = 0$ in the sense of distribution in ^{n}.

Set $u(z) = (2\pi i)^{-1} \quad (\xi - z_1)^{-1} f_1 (\xi, z_2, ..., z_n)\, d\xi \wedge d\xi$,

then u satisfies (3.1) and (3.2).

Regularize f coefficientwise:

$$f_m = \sum_{j=1}^{n} (f_j)_m \, d\bar{z}_j,$$

$$(f_j)_m (z) = \quad f_j(z - \frac{\xi}{m})\, \varphi(\xi)\, d\lambda(\xi) = m^{2n} \int f_j(\xi) \varphi(m)(z - \xi))\, d\lambda(\xi), \qquad (3.4)$$

where $\varphi \in C_0^\infty(\quad)$, $\quad \varphi \, d\lambda = 1$, $\varphi \geq 0$ sup $\varphi = \{z : |z| \leq 1\}$, and λ is the Lebesgue measure.

Then $\| f_m \|_{L^1_{(0,1)}}$ and $f_m \to f$ in $L \quad$ (Ω) as $m \to \infty$ and f_m is -closed in ^{n}.

$$\text{Set } u_m(z) = (2\pi i)^{-1} \quad (\xi - z)^{-1} (f_1)_m (\xi - z_2, ..., z_n)\, d\xi \wedge d\bar{\xi}. \qquad (3.5)$$

$$\text{then } u_m(z) = -(2\pi i)^{-1} \quad \xi^{-1} (f_1)_m (z_1 - \xi, z_2, ..., z_n)\, d\xi \wedge d\bar{\xi}. \qquad (3.6)$$

From (3.5) and (3.6)

$$\frac{\partial u_m(z)}{\partial} = (2\pi i)^{-1} \quad (\xi - z_1)^{-1} \frac{\partial (f_\ell)_m (\xi, z_2, ..., z_n)}{\partial}\, d\xi \wedge d$$

$$= (f\ )_m(z) \qquad (3.7)$$

Therefore, $u_m = f_m$, and since $f_m \to f$ in L (Ω), and $u_m \to u$ in L^1 (Ω) we have $u =$ f and from (3.3).

$$||u||_{z(\Omega)} \leq C\ ||f||_{z(\Omega).}$$

for some C depending on Ω.

(Note that if f_1 0 and f 0 then there is f_{jo} 0 (jo > 1 which we can use in (3.3) in place of f_1 witht the appropriate changes.)

Global Estimates

(1) We first prove the following:

Lemma 4.1. Let Ω be a bounded domain in with boundary $\partial\Omega$ of Lebesque measure zero. If f is a -closed (0, 1) -form on Ω with $||f||_{z(\Omega)} < \infty$, then there is $u \in Z$ (Ω) such that $u = f$ and $||u||_{z(\Omega)}, < C\ ||f||_{z(\Omega)}$, where C depends on Ω.

Proof. If we extend f by zero outside Ω into a polidisk Ω' in which Ω is relatively compact, then f is still -closed in Ω' and $||f||_{z(\Omega')} < \infty$.

Therefore, Lemma 4.1 follows from Lemma 3.1.

(2) Now to prove Theorem 2.1, let Ω and f satisfy the hypothsis of that theorem. Fix $x_0 \in$ $\partial\Omega$ the boundary of Ω and choose a neighbourhood V of x_0 such that V is relatively compact in some coordinate neigbourhood of x_0 and such that $V \cap \Omega$ is biholomorphic to a bounded domain in ^{n} with boundary of Lebesgue measure zero. Then from Lemma 4.1 since $f = 0$ on $V \cap \Omega$, there is a $u_1 \in Z(V \cap \Omega)$ such that $u_1 = f$ on $V \cap \Omega$ and

$$||u_1||_{z(V\cap\Omega)} \leq C\ ||f||_{z(V\cap\Omega)} \quad (4.1)$$

for some C depending on $V \cap \Omega$. Choose $\varphi \in C$ (V) with φ 1 in a neighbourhood of x_0. Then $f_1 = f -$ (φu_1) is zero near x_0, and, hence, can be trivially extended to a domain Ω_1 which contains x_0. Ω_1 is obtained by a small perturbation of Ω_1 and, hence, can be assumed to have a piecewise strictly psendoconvex houndary. By repeating this step finitely many times, one can find an (0, 1)-form f' on a domain Ω' with a piecewise strictly pseudoconvex boundary such that $\Omega \subset\subset \Omega'$, and a function u' on Ω which stafifies

$$||f||_{z(\Omega')} \leq \mathrm{C}\ ||f||_{z(\Omega)} \quad (4.2)$$

$$||u'||_{z(\Omega)} \leq \mathrm{C}\ ||f||_{z(\Omega)} \quad (4.3)$$

$$f' = f - \delta u' \text{ on } \Omega. \quad (4.4)$$

3. Let O be the Sheaf of germs of holomorphic functions on X. Then there is a fundamental system of neighbourhoods $\{\Omega''\}$ of Ω such that each Ω'' is a

dommain with a piecewise strictly pseudoconvex boundary, and the restriction maps

$$(\Omega'', O) \rightarrow \quad (\Omega, O)$$

induce isomorphisms

$$\gamma_q: H^q(\Omega'', O) \rightarrow H^q(\Omega, O), \text{ for } q \geq 1. \quad (4.5)$$

The cohomology groups $H^q(\Omega''\ O)$ and $H^q(\Omega, O)$ can be computed from fine resolutions of O on Ω'' and Ω, respectively, by the Sheaf of germs of (O,γ)-forms locally in $Z(\Omega'')$ and $Z(\Omega)$.

By choosing Ω'' so that $\Omega \subset\subset \Omega''$ C Ω', from (4.4) above, f' is -exact on Ω and locally in $Z(\Omega'')$, and, therefore, denoting by $[f']$ the cohomology class in $H^1(\Omega'', O)$ represented by f', we have $\gamma_i [f] = 0$, by [3, Theorem 1]. Since γ_1 is injective, $[f_,] = 0$, i.e. $f' = \ v'$ for some v' locally in $Z(\Omega'')$. By shrinking Ω'', one can get $v' \in Z(\Omega'')$, and by interior regularity of elliptic partial differential operators, v' can be chosen so that

$$||v'||_{z(\Omega)} \leq C\ ||f'|_{z(\Omega)}, \text{ for some C.}$$

The function $u = u' + v'$ satisfies $u = f$ and

$$||u||_{z(\Omega)} \leq C\ ||f||_{z(\Omega)}, \text{ for some } C.$$

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