Ghana J. Sci. **50** (2010), 13-16

HÖLDER AND L^p ESTAMTES FOR THE $\bar{\partial}$-OPERATOR ON POLYCYLINDERS

P. W. DARKO
Department of Mathematics, Delaware State University, 1200 N. Dupont Highway, Dover, DE 19901–2277. E-mail: pdarko10@agl.com

Abstract

Hölder estimates are used to obtain L^p estmates for the $\bar{\partial}$-operator and to solve the Hölder-Gleason problem on admissible polycylinders.

Introduction

In trying to obtain L^p estimates for the -operator on polycylinders, we were led to the estimates in [1]. Those estimates proved to be very useful, but they were not of the usual kind. Here $1 < p < \infty$ we obtain L^p estimates for the -operator of the usual kind on polycylinders. This is done by using he Holder estimates for the -operator on the so called admissible poylcylinders in [2]. The Hölder estimates are again used to solve the Hölder-Gleason problem.

$\sum_{0\le|\gamma|\le k}$

Preliminaries

Let U be open in , $0 < \alpha < 1$, $k \geq 0$ an integer. We define $C^{k,\acute{a}}(U)$ to be the space of functions f on U such that

$$|f|_{C^{k,\alpha}(U)} := \sup_u |f| + \qquad \sup_u \frac{|D^\gamma f(x) - D^\gamma(y)|}{|x-y|^\alpha} < \infty,$$

where D^r is a derivative of order $|\gamma| = \sum_{j=1}^{n} \gamma_j$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_j \geq 0$

If $U \subset \mathbb{C}^n$ is open, we use the real underlying coordinates of considered as to define $C^{k,\alpha}(U)$.

If

$$\sum_{\substack{(i_1,\ldots,i_r)\\(j_i,\ldots,j_q)}} f_{i1,\ldots,ir,\,ji,\ldots,jq}\, dz_{i_1} \wedge \ldots dz_{ir} \wedge d\bar{z}_{ji} \wedge \ldots \wedge d\bar{z}_{jq}$$

is an (r,q)-form on U, where $\sum'$ means the summation is over increasing multi-indices, we write f as $\sum_I{}_J f_{I,J} dz^I \wedge d\bar{z}^J$ for short, $I = (i_1,\ldots, i_r)$, $J = (j_r,\ldots, j_q)$, and set

$$\left| f \right| C^{k,\alpha}_{r,q} = \max_{I,J} \left| f_{I,J} \right| C^{k,\alpha}(U)$$

A polycylinder is called admissible if it has factors with boundaries of plane measure zero.

The Hölder estimates theorem that we use is

Theorem 1. Let Ω be an admissible polycylinder in and $0 < \alpha < 1$, $k > 0$ an integer. There is $K > 0$ such that if f is a -closed $(r,q+1)$ - form on Ω with $|f|_{(\Omega)} < \infty$, then there i an (r, q)-form on Ω with $u = f$ and

$$|u|_{(\Omega)} \; \Omega \leq K|f|_{(\Omega)}$$

The (0,1) case of this theorem is proved in [2] and there is an indication there how the (0,q) case should be proved ($q > 1$). The formalism for proving the (0, q) case works for the (r, q) case.

Let $f = \sum_{I,J} f_I, \mathrm{j} dz^I \wedge dz^J$ be an (r,q)-form as above; we define

$$|f|_{(U)} := \max_{i,j} \|f\|_{L^p(U)}, \; (1 \leq p \leq \infty)$$

We then have

Theorem 2. Let Ω be any polycylinder in ^{n} and $1 < p < \infty$. There is K > 0 such that if f is a -closed $(r,q+1)$ - form on Ω with $\|f\|_{(\Omega)} < \infty$ then there is an (r,q) - form u on Ω with $u = f$ and

$$|u|_{(\Omega)} \leq K\| f \|_{(\Omega)}$$

With $A(\Omega)$ the space of holomorphic functions on Ω, $A^{k,\alpha}(\Omega) = A(\Omega)\, C^{k,\alpha}(\Omega)$,for Ω open in ^{n}.

The Hölder-Gleason theorem is then

Theorem 3 Let Ω be an admissible polycylinder in ^{n}, Let w ∈ Ω and $f_1, \ldots, f_N \in A^{k,\alpha}(\Omega)$ with $\{\omega\} = \{z \in \; : f_1(z) = \ldots = f_N(z) = 0\}$. Moreover, let $f \in A^{k,\alpha}(\Omega)$ ($k > 0$) such that here exist a neighbourhood U of ω and holomorphic functions $_1, \ldots \; _N$ on U with

$$f = \sum_{j=1}^{N} f_j \; _j \quad \text{on } U$$

There exist $g_1, \ldots, g_N \in A^{k,\alpha}(\Omega)$ with

$$f = \sum_{j=1}^{N} f_j \; _j$$

We do not prove Theorem 3 here, because the formalism for proving the Ck version on strictly pseudoconvex domains in [4] works here also.

L^p Estimates

We give a proof of the (0,1) case of Theorem 2 in this section, the (*r*,*q*)-case following similarly.

Let *f* satisfy the hypothesis of Theorem 2, then the following is true.

Lemma 4. There is a sequence of admissible polycylinders $\Omega_1 \subset\subset \Omega_2$ ... such that

$\Omega_v = \Omega$ and a sequence $\{u_v\}\}_{v=1}^{\infty}$ of function with $u_v \in L^p(\Omega_v)$, $\bar{\partial} u_v = f$ in Ω and

$$\| u_v \|_{L^P(\Omega v)} \leq K \|f\|_{L^P(\Omega)},$$

where K is the same for all *v*, $1 \leq p \leq \infty$.

Proof. Let $f =$ $f_1 \not\equiv 0$, where *f* satisfies the hypothesis of Theorem 2 in the (0, 1)-case. Define *f* to be zero outside Ω and still call it *f*. That, there is a sequence of admissible polycylinders $(\Omega_v)_{v \cong 1}^{\infty}$ exhausting Ω,is clear.

$\sum_{j=1} f_j d\bar{z}_j,$

Let *v* be fixed and regularize *f* coefficientwise:

$$f_m = (f_j)_m dz_j,$$

$$(f_j)_m = \int_{\mathbb{C}^n} f_j(z - \xi/m)\varphi(\xi)d\lambda\ \xi$$

$$= m^{2n} \int_{\mathbb{C}^n} f_j(\xi)\varphi\ (m)\ (z - \xi)d\lambda(\xi)$$

where $\varphi \in C_0^{\infty}(\mathbb{C}^n)$ $\varphi d\lambda = 1$, $\varphi > 0$, $\sup \varphi = \{z: |z| < 1\}$, and λ is Lebesgue measure, Then $f_m \in$ (Ω_v), $0 < \alpha < 1$, for m sufficiently large and $\bar{\partial} f_m = 0$ in Ω_v. For such an m (sufficiently large) define

$$g_m := \begin{matrix} f_m & \text{in } \Omega_v \\ 0 & \text{outside } \Omega_v. \end{matrix}$$

Then

$$g_m = \sum_{j=1}^{n} g_{i,m}\, d\ _i, \text{ and by Lemma 3 in [2], if}$$

$$uv,m = (2\pi i)^{-1} (\xi - z_1)^{-1} g_1,m\ (\xi, z_2, ..., z_n)\ d\xi \wedge d\xi$$

$$\bar{\partial} u_{\xi,m} = f_m \text{ in } \Omega_v \text{ and}$$

$$\|u_{v,m}\|_{L^P(\Omega v)} \leq K \| f \|_{(\Omega v)}$$

Now it is clear that as m $\to \infty$ $g_m \to f$ in $L^p_{(0,1)}(\Omega v)$ and $u_{v,m} \to$ some u_v in $L^p(\Omega_v)$, $u_v = f$ in Ω_v and

$$||u_{v,m}||_{L^p(\Omega v)} \leq K|| f || \quad _{(\Omega v)} \qquad (1)$$

Now define u_v as zero outside Ω_v, then since $L^p(\Omega)$ is reflexible for $1 < p < \infty$, by the Banach-Alaoglu Theorem, there is a u e $L^p(\Omega)$ with

$$||u||_{L^p(\Omega v)} \leq K|| f || \quad _{(\Omega)} (1 < \pi < \infty), \qquad (2)$$

and a subsequence $\{u_{v_N}\}$ of $\{u_v\}$ such that $u_{v_N} \to u$ weakly in $L^p(\Omega)$ as $N \to \infty$. In particular $u_{v_N} \to u$ in distribution theory, as $N \to \infty$. Therefore

$$u = f \text{ in } \Omega$$

and with (2) Theorem 2 is proved.

If f 0 and $f_1 \equiv 0$ there is a j_o such that f_{jo} 0, and we can use f_{jo} instead of f_1 in the above.

Conclusion

The results in the present paper together with those in [2] and [3] go a long way to address Professor Tutschke's comments in [5].

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Received 29 Oct 08; revised 16 Jan 09.