Ghana J. Sci. **50** (2010), 9-12

REGULARITY THEORY FOR FULLY NONLINEAR UNIFORMLY ELLIPTIC EQUATIONS

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Abstract

With the aim of obtaining at least Cordes-Nirenberg, Schauder and Calderon-Zygmund estimates for solutions of Fully Nonlinear Uniformly Elliptic Equations, we arrive at $W^{2,p}$, $C^{1,\alpha}$(}, $C^{2,\alpha}$ regularity estimates for those equations, improving the existing estimates.

Introduction

Consider fully nonlinear second order uniformly elliptic equations of the form

$\forall$
$$F(D^2u, x) = f(x) \tag{1}$$

where $x \in \Omega$ and u and f are functions define in a bounded domain Ω of R^n, and $F(M, x)$ is a real valued function defined on $S \times \Omega$, where S is the space of real $n \times n$ symmetric matrices. Assume F is uniformly elliptic in the sense that there are positive constants) $\lambda < \wedge$ such that for any M $\in S$ and $x \in \Omega$,

$$\lambda||N|| \leq F\,(M + N, x) - F(M, x) \leq \Lambda\,||N||\;\forall N \geq O. \tag{2}$$

When N is a symmetric matrix, $N > 0$ means, i.e. is non-negative definite. $||M||$ denotes the (L^2, L^2)-*norm* of M (i.e. $||M|| = \sup_{|x|=1} |Mx|$; therefore $||N||$ is equal to the maximum eigenvalue of N wherever $N \geq 0$).

Recalling that any $N \in S$ can be uniquely decomposed as $N = N^+ - N^-$, where $N^+, N^- > 0$ and $N^+ N^- = 0$, it follows that F is uniformly elliptic if and only if

$$F(M + N \leq F\,(M), \mathrm{x} + \wedge\,||N^+|| - \lambda||N^-|| \quad M, N \in S, x \in \Omega. \tag{3}$$

It also follows from (3) that if F is uniformly elliptic, then

$$\lambda|\hat{e}| \leq |F|(M, x) \leq +\,|F(0, x)| \quad M \in S, x \in \Omega. \tag{4}$$

where $|\hat{e}| = \max\{|e_1|, \ldots, |e_n|\}$, the e_j $(1 \leq j \leq n)$ being the eigenvalues of M.

In [1], rather detailed regularity estimates were obtained for solutions of (1), where Ω was the unit ball and *F(M, x)* was convex or concave in *M.* As is normal, it is natural to ask whether we can remove the convexity conditions on *F* and make Ω a general bounded domain in]^n. We show here that we can, without even assuming that Ω is F-convex in the sense of [2].

Our methods are not direct generalization of those of [1]. Rather we use only the philosophy that the most useful square matrix is a diagonal one, the approach being frontal. We obtain sharp Hölder and Sobolev regularity results, and from the Hölder esitmates, show that once *f* in (1) is continuous and *F(0,·)* is locally integrable in ($L^\infty(\Omega)$), every solution of (1) is a viscosity solution.

We consider in this paper only those solutions of *u* of (1) such that the distributional derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ are actual functions on Ω, and we also assume that the boundary of Ω has Lebesgue measure zero.

Our results are as follow:

Theorem 1. If $F(0, \cdot)$ and f are in $L^p(\Omega)$, $1 \le p \le \infty$, then $u \in W^{2,p}(\Omega)$ and there is a constant K independent of F such that

$$||u||_{W^{2,p}(\Omega)} \le K \quad ||F(0,\cdot)||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \ .$$

$$\sum_{i,j} M^2_{ij}$$

Theorem 2. If $F(0, \cdot)$ and f are in $L^{n/1-\alpha}(\Omega)$, $O < \alpha < 1$, then $u \in C^{1,\alpha}(\Omega)$ and there is constant K independent of F such that

$$||u||_{C^{1,\alpha}(\Omega)} \le K \ ||F(0, .)||_{L^{n/1-\alpha}(\Omega)} + ||f||_{L^{n/1-\alpha}(\Omega)} \ .$$

Theorem 3. If $F(0, \cdot)$ and f are in $L^\infty(\Omega)$, then $u \in C^{2,\alpha}(\Omega_o)$ for every domain $\Omega_o \subset\subset \Omega$ and there is a constant $K = K(\Omega_o, \Omega)$ such that

$$||u||_{C^{2,\alpha}(\Omega_o)} \le K \ ||F(0, \cdot)||_{L^\infty(\Omega)} + ||f||_{L^\infty(\Omega)} \ .$$

Theorem 4. If f is continuous and $F(0,\cdot) \in L^\infty_{1oc}(\Omega)$, then every solution of (1) is a viscosity solution.

Proof of Theorems

First if $M = (M_{ij})$ is in S, we define $|M| := \quad {}^{1/2}$. It then follows, using the fact that $M = ODO^t$, where $D_{ij} = e_i\delta_{ij}$ (e_i being the eigenvalues of M) and O is an orthogonal matrix, and Cauchy-Schwartz Inequality that

$$|M| \leq n \quad e_j^2 \quad ^{1/2}.$$

Therefore, since $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right.$, we have from (4) and (5) that

$$D^2u \leq K\,(|F(0,\,.)| + |\,f\,|),$$

for some $K > 0$.

Putting $u = 0$ outside n and using Poincaré Inequality (noting that the boundary of Ω has Lebesgue measure zero) we get Theorem 1 from (6).

To prove Theorem 2 we use (from [3] p. 123) Lemma 5. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and assume that $\partial_j u \in L^p(\ \ ^n)$, $j = 1, \ldots, n$, where $p > n$. Then u is continuous and with $\gamma = 1 - n/p$, we have

$$\mathbb{R}\sum_{j=1}^{n} \quad \sup_{x \neq y} |u|(x) - u\,(y)/\,|x - y|^{\gamma}| \leq C \quad ||\partial_j u||_p,$$

for some $c > 0$.

Putting again $u = 0$ outside Ω and noting that the boundary of Ω has Lebesgue measure zero, we get Theorem 2 from Lemma 5, Theorem 1 and Poincaré Inequality.

To prove Theorem 3, we note that there is a constant $K = K(\Omega_o, \Omega)$ such that

$$\sup_{\substack{x \neq y \\ x,y \in \Omega_o}} \frac{\left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_j \partial x_j}\right|}{|x - y|} \leq \quad |\,\partial_j u\,\|_{L^\infty(\Omega)}$$

$1 \leq i \leq n$, and then use Theorem 1.

To prove Theorem 4, we note that fromt the hypothesis of Theorem 4, Theorem 2 holds on any domain $\Omega_o \subset\subset \Omega$.

References

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Received 29 Nov 08; revised 16 Jan 09