Ghana J. Sci. **50** (2010), 3-6

HÖLDER ESTAMTES FOR THE $\bar{\partial}$-OPERATOR ON BOUNDED DOMAINS IN C^n

P. W. DARKO

Department of Mathematics, Delaware State University, 1200 N. Dupont Highway, Dover, DE 19901–2277. E-mail: pdarko10@agl.com

Abstract

Holder estimates are obtained for the $\bar{\partial}$-operator on bounded domains in C^n with boundaries of Lebesgue zero.

Introduction

The pioneering work on the type of Hölder estimates for the δ-operator that we consider was done by ALY [1] and SIU [5] for the (0,1)-forms and for (0,q)-forms by Lieb Range [4]. Since then various Holder estimates for the δ-operator have appeared (see references) [2] and [3]. Most of the results for Hölder estimates for the δ-operator mentioned above have been for strongly pseudoconvex domains or pseudoconvex domains of finite type.

Working with the Bochner-Martinelli-Koppelman kernel it dawned on us that we could get a generalization of Alt-Sui-Lieb-Range results to all bounded domains in C^n with boundaries of Lebesgue measure zero (at least for the range of Holder estimates we consider here). This short paper shows that we are right.

Preliminaries

Let U be open in $\mathbb{R}^n$, $0 < \alpha < 1$, $k \geq 0$ an integer. We define $C^{k,\alpha}$ (U) to be the space of functions f on U such that

$$|f|_{C^{k,a}(U)} := \sup_{\Omega} |f| + \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|D^r f(x) - D^r f(y)|}{|x-y|^{\alpha}}$$

is finite, where D^γ is a derivative or order $|\gamma|$, $\gamma_1, \ldots, \gamma_n)$, $\gamma_j \geq 0$. If $U \subset \mathbb{C}^n$ is open, we use the real underlying coordinates of ^{n} considered as ^{2n} to define $C^{k,a}(U)$. $C^{k,a}(U)$ is defined similarly.

If $f = \quad f_{(i1,\ldots,iq)} d\bar{z}_{i1} \wedge \ldots . \wedge d_{\quad iq}$ is an (0,q)-form on U, where means the summation is over increasing multi-indices, we write f as $\sum' f_I d\bar{z}^I$ for short, $I = (i_1, \ldots, i_q)$ and set

$$|f|_{C^{k,\alpha}_{(0,q)}}(U) = \max_I |f_I|_{C^{k,\alpha}(U)}$$

Our result is then

Theorem 1, Let Ω be a bounded domain in ^{n} with boundary of Lebesgue measure zero and $0 < \alpha < 1$ and $k = 1, 2, \ldots$, and let $f \in C^{k,\alpha}_{(0,q+1)}$ be $\bar{\partial}$-closed then there is $u \subset C^{k,\alpha}_{(0,q)}(\Omega)$ such that

$\delta u = f$ in the sense of distributions and

$$|u|_{C^{k,\alpha}_{(0,q)}}(\Omega) \leq |f|_{C^{k,\alpha}_{(0,q+1)}}(\Omega)$$

where δ does not depend on f

Bochner-Martinelli-Koppelman Formula and -u = *f*

Theorem 2 (Bochner-Martinelli-Koppelman). Let Ω be a bounded domain in ^{n} with C^1 boundary. For $f \in C^{1,\alpha}_{(0,q)}(\ )$, $0 \leq q \leq n$, we have

$$f(z) = \quad Bq(., z) \wedge f + \int_{\Omega} Bq\,(., z) \wedge \bar{\partial}_{\xi} f$$

$$+\ \bar{\partial}_{z} \quad Bq - 1(., z) \wedge f,\ z \in \Omega \qquad (1)$$

where $\beta q\,(\xi, z)$ is the Bochner-Martinelli-Koppelman kernel of degree $(0,q)$ in z and of degree $(n, n - q - 1)$ in ξ. Recall, with $\beta = |\xi - z|^2$,

$$\beta q(\xi, z) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \begin{matrix} n-1 \\ q \end{matrix} \quad \beta^{-n}\, \partial_{\xi}\beta \wedge (\partial_{\xi}\partial_{\xi}\beta)^{n-q-1} \wedge (\bar{\partial}_{z}\partial_{\xi}\beta)^{q} \qquad (2)$$

Lemma 3. With f as in Theorem 1, if

$$u(z) \quad \beta_{q}(., z) \wedge f,\ z \in \Omega, \qquad (3)$$

then $\bar{\partial} u = f$.

Proof With $f -\ \ '\ f_J d\bar{z}^J$ defined as zero outside Ω, regularize f coeffienctwise: $f_m =$ $(f_J)_m d\ ^J$ where

$$(f_J)(\ ) = \int_{\mathbb{C}^n} f_J(z - \xi/m)\varphi(\xi)d\lambda(\xi) \qquad (4)$$

$$= m^{2n} \ {}_{\mathbb{C}^n}\ fJ(\xi)\varphi\,(m(z - \xi))\ \delta_1(\xi)$$

and $\varphi \in\ ^{\infty}_{0}(\ ^{n})$, $f\varphi\, d\lambda = 1$, $\varphi \geq 0$, sup $\varphi = \{z \in C^n : |z| \leq 1\}$ and λ is Lebesgue measure.

Then $||fm\,||L^1_{(0,q+1)}(\ ^{n}) \leq ||\,f\,||\,L^1_{(0,q+1)}(\ ^{n})$, $f_m \to f$ in $L^1_{(0,q+1)}(\Omega)$ as $m \to \infty$, and f_m is -closed in ^{n} in the sense of distributions.

Now let

$$u_m(z) = \int_{\mathbb{C}^n} B_q(., z) \wedge f_m, \qquad (5)$$

then from Theorem 2,

$$\bar{\partial} u_m = f_m,$$

and since $f_m \to f$ in $L^1_{(0,q+1)}(\Omega)$, we have $u_m \to u$ in $L^1_{(0,q)}(\mathbb{C}^n)$ and $u = f$.

Note that in the proof of Lemma 3, since $f \in C^{1,a}_{(0,q)}(\Omega)$ it follows that f belongs to the Sobolev space $W^{1,\infty}_{(0,q+1)}(\Omega)$ and, therefore, f extended by zero outside Ω belongs to $W^{1,\infty}_{(0,q+1)}(\mathbb{C}^n)$, even though it man not belong to $C^{1,\infty}_{(0,q+1)}(\mathbb{C}^n)$, and since all derivatives are taken in the distribution sense, that is all we need!

Holder Estimates

In this section we finish the proof of Theorem 1:

From (5), we get

$$\partial^{\alpha} u_m(z) = \int_{\mathbb{C}^n} \beta_q(., z) \wedge \partial^{\alpha} f_m \qquad (6)$$

where

$$\partial^{\alpha} u_m = \frac{|\partial|^{|\alpha|}}{\partial x_1^{\alpha_1} \partial y_1^{\alpha_2} \ldots \partial x_n^{\alpha_{2n-1}} \partial y_n^{\alpha_{2n}}},$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}),\ z = (x_1 + iy_1, \ldots, x_n + iy_n),\ i = \sqrt{-1},$$

and the derivatives are taken coefficientwise. From (4), $\alpha < k$, as $m - \infty$ $\partial^{\alpha} f_m \to \partial^{\alpha} f$ in $L^1_{(0,q+1)}(\Omega)$ and so from (6) $\partial^{\alpha} u_m \to \partial^{\alpha} u$ in $L^1_{(0,q)}(\Omega)$ and

$$\partial^{\alpha} u(z) = \int_{\Omega} \beta q(., z) \wedge \partial^{\alpha} f,\ z \in \Omega. \qquad (7)$$

Now from known properties of Bq (ξ, z) (see for example [2], page 269), we get the estimate from (7).

$$u C^{k,\alpha}_{(0,q)}(\Omega) \le \delta\, |f|\, C^{k,\alpha}_{(0,q+1)}(\Omega)$$

$(0 < \alpha < 1,\ k \ge 1)$.

Reference

[1.] ALT, W. (1974) Holderabschatzungen von Losungen den Gleischung $\bar{\partial} u = f$ bei streng pseudokonvexem Rand. *Man. Math.* **13**, 381–414.

[2.] CHEN, S.-C. & SHAW, M.-C. (2001). Partial Differential Equations in Several Complex Variable. *Studies in Advanced Mathematics* No. 19. AMS-International Press.

[3.] Lieb, I. & MICHEL J. (2002). The Cauchy-Riemann Complex (Integral Formulae and Neumann Problem). *Aspects of Mathematics* **E34**. Vieweg.

[4.] LIEB, I. & RANGE, R. M. (1980). Ein Losungsoperator fur den Cauchy-RiemannKomplex mit Ck-Abschatzugen. *Math. Ann.* **253**, 145–164.

[5.] SIN, Y. T. (1974). The -problem with uniform bounds on derivations. *Math. Ann.* **207**, 163–176.

Received 14 Apr 08; revised 15 Aug 08.