Ghana J. Sci. **29-30**, 3-6

A NOTE ON GLOBAL REGULARITY WITH GROWTH FOR δ ON PSEUDOCONVEX DOMAINS

P. W. DARKO
Department of Mathematics, University of Ghana, Legon, Ghana

Summary

A regularity result is obtained for solutions of the δ-equation in the presence of growth conditions on pseudoconvex domains in ¢ⁿ. The solutions for which regularity is obtained here were obtained by Hormander-L^2 - estimates and existence theorems for the δ -operator. The result here adds to results by Darko - On regularity of solutions of the δ-equation with growth on bounded pseudoconvex domains.

Ever since Kohn's classic solution of the ∂ - Neumann problem [8], [9], [3] and Hormander's celebrated estimates for the ∂–operator [5], [6] , ∂-Neumann and ∂ related results have been obtained year after year by many people of many nations. The direction of most of the recent results can be found in [4], [10], and [11].

In this note we go back to a problem in [5] which was left open. This is regularity of solutions of the δ -equation in the presence of growth conditions. Regularity in all these cases was considered in [6], except for this one.

In [1] we look at conditions on the plurisubharmonic function that give the growth condition, that make it possible to have regularity of the solutions. The condition that we arrived at was derived from a condition in [7] and it includes the important case of polynomial growth.In [1] the pseudoconvex domain is bounded and the condition on the function that gives the growth condition was obtained before it was noticed that it included the case of ploynomial growth. In this note the pseudoconvex domain is not necessarily bounded so it seemed reasonable to start with the case of polynomial growth as given in [2]. We obtained regularity of the solutions of the ∂ -equation in this case and the condition we give here on the function that gives the growth condition is a generalization of the case of polynomial growth.

Let λ be lebesgue measure and $\delta_0(z) = (1+|z|^2)^{-1/2}$ for $z \varepsilon$ ¢ⁿ .

Our starting point is the following result of Hormander:

Theorem.

Let ϕ be a plurisubharmonic function in $G \subset$ ¢ⁿ, and *r* a non-negative interger. For every differential form *v* of type (o, r+1) in G which is square integrable with respect to measuer $e^{-\phi}\, d\lambda$ and satisfies $\partial v = 0$ in the distribution sense, there exists a locally integrable differencial form *u* of type (o, r) in G such that $\partial u = v$ and

$$\int_G |u|^2 e^{-\phi}\delta_0^4 d\lambda \le \int_G |v|^2 e^{-\phi} d\lambda$$

Our question is, if *v* is C^∞, can *u* be chosen to be C^∞ with *u* still satisfying the estimate (1)? The answer we give here is yes, when ϕ satisfies the following additional condition:

(H_o) (i) $\exp(-\phi(z) \le \delta_0(z)$ for $z \varepsilon G$

(ii) There are constants $K_1 > 0$, K_2, $K_3 > 0$ such that

$$z \varepsilon G \text{ and } |z-\zeta| \leq \exp(-K_1 \phi(z) - K_2) \text{ imply}$$
$$\zeta \varepsilon G \text{ and } \phi(\zeta) \leq \phi(z) + K_3$$

Regularization of functions satisfying condition (H_o)

We proceed nearly exactly as in [1].

If d(z) is the distance of z in G to the complement of G,

then $d(z) \geq \exp(-K_1\phi(z) - K_2)$, that is $\phi(z) \geq (\log(1/d(z)) - K_2)/K_1$, and

$\phi(z) \rightarrow \infty$ as z tends to a boundary point of G. Therefore

$\delta(z) = \exp(-\phi(z)) \rightarrow 0$ as z tends to a boundary point of G.

Now $\sup_{z \varepsilon G} \delta(z) \leq \sup_{z \varepsilon G} \delta_o(z) < \infty$.

Let $\alpha_0 = \sup_{z \quad G} \delta(z)^{k_1}$, $\alpha = e^{-k_3 k_1}$, $\alpha_m = \alpha_o(\alpha)^m$, $(m \geq 1)$,

$\beta_o = \alpha_o^{1/k_1}$ $\beta = e^{-k_3}$, $\beta_m = \beta_o(\beta)^m$, $(m \geq 1)$,

then $\beta_m^{k_1} = \beta_o^{k_1} (\beta^{k_1})^m = \alpha_o(\alpha)^m = \alpha_m$, $(m \geq 1)$, and

$\alpha_m \rightarrow 0$, $\beta_m \rightarrow 0$ as $m \rightarrow \infty$, $\alpha_m / \alpha_o < 1$, $\beta_m / \beta_o < 1$, $(m \geq 1)$

We then have the following

Lemma. There is a C^∞ function δ^1 on G such that for all z ε G

$$\frac{\alpha^{2/k_1}}{(1-\beta)} \delta(z)\ \delta_o(z)^4 \leq \delta^1(z) \leq \frac{1}{\alpha^{1/k_1}(1-\beta)} \delta(z)\,\delta_o(z)^4$$

Proof.

Let θ be a $\mathbb{C}^\infty$ function on $\mathbb{C}^n$ with support in $\{z \varepsilon \mathbb{C}^n : |z| \leq 1\}$, such that $\int_{\mathbb{C}^n} \theta(\zeta)\, d\lambda(\zeta) = 1$.

Set $\theta_m(z) = (e^{-k_{2\alpha}} m)^{-2n} \quad \theta(z / e^{-k_{2\alpha}} m)$.

Let S_m be the set $\{ z \varepsilon G : \delta(z)^{k_1} \geq \alpha_m \}$ $(m \geq 1)$, and χ_m the characteristic function of S_m, and $\chi_m * \delta_m$ the convolution of χ_m and θ_m.

Then $\quad |\chi_m * \theta_m| \leq 1$.

Define $\delta^1(z) / \delta_o(z)^4 = \sum_{m=1}^{\infty} \beta_m \chi_m * \theta_m(z)$.

If D is a derivative of any order

$$D(\delta^1(z) / \delta_o(z)^4) = \sum_{m=1}^{\infty} \beta_m \chi_m * (D\theta'_m(z)),$$

the infinite series being convergent locally uniformly. It follows that δ^1 is a C^∞ function.

Suppose that $z \in S_N$ so that $\delta(z)^{k_1} \geq \alpha_N$, then if $| z - z^1 | \leq e^{-k_2}\alpha_N$,

$| z - z^1 | \leq e^{-k_2}\, \delta(z)^{k_1}$ and from condition (H_o) $\delta(z^1) \leq \delta(z)\, e^{-k_3}$ which implies that

$\delta(z^1)^{k_1} \geq \delta(z)^{k_1}\, e^{-k_3 k_1} \geq \alpha_N e^{-k_3 k_1} = \alpha_{N+1}$.

Therefore, if $z \in S_N$ then $\chi_m * \theta_m(z) = 1$ for $m \geq N+1$ since $| z - z^1| \leq e^{-k_2}\alpha_m$ for all

z^1 in the support of θ_m. Hence if $\delta(z)^{k_1} \geq \alpha_N$ then

$$\delta^1(z) / \delta_o(z)^4 \geq \sum_{m \geq N+1} \beta_m = \frac{\beta^{N+1}}{1-\beta}$$

or $$(\delta^1(z) / \delta_o(z)^4)^{k_1} \geq \frac{\alpha_{N+1}}{(1-\beta)^{k_1}}$$

Similarly if $(\delta(z))^{k_1} < \alpha_{N-1}$ then $\chi_m * \theta_m(z) = 0$ for $m \leq N-2$ and

$$(\delta^1(z) / \delta_o(z)^4) \leq \sum_{m \geq N-1} \beta_m = \frac{\beta^{N-1}}{(1-\beta)}$$

or $$(\delta^1(z) / \delta_o(z)^4)^{k_1} \leq \frac{\alpha_{N+1}}{(1-\beta)^{k_1}},$$

Therefore for $m \geq 1$, $\alpha_m \leq \delta(z) < \alpha_{m-1}$ implies that

$$\frac{\alpha_{m+1}}{(1-\beta)^{k_1}} \leq (\delta^1(z) / \delta_o(z)^4)^{k_1} \leq \frac{\alpha_{m-1}}{(1-\beta)^{k_1}}$$

hence $\delta(z)^{k_1}\, \alpha^2 < \alpha_{m+1}\, \alpha^2 = \alpha_{m+1} < (1-\beta)^{k_1} (\delta^1(z) / \delta_o(z)^4)^{k_1} \leq \alpha_{m-1}$

$$= \alpha_{m/\alpha} \leq 1/\alpha\, \delta(z)^{k_1}$$

That is to say, for all $z \in G$

$$\frac{\alpha^2}{(1-\beta)^{k_1}}\, \delta(z)^{k_1} \leq (\delta^1(z) / \delta_o(z)^4)^{k_1} \leq \frac{\delta(z)^{k_1}}{\alpha(1-\beta)^{k_1}}$$

and the lemma follows.

Regularity of solutions of δ

Let $H^r_\psi = \{ (o, r)\text{ - form } g: \int_G |g|^2 e^{-\Phi \delta_0^4} d\lambda < \infty \}$

$H^r_\Phi = \{ (o, r)\text{ - form } g: \int_G |g|^2 \delta^1 d\lambda < \infty \}$

and $S : H^r_\psi \quad H^{r+1}_\psi$

the densely define operators induced by ∂.
Let S* be the Hilbert space adjoint of S and [RS*] the closure in H^r_ψ of the range of S* .

By the lemma, $H^r_\Phi = H^r_\psi$. Hence if we let *w* be the projection of *u* onto [RS*], the $\partial w = v$. Since δ^1 is a C^∞ function it can be shown [6] that *w* is a C^∞ (o, r) -form.
This answers our question in the affirmative.
Remark : Of course, when Φ is a C^∞ function, regularity of the solutions we have considered is no problem.

References

DARKO, P. W. (1982) On regularity of solutions of the ∂-equation with growth on bounded pseudoconvex domains. *Rc. Mat. Applic.* **2**, 67-612.

FERRIER, J. P. (1973) Spectral theory and complex analysis. North-Holland Publishing Company.

FOLLAND, G. B. & KOHN, J. J. (1972) The Neumann problem for the Cauchy-Riemann complex. *Ann. Math. Stud.*, No. 75. Princeton University Press.

GRETNER, P. C. & STEIN E. M. (1977) Estimates for the Neumann problem. Princenton Mathematical Notes No. 19. Princeton University Press.

HORMANDER, L. (1965) L^2-estimates and existence theorems for the ∂-operator. *Acta. math.* **113**, 85-152.

HORMANDER, L. (1966) *An introduction to complex analysis in several variables*. Princeton, New Jersey: D. Van Nostrand.

HORMANDER, L. (1967) Generators for some rings of analytic functions. *Bull. Am. math. Soc.* pp. 943-949.

KOHN, J. J. (1963) Harmonic integrals on strongly pseudoconvex manifolds. I. *Ann. Math.* **78**, 206-213

KOHN, J. J. (1964) Harmonic integrals on strongly pseudoconvex manifolds. II. *Ann. Math.* **79**, 450-472.

KOHN, J. J. (1979)Subelliptic estimates of the ∂-Neumann problem on pseudoconvex domains: sufficient conditions, *Acta. math.* **142**, 79-122 .

FORNAESS J. E. (ed.) (1981) Recent developments in several complex variables. *Ann. Math. Stud.*, No. 100. Princeton University Press.

Received 12 Jun 87; revised 12 Jan 89.