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COMPUTATIONAL SOLUTION OF ADVANCED STOCHASTIC TIME-DELAY DIFFERENTIAL EQUATIONS USING HYBRID BLOCK EXTENDED ADAMS MOULTON METHODS FOR CUSTOMERS SATISFACTION WHEN USING ELECTRONIC PAYMENT SYSTEMS IN NIGERIAN BANKS

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ABSTRACT

This study examined the implementation of Hybrid Block Extended Adams Moulton Methods (HBEAMM) for the computational solution of Advanced Stochastic Time-Delay Differential Equations (ASTDDEs) through the use of various electronic payment systems such as ATM, POS and internet banking offered by the Nigerian banks. This work attempts to proffer solution to customer challenges in using the ATM of most banks in Nigeria. The challenges are considered in this work as capable of resulting to advanced time-delay and volatility noise. It is therefore modeled as advanced stochastic time-delay differential equation (ASTDDE) and solved using Hybrid Extended Block Adams Moulton Methods (HEBAMMs).The discrete schemes of the proposed method (HBEAMM) were obtained by continuous formulations of multistep collocation method by matrix inversion approach. The necessary and sufficient conditions for convergence and stability of the method were analyzed and proved satisfactory. The discrete schemes obtained were applied in solving some advanced stochastic time-delay differential equations and the results obtained revealed the degree of customers' satisfaction in the use of e-payment systems. The accuracy of the method is compared with other existing methods and presented graphically to prove its superiority.

Keywords: Advanced stochastic time-delay differential equations, hybrid block extended Adams Moulton methods, customers, electronic payment systems, Nigerian banks.

Introduction

Electronic payment system has been defined by Elisha (2010), Gao *et al.* (2008) as the use of internet and digital stored value systems, personal computers, ATM, POS, internet and other electronic channels to process financial transactions without the account holders being present in the bank. It gives the customers fastest and easiest ways to transfer money, pay bills, make purchase and carry out other financial transactions. Despite the efforts made by the banks to ensure the customers enjoy the benefits of electronic payment systems, most Nigerian banks frequently receive complaints from customers on frequent network failure, account debit without dispensing of cash, incomplete money transfer, withholding of ATM cards due to machine failure, online theft and fraud challenges as studied and revealed by Ogunlowore *et al.* (2014). These challenges may result to advanced time-delay and volatility noise in investigating and reversal of wrongly debited money without dispensing of cash. If these challenges are not properly addressed, customers' attitude towards the use of e-payment systems may change. Advanced stochastic time-delay differential equation (ASTDDE) is a stochastic differential equation where the increment of the process depends not only on current state but also on the advanced/future part of the system being modeled which contains the random values of the volatility noise term. The applications of SDDEs can be seen in applied sciences, economics and engineering. Several authorssuch as Evelyn (2000), Zhang *et al.* (2009), Bahar (2019), Wang *et al.* (2011), Kazmerchuk (2005), Kazmerchuk *et al.* (2004), Akhtari *et al.* (2015) used Euler-Maruyama scheme to formulate continuous split-step schemes of SDDE on a continuous interval for the numerical solutions and encountered some challenges in the use of interpolation techniques in evaluating their delay terms as studied by Majid *et al.* (2013). In order to circumvent these challenges, we applied Hybrid Block Extended Adams Moulton Methods (HBEAMM) as a linear multistep collocation method to discretize ASTDDEs on a discrete interval $[t_0, t_a)$. This was done, in order to obtain its discrete schemes for its numerical solution for customers' satisfaction through the use of electronic payment systems in Nigeria banks without interpolation techniques.

From Kazmerchuk *et al.* (2004), advanced stochastic time-delay differential equation (ASTDDE) can be express as:

$$dy(t) = \alpha(y(t), y(t+\tau), t)dt + \beta(y(t), y(t+\tau), t)ds(t) \text{ for } t > t_0, \tau > 0$$
$$y(t) = \varphi(t), \text{ for } t \le t_0 \quad (1)$$

where $\varphi(t)$ is the initial function, $y(t)$ is the stochastic process of the current state, t is the time, τ is called the delay, $(t+\tau)$ is the advanced/future time-delay term and $y(t+\tau)$ is the solution of the advanced time-delay term on the drift part of (1). $S(t)$ is the Standard Brownian Motion with its differential equivalence as $ds(t)$. This is the volatility noise term or Wiener process together with solution of the advanced time-delay term as $y(t+\tau)ds(t)$ on the volatility or diffusion part of (1). The drift part of the Equation (1) which is $dy(t) = \alpha(y(t), y(t+\tau), t)dt$ is deterministic and takes care of the average time rate of reversal of wrongly debited money without dispensing of cash or any risk involved. The volatility or diffusion part $dy(t) = \beta(y(t), y(t+\tau), t)ds(t)$ is stochastic, which takes care of the random change and the risk involved in the time it takes for the reversal of wrong transactions caused by frequent network failure. These include account debit without dispensing of cash, incomplete money transfer, withholding of ATM cards due to machine failure, online theft and fraud.

Experimental

Derivation of the method and analysis of basic properties of the method

Applying the discrete schemes of the Hybrid Block Extended Adams Moulton Methods (HBEAMM) derived by Chibuisi *et al.* (2022) using *k*-step multistep collocation method developed by Onumanyi *et al.* (1994) for step numbers k = 2 and 3 which are presented as:

$$y(x) = \sum_{j=0}^{z-1} \alpha_j(x) y_{n+j} + d\sum_{j=0}^{v-1} \beta_j(x) f(x, y(x)) \qquad (2)$$

$$\alpha_j(x) = \sum_{i=0}^{z+v-1} \alpha_{j,i+1} x^i \text{ for } j = \{0,1,\ldots,z-1\} \qquad (3)$$

$$d\beta_j(x) = \sum_{i=0}^{z+v-1} d\beta_{j,i+1} x^i \text{ for } j = \{0,1,\ldots,v-1\} \qquad (4)$$

where $x_0, \ldots, x_{v-1}$ are the $v$collocation points and X_{m+j} , $j = 0, 1, 2, \ldots, z-1$ are the z arbitrarily chosen interpolation points while d is the constant step size.

To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena (1997) arrived at a matrix equation of the form

$$AB = I \qquad (5)$$

where I is the identity matrix of dimension (z+v) x (z+v)while A and B are matrices defined as

$$A = \begin{bmatrix} 1 & x_m & x_m^2 & \cdots & x_m^{z+v-1} \\ 1 & x_{m+1} & x_{m+1}^2 & \cdots & x_{m+1}^{z+v-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{m+z-1} & x_{m+z-1}^2 & \cdots & x_{m+z-1}^{z+v-1} \\ 0 & 1 & 2x_0 & \cdots & (z+v-1)x_0^{z+v-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{v-1} & \cdots & (z+v-1)x_{v-1}^{z+v-2} \end{bmatrix} \qquad (6)$$

$$B = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{z-1,1} & d\beta_{0,1} & \cdots & d\beta_{v-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{z-1,2} & d\beta_{0,2} & \cdots & d\beta_{v-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,z+v} & \alpha_{1,z+v} & \cdots & \alpha_{z-1,z+v} & d\beta_{0,z+v} & \cdots & d\beta_{v-1,z+v} \end{bmatrix} \qquad (7)$$

For $k = 2$ of HBEAMMs

$$y_m = y_{m+1} - \frac{67031}{249480} df_m - \frac{6037}{3528} df_{m+1} + \frac{47129}{2520} df_{m+2} - \frac{130048}{2835} df_{m+\frac{9}{4}} + \frac{14296}{315} df_{m+\frac{5}{2}} - \frac{510464}{24255} df_{m+\frac{11}{4}} + \frac{28817}{7560} df_{m+3}$$

$$y_{m+2} = y_{m+1} - \frac{799}{249480} df_m + \frac{543}{1960} df_{m+1} + \frac{3307}{840} df_{m+2} - \frac{4096}{567} df_{m+\frac{9}{4}} + \frac{216}{35} df_{m+\frac{5}{2}} - \frac{20992}{8085} df_{m+\frac{11}{4}} + \frac{3313}{7560} df_{m+3}$$

$$y_{m+\frac{9}{4}} = y_{m+1} - \frac{40825}{12773376} df_m - \frac{250055}{903168} df_{m+1} + \frac{520375}{129024} df_{m+2} - \frac{63755}{9072} df_{m+\frac{9}{4}} + \frac{12325}{2016} df_{m+\frac{5}{2}} - \frac{200125}{77616} df_{m+\frac{11}{4}} + \frac{168625}{387072} df_{m+3}$$

$$y_{m+\frac{5}{2}} = y_{m+1} - \frac{43}{13440} df_m + \frac{1737}{6272} df_{m+1} + \frac{18021}{4480} df_{m+2} - \frac{724}{105} df_{m+\frac{9}{4}} + \frac{219}{35} df_{m+\frac{5}{2}} - \frac{636}{245} df_{m+\frac{11}{4}} + \frac{1961}{4480} df_{m+3}$$

$$y_{m+\frac{11}{4}} = y_{m+1} - \frac{5831}{1824768} df_m + \frac{567}{2048} df_{m+1} + \frac{123823}{30720} df_{m+2} - \frac{44933}{6480} df_{m+\frac{9}{4}} + \frac{1029}{160} df_{m+\frac{5}{2}} - \frac{6559}{2640} df_{m+\frac{11}{4}} + \frac{119707}{276480} df_{m+3}$$

$$y_{m+3} = y_{m+1} - \frac{20}{6237} df_m + \frac{611}{2205} df_{m+1} + \frac{1264}{315} df_{m+2} - \frac{19456}{2835} df_{m+\frac{9}{4}} + \frac{1984}{315} df_{m+\frac{5}{2}} - \frac{54272}{24255} df_{m+\frac{11}{4}} + \frac{487}{945} df_{m+3} \qquad (8)$$

For $k=3$ of HBEAMMs

$$\begin{aligned}
y_m &= y_{m+2} - \frac{3841}{13230}df_m - \frac{5506}{3465}df_{m+1} + \frac{962}{2205}df_{m+2} - \frac{2582}{945}df_{m+3} + \frac{14464}{2205}df_{m+\frac{7}{2}} - \frac{434176}{72765}df_{m+\frac{15}{4}} + \frac{997}{630}df_{m+4} \\
y_{m+1} &= y_{m+2} + \frac{127}{13230}df_m - \frac{3461}{9240}df_{m+1} - \frac{5659}{5880}df_{m+2} + \frac{9307}{7560}df_{m+3} - \frac{1864}{735}df_{m+\frac{7}{2}} + \frac{159232}{72765}df_{m+\frac{15}{4}} - \frac{467}{840}df_{m+4} \\
y_{m+3} &= y_{m+2} + \frac{2}{1323}df_m - \frac{551}{27720}df_{m+1} + \frac{6967}{17640}df_{m+2} + \frac{8963}{7560}df_{m+3} - \frac{3032}{2205}df_{m+\frac{7}{2}} + \frac{77312}{72765}df_{m+\frac{15}{4}} - \frac{629}{2520}df_{m+4} \\
y_{m+\frac{7}{2}} &= y_{m+2} + \frac{181}{125440}df_m - \frac{471}{24640}df_{m+1} + \frac{24429}{62720}df_{m+2} + \frac{391}{280}df_{m+3} - \frac{927}{980}df_{m+\frac{7}{2}} + \frac{2416}{2695}df_{m+\frac{15}{4}} - \frac{3921}{17920}df_{m+4} \\
y_{m+\frac{15}{4}} &= y_{m+2} + \frac{6419}{4423680}df_m - \frac{77861}{4055040}df_{m+1} + \frac{287567}{737280}df_{m+2} + \frac{1536983}{1105920}df_{m+3} - \frac{9359}{11520}df_{m+\frac{7}{2}} + \frac{1526}{1485}df_{m+\frac{15}{4}} - \frac{335111}{1474560}df_{m+4} \\
y_{m+4} &= y_{m+2} + \frac{19}{13230}df_m - \frac{2}{105}df_{m+1} + \frac{286}{735}df_{m+2} + \frac{1322}{945}df_{m+3} - \frac{128}{147}df_{m+\frac{7}{2}} + \frac{8192}{6615}df_{m+\frac{15}{4}} - \frac{29}{210}df_{m+4} \qquad (9)
\end{aligned}$$

Analysis of basic properties of the method

The order, error constant, consistency, zero stability and region of absolute stability of (8) and (9) are analyzed using the conditions proposed by Lambert (1973) and Dahlquist (1956).

Order and error constant

According to Lambert (1973), the Linear Multistep Method is said to be of order e if $c_0 = c_1 = 0, \ldots, c_e = 0 \quad c_{e+1} \neq 0$ and c_{e+1} yields the error term.

The order and error constants for (8) are obtained as follows:

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

as

$$-\frac{3751}{3386880}, -\frac{89}{1128960}, -\frac{872075}{11098128384}, -\frac{79}{1003520}, -\frac{5929}{75497472}, -\frac{67}{846720}$$

Therefore, (8) has order $e = 7$ and error constants,

$$\left(-\frac{787}{211680}, \frac{317}{376320}, \frac{773}{3386880}, \frac{1721}{8028160}, \frac{244853}{1132462080}, \frac{1}{4704}\right)^T$$

Applying the same approach to (9), we obtained

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

but

$$-\frac{787}{211680}, \frac{317}{376320}, \frac{773}{3386880}, \frac{1721}{8028160}, \frac{244853}{1132462080}, \frac{1}{4704}$$

Therefore, (9) has order $e = 7$ and error constants,

Consistency

In Lambert (1973), a numerical method is said to be consistent if the order e is greater than 1 i.e. $e \geq 1$. Since, the order of our proposed method HBEAMMs as analyzed for the discrete schemes (8) and (9) is greater than 1 i.e. $e \geq 1$, the method is consistent.

Zero stability analysis

In Dahlquist (1956), a computational method is said to be zero stable if the roots $r_s, s = 1, 2, 3, \ldots, n$ of the first characteristic polynomial $\psi(g)$ expressed as $\psi(g) = \det(gW_2^{(1)} - W_1^{(1)})$ satisfies $|g_s| \leq 1$ and the roots $\left|g_s\right|$ is simple or distinct.

The zero stability for (8) is examined as follows:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+\frac{9}{4}} \\ y_{m+\frac{5}{2}} \\ y_{m+\frac{11}{4}} \\ y_{m+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{m-\frac{11}{4}} \\ y_{m-\frac{5}{2}} \\ y_{m-\frac{9}{4}} \\ y_{m-2} \\ y_{m-1} \\ y_{m} \end{pmatrix}$$

$$+d \begin{pmatrix} -\frac{6037}{3528} & \frac{47129}{2520} & -\frac{130048}{2835} & \frac{14296}{315} & -\frac{510464}{24255} & \frac{28817}{7560} \\ \frac{543}{1960} & \frac{3307}{840} & -\frac{4096}{567} & \frac{216}{35} & -\frac{20992}{8085} & \frac{3313}{7560} \\ \frac{250055}{903168} & \frac{520375}{129024} & -\frac{63755}{9072} & \frac{12325}{2016} & -\frac{200125}{77616} & \frac{168625}{387072} \\ \frac{1737}{6272} & \frac{18021}{4480} & -\frac{724}{105} & \frac{219}{35} & -\frac{636}{245} & \frac{1961}{4480} \\ \frac{567}{2048} & \frac{123823}{30720} & -\frac{44933}{6480} & \frac{1029}{160} & -\frac{6559}{2640} & \frac{119707}{276480} \\ \frac{611}{2205} & \frac{1264}{315} & -\frac{19456}{2835} & \frac{1984}{315} & -\frac{54272}{24255} & \frac{487}{945} \end{pmatrix} \begin{pmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+\frac{9}{4}} \\ f_{m+\frac{5}{2}} \\ f_{m+\frac{11}{4}} \\ f_{m+3} \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{67031}{249480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{799}{249480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{40825}{12773376} \\ 0 & 0 & 0 & 0 & 0 & -\frac{43}{13440} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5831}{1824768} \\ 0 & 0 & 0 & 0 & 0 & -\frac{20}{6237} \end{pmatrix} \begin{pmatrix} f_{m-\frac{11}{4}} \\ f_{m-\frac{5}{2}} \\ f_{m-\frac{9}{4}} \\ f_{m-2} \\ f_{m-1} \\ f_{m} \end{pmatrix}$$

where $W_2^{(1)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, W_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

and $Z_2^{(1)} = \begin{pmatrix} -\frac{6037}{3528} & \frac{47129}{2520} & -\frac{130048}{2835} & \frac{14296}{315} & -\frac{510464}{24255} & \frac{28817}{7560} \\ \frac{543}{1960} & \frac{3307}{840} & -\frac{4096}{567} & \frac{216}{35} & -\frac{20992}{8085} & \frac{3313}{7560} \\ \frac{250055}{903168} & \frac{520375}{129024} & -\frac{63755}{9072} & \frac{12325}{2016} & -\frac{200125}{77616} & \frac{168625}{387072} \\ \frac{1737}{6272} & \frac{18021}{4480} & -\frac{724}{105} & \frac{219}{35} & -\frac{636}{245} & \frac{1961}{4480} \\ \frac{567}{2048} & \frac{123823}{30720} & -\frac{44933}{6480} & \frac{1029}{160} & -\frac{6559}{2640} & \frac{119707}{276480} \\ \frac{611}{2205} & \frac{1264}{315} & -\frac{19456}{2835} & \frac{1984}{315} & -\frac{54272}{24255} & \frac{487}{945} \end{pmatrix}$

$$\psi(g) = \det\left(gW_2^{(1)} - W_1^{(1)}\right) = \left|gW_2^{(1)} - W_1^{(1)}\right| = 0. \quad (10)$$

We have $\psi(g) = \left| g\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} -g & 0 & 0 & 0 & 0 & 0 \\ -g & g & 0 & 0 & 0 & 0 \\ -g & 0 & g & 0 & 0 & 0 \\ -g & 0 & 0 & g & 0 & 0 \\ -g & 0 & 0 & 0 & g & 0 \\ -g & 0 & 0 & 0 & 0 & g \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right|$

$$\Rightarrow \psi(g) = \begin{pmatrix} -g & 0 & 0 & 0 & 0 & -1 \\ -g & g & 0 & 0 & 0 & 0 \\ -g & 0 & g & 0 & 0 & 0 \\ -g & 0 & 0 & g & 0 & 0 \\ -g & 0 & 0 & 0 & g & 0 \\ -g & 0 & 0 & 0 & 0 & g \end{pmatrix}.$$

Using Maple (18) software, we obtain

$$\psi(g) = -g^5\left(g+1\right)$$

$$\Rightarrow -g^5\left(g+1\right) = 0$$

$\Rightarrow g_1 = -1, g_2 = 0, g_3 = 0, g_4 = 0, g_5 = 0, g_6 = 0$ .Since the determinant of the first characteristic polynomial $\psi(g$ expressed as $\psi(g) = \det(gW_2^{(1)} - W_1^{(1)})$ satisfies $|g_s|$ and the roots $|g_s|$ are simple or distinct for $|g_s| < 1,\ s = 1,2,3,4,5,6$, the discrete scheme (8) is zero stable.

Using the same approach, then **(9)** is presented as

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+3} \\ y_{m+\frac{7}{2}} \\ y_{m+\frac{15}{4}} \\ y_{m+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{m-\frac{15}{4}} \\ y_{m-\frac{7}{2}} \\ y_{m-3} \\ y_{m-2} \\ y_{m-1} \\ y_m \end{pmatrix}$$

$$+d\begin{pmatrix} -\frac{5506}{3465} & \frac{962}{2205} & -\frac{2582}{945} & \frac{14464}{2205} & -\frac{434176}{72765} & \frac{997}{630} \\ -\frac{3461}{9240} & -\frac{5659}{5880} & \frac{9307}{7560} & -\frac{1864}{735} & \frac{159232}{72765} & -\frac{467}{840} \\ -\frac{551}{27720} & \frac{6967}{17640} & \frac{8963}{7560} & -\frac{3032}{2205} & \frac{77312}{72765} & -\frac{629}{2520} \\ -\frac{471}{24640} & \frac{24429}{62720} & \frac{391}{280} & -\frac{927}{980} & \frac{2416}{2695} & -\frac{3921}{17920} \\ -\frac{77861}{4055040} & \frac{287567}{737280} & \frac{1536983}{1105920} & -\frac{9359}{11520} & \frac{1526}{1485} & -\frac{335111}{1474560} \\ -\frac{2}{105} & -\frac{286}{735} & \frac{1322}{945} & -\frac{128}{147} & \frac{8192}{6615} & -\frac{29}{210} \end{pmatrix}\begin{pmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+3} \\ f_{m+\frac{7}{2}} \\ f_{m+\frac{15}{4}} \\ f_{m+4} \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{3841}{13230} \\ 0 & 0 & 0 & 0 & 0 & \frac{127}{13230} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{1323} \\ 0 & 0 & 0 & 0 & 0 & \frac{181}{125440} \\ 0 & 0 & 0 & 0 & 0 & \frac{6419}{4423680} \\ 0 & 0 & 0 & 0 & 0 & \frac{19}{13230} \end{pmatrix}\begin{pmatrix} f_{m-\frac{15}{4}} \\ f_{m-\frac{7}{2}} \\ f_{m-3} \\ f_{m-2} \\ f_{m-1} \\ f_{m} \end{pmatrix}$$

Where
$$W_2^{(2)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}, W_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and
$$Z_2^{(2)} = \begin{pmatrix} -\frac{5506}{3465} & \frac{962}{2205} & -\frac{2582}{945} & \frac{14464}{2205} & -\frac{434176}{72765} & \frac{997}{630} \\ -\frac{3461}{9240} & -\frac{5659}{5880} & \frac{9307}{7560} & -\frac{1864}{735} & \frac{159232}{72765} & -\frac{467}{840} \\ -\frac{551}{27720} & \frac{6967}{17640} & \frac{8963}{7560} & -\frac{3032}{2205} & \frac{77312}{72765} & -\frac{629}{2520} \\ -\frac{471}{24640} & \frac{24429}{62720} & \frac{391}{280} & -\frac{927}{980} & \frac{2416}{2695} & -\frac{3921}{17920} \\ -\frac{77861}{4055040} & \frac{287567}{737280} & \frac{1536983}{1105920} & -\frac{9359}{11520} & \frac{1526}{1485} & -\frac{335111}{1474560} \\ -\frac{2}{105} & -\frac{286}{735} & \frac{1322}{945} & -\frac{128}{147} & \frac{8192}{6615} & -\frac{29}{210} \end{pmatrix}$$

$$\begin{aligned} \psi(g) &= \det\left(gW_2^{(2)} - W_1^{(2)}\right) \\ &= \left|gW_2^{(2)} - W_1^{(2)}\right| = 0. \end{aligned} \tag{11}$$

We have

$$\psi(g)=\left|g\begin{pmatrix}0&-1&0&0&0&0\\1&-1&0&0&0&0\\0&-1&1&0&0&0\\0&-1&0&1&0&0\\0&-1&0&0&1&0\\0&-1&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right|=\left|\begin{pmatrix}0&-g&0&0&0&0\\g&-g&0&0&0&0\\0&-g&g&0&0&0\\0&-g&0&g&0&0\\0&-g&0&0&g&0\\0&-g&0&0&0&g\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right|$$

$$\Rightarrow\psi(g)=\begin{pmatrix}0&-g&0&0&0&-1\\g&-g&0&0&0&0\\0&-g&g&0&0&0\\0&-g&0&g&0&0\\0&-g&0&0&g&0\\0&-g&0&0&0&g\end{pmatrix}.$$

Using Maple (18) software, we obtain:

$\psi(g)=g^5(g+1)$

$\Rightarrow g^5(g+1)=0$

$\Rightarrow g_1=-1, g_2=0, g_3=0, g_4=0, g_5=0, g_6=0$

.Since the determinant of the first characteristic polynomial $\psi(g)$ expressed as $\psi(g)=\det(gW_2^{(1)}-W_1^{(1)})$ satisfies $|g_s|\le 1$ and the roots $|g_s$ are simple or distinct for $|g_s|<1,\ s=1,2,3,4,5,6$, the discrete scheme (9) is zero stable.

Convergence

Theorem 1: The necessary and sufficient condition for a linear multistep method to be convergent as stated by Dahlquist G. (1956) is that it must be consistent and zero stable. Since, the discrete schemes (8) and (9) are both consistent and zero stable, therefore the method is convergent.

Region of absolute stability

The regions of absolute stability of the numerical methods for ASTDDEs are considered. We considered finding the P-and Q-stability by applying (8) and (9) to the ASTDDEs of this form

$$dy(t)=\alpha(y(t)+y(t+\tau))dt+\beta(y(t)+y(t+\tau))ds(t)$$

for $t>t_0, \tau>$

$$y(t)=\varphi(t), \text{for } t\le t_0 \qquad (12)$$

where $\varphi(t$ is the initial function, α,β are complex coefficients, $\tau=vd, v\in Z^+$, d is the step size and $v=\frac{\tau}{d}, v$ is a positive integer. Let $P_1=d\alpha$ and $P_2=d\beta$, then the P-and Q-stability of (8) and (9) for $v=1$ are investigated, plotted using Maple 18 and MATLAB and presented in Figs. 1 to 4.

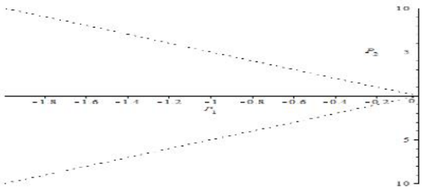

Fig. 1: Region of P-stability (HEBAMM) in (8)

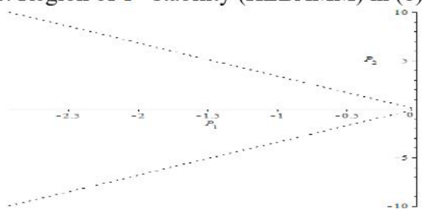

Fig. 2: Region of P-stability (HEBAMM) in (8)

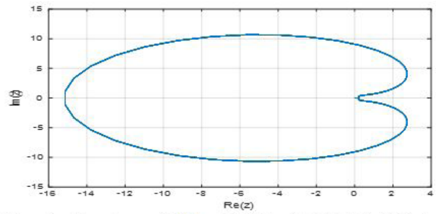

Fig. 3: Region of Q-stability (HEBAMM) in (9).

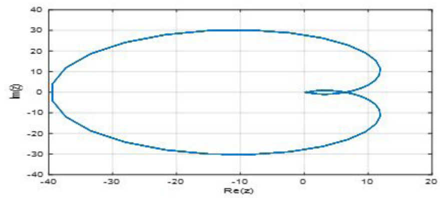

Fig. 4: Region of Q-stability (HEBAMM) in (9).

The P-stability regions in Figs. 1 to 2 lie inside the open-ended region while the Q-stability regions in Figs. 3 to 4 lie inside the enclosed region.

Evaluations of delay terms and numerical computations

Evaluations of delay terms

Here, we shall formulate two accurate and efficient mathematical expressions for the evaluation of the advanced time-delay terms on the drift and the volatility noise term on the diffusion part of the stochastic time-delay differential equations. The advanced time-delay term $(t+\tau)$ shall be evaluated with the accurate and efficient formula of this form

$$\delta_{m+j}(t) = \frac{m}{c}\left(\left(cq+\left(m+j-r-1\right)d\right)\right), c \neq 0 \quad (13)$$

Using normalized Brownian Motion,we formulated an expression to evaluate the volatility noise term $ds(t)$ such that the distribution are Gaussian with $N(0,1)$ whose mean μ is 0 and the standard deviation σ is 1. The random process is expressed as

$$S(t) = \frac{1}{\sqrt{((m+j-r-1)d)\pi}} e^{-t^2/(m+j-r-1)d} \text{ for } t \geq 0 \quad \text{for (14)}$$

Then by differentiating (14), we have

$$dS(t) = \frac{2t}{(m+j-r-1)d\sqrt{((m+j-r-1)d)\pi}} e^{-t^2/(m+j-r-1)d} \text{fort} \geq 0 \quad \text{for (15)}$$

where $j \in (-k,k), k$ is a step number, $v = \frac{\tau}{d} \in Z, \tau = vd; \tau$ is the delay term, $z = 0,1,2,...,Z-1$ and Z is the number of solutions in the give interval which is implemented to evaluate the volatility noise term *dS(t)*

Numerical computations

In this section, the evaluated advanced time-delay term and the volatility noise term using the two expressions (13) and (15) developed, shall be incorporated into some advanced stochastic time-delay differential equations before its evaluation with the discrete schemes (8) and (9) at constant step size d = 0.01 to obtain the computational solutions of $dy(t)$ i.e. we present some .

Numerical problems

Problem 1

$$dy(t) = -1000(y(t) + y(t + (In(1000+1)))dt + (y(t) + y(t + (In(1000+1))ds(t),$$
$$0 \le t \le 3$$

$$\varphi(t) = e^{-t}, t \ge 0$$

Exact solution $\varphi(t) = e$

Problem 2

$$dy(t) = \cos(t)((y(y(t)+2)dt + (y(y(t)+2))ds(t),$$
$$0 \le t \le 3$$

$$\varphi(t) = 1, t \ge 0$$

Exact solution $\varphi(t) = 1 + \sin(t)$

Results

Results, graphical presentations and discussions

Problems 1 & 2 were solved using the discrete schemes (8) and (9) generated by Hybrid Extended Adams Moulton Methods (HEBAMMs) and the results of the absolute random errors obtained are presented in Tables 1 to 2

TABLE 1

Absolute random errors of HEBAMMs for k = 2 and 3 using Problem 1.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error
1	0.565751199	0.526170595
2	0.842809479	0.862157617
3	0.911592118	0.905342205
4	1.002351227	0.929347474
5	0.864749645	0.871647786
6	0.913250098	0.860524042
7	1.146237691	0.850835748
8	0.646948079	0.866623951
9	0.84348562	0.666338564
10	1.319855341	0.645648321
11	0.362816087	0.644382402
12	0.754941177	0.726480697
13	1.481873881	0.44237991
14	0.082884249	0.401695385
15	0.660909689	0.420132842
16	1.587381389	0.567087895
17	0.143689956	0.267910683
18	0.56993907	0.197913565
19	1.611965638	0.242315408
20	0.293662272	0.423528521
21	0.48642232	0.16047006
22	1.553277394	0.063060018
23	0.367574147	0.131585718
24	0.411346837	0.310742833
25	1.425451969	0.104652176
26	0.380887536	0.008028775
27	0.343946862	0.075032333
28	1.251127777	0.227251551
29	0.354163289	0.076427669
30	0.283216727	0.035628969

CPU time of HEBAMMs for k = 2 is 0.05s and k = 3 is 0.10s.

TABLE 2

Absolute random errors of HEBAMMs for k = 2 and 3 using Problem 2.

t	K = 2 Absolute Random Error	K = 3 Absolute Random Error
1	0.845719236	0.845719236
2	0.917929971	0.917929971
3	0.154271898	0.154271898

4	0.749531817	0.738997221
5	0.944923733	0.936881179
6	0.260472177	0.251682861
7	0.641732346	0.690204361
8	0.986803841	1.028003311
9	0.415212543	0.454234715
10	0.613232709	0.49338445
11	1.048601487	0.942005002
12	0.578623583	0.471703454
13	0.274896696	0.486723777
14	0.872323399	1.068803713
15	0.539539553	0.737431921
16	0.498281629	0.192244135
17	1.145443495	0.867394105
18	0.926684308	0.644419384
19	0.054269463	0.26442725
20	0.720869531	1.036821909
21	0.653305775	0.954966798
22	0.059092722	0.11750989
23	0.916341665	0.717674469
24	0.967245579	0.768929256
25	0.176929452	0.001728019
26	1.006426375	0.893273334
27	1.207809439	1.080526603
28	1.136693034	0.399595691
29	0.093831028	0.510970701
30	0.22410745	0.853166636

CPU time of HEBAMMs for k = 2 is 0.06s and k = 3 is 0.09s.

Graphical presentation of results

Using Microsoft Excel, the Absolute Random Error Results of HEBAMMs for Problem 1 and 2 in Table 1 and 2 are presented as;

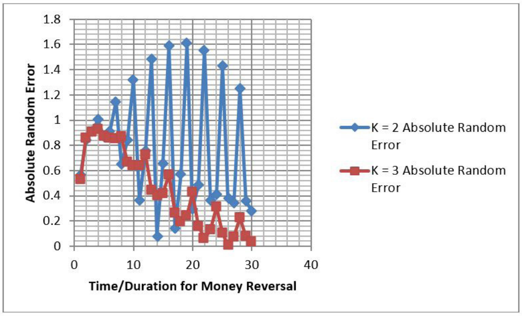

Fig. 5: Advanced Stochastic Time-Delay DDEs absolute random error results for Problem 1.

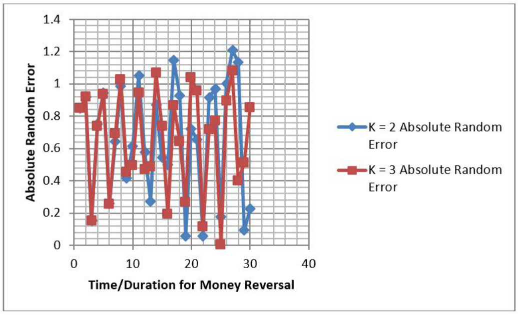

Fig. 6: Advanced Stochastic Time-Delay DDEs absolute random error results for Problem 2

Comparison of results

In order to determine the accuracy, efficiency and advantage of our method HEBAMM, we compared the absolute maximum errors of our method with other existing methods in Evelyn (2000); Bahar (2019) and Osu *et al.* (2021).

TABLE 3

Comparison between the Maximum Absolute Random Errors (MARE) of our method HEBAMM for k = 2 and 3 with Evelyn (2000); Bahar (2019) and Osu et al. (2021) for constant step size d=0.01 for Problem 1.

Numerical Method	COMPARED MAREs with [5,7,18]
HEBAMM MARE for k = 2	1.611965638
HEBAMM MARE for k = 3	0.929347474
CSSEMM MARE for k = 2	4.76E-02
CSSEMM MARE for k = 3	9.17E-02
CSSEMM MARE for k = 4	1.62E-01
EMM MARE for k = 2	1.84E-02
EMM MARE for k = 3	4.04E-03
EMM MARE for k = 4	9.73E-04
BSM MARE for k = 2	7.04E-01
BSM MARE for k = 3	7.04E-01
BSM MARE for k = 4	7.04E-01

TABLE 4

Comparison between the Maximum Absolute Random Errors (MARE) of our method HEBAMM for k = 2 and 3 with Evelyn (2000); Bahar (2019) and Osu et al. (2021) for constant step size d= 0.01 for Problem 2.

Numerical Method	COMPARED MAREs with [5,7,18]
HEBAMM MARE for k = 2	1.207809439
HEBAMM MARE for k = 3	1.080526603
CSSEMM MARE for k = 2	3.18E-02
CSSEMM MARE for k = 3	5.90E-02
CSSEMM MARE for k = 4	1.37E-01
EMM MARE for k = 2	1.09E-01
EMM MARE for k = 3	4.91E-02
EMM MARE for k = 4	2.44E-02
BSM MARE for k = 2	6.96E-01
BSM MARE for k = 3	6.96E-01
BSM MARE for k = 4	6.96E-01

Graphical presentation for compared results

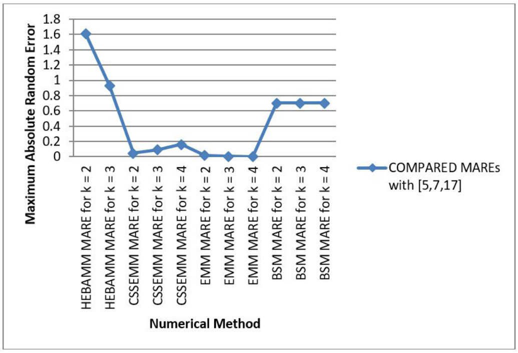

Fig. 7: Compared MAREs of HEBAMM with Evelyn (2000); Bahar (2019) and Osu et al. (2021) for Problem 1

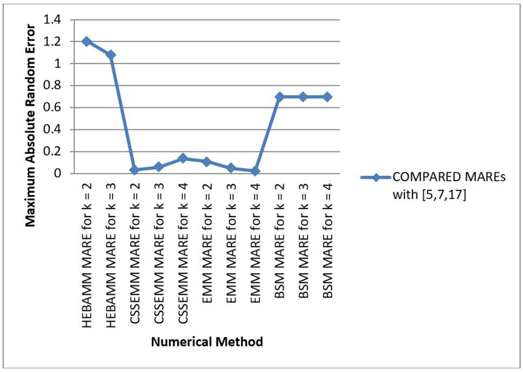

Fig. 8: Compared MAREs of HEBAMMs with Evelyn (2000); Bahar (2019) and Osu et al. (2021) for Problem 2.

Conclusion

In this study, we have demonstrated that Hybrid Extended Block Adams Moulton Methods (HEBAMMs) are suitable for computational solution of Advanced Stochastic Time-Delay Differential Equations (ASTDDEs) without the application of interpolation techniques in the evaluation of the advanced delay term and volatility noise term. The incorporation of the evaluated mathematical expressions (13) and (15) for evaluation of the advanced delay term and volatility noise term give better and faster results as shown in Tables 1 to 4 and Figs. 5 to 8. The numerical results revealed that there are stochastic swings or uncertainties in the satisfaction that customers or users derive from the use of electronic payment systems in performing financial transactions as a result of the effect of advanced time-delay. To reduce these uncertainties in the customers' satisfaction, prompt action should be taken by the banks to curb the advanced time-delay that may occur in tracing and reversal of wrongly debited money, malfunctioning of the automated teller machines (ATMs), network failure in receiving transaction alerts and any other challenges that can lead to decline in customers' satisfaction. This can be done by regular servicing of the ATMs, developing a new online electronic payment App which can handle any future challenges and quick restoration of network by the ICT unit of the banks for easy and fast transactions. Also, it was observed in Tables 1 to 4 that the discrete schemes of lower step number $k=2$ of HEBAMMs performed slightly better and faster than the higher step number $k=3$ when compared with other existing methods. Further research should be carried-out for step numbers $k=4,5,6...$ on the computational solutions of ASTDDEs using HEBAMMs.

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