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ON UNIFORMLY ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

P. W. Darko
Department of Mathematics and Computer Science, 1570 Baltimore Pike, P. O. Box 179, Lincoln University, PA 19352.
Email:pdarko10@aol.com

Abstract

Using fundamental solutions, exact solutions are constructed for uniformly elliptic equations in non-divergence form. Distributional and viscosity solutions and Lp estimates are obtained and solutions are derived for the Dirichlet problem.

Introduction

In the AMS colloquium publication (Caffarelli & Cabre, 1995), the regularity theory of fully non-linear elliptic equations is reduced to solutions of uniformly elliptic equations in non-divergence form. Fully non-linear elliptic equations are very useful in applied mathematics. They arise in gas dynamics, in control theory, in optimisation, in elastic thin shells, in stochastic theory and Monge-Kantorvitch mass transfer problem. Because of this, uniformly elliptic equations in non-divergence form are also very useful. However, the regularity theory for these linear equations (Chen & WV, 1998) is ageing. Therefore, it pays to take a closer fresh look at these linear equations. In the paper, fundamental solutions were used to construct exact solutions of uniformly elliptic equations in non-divergence form. Distributional solutions, viscosity solutions, and Lp estimates were obtained. The Dirichlet problem for these linear equations were solved.

Uniformly elliptic equations in the form

$L(u) = a_{ij}\, \partial_{ij(u)} = f$ were considered. (1.1)

In a bounded domain Ω inR^n, where there are constants $0 < \lambda \leq \wedge$ such that

$\lambda|\xi|^2 \leq$ a_ij $(x)\, \xi_i\, \xi_j \leq \wedge||\xi|^2$ for all $x\ \varepsilon\ \Omega$ and $\xi\ \varepsilon\ R^n$ (1.2)

and the summation convention was used. This implies that $\lambda \leq a_{jj}\ (x) \leq \wedge$ for all $x\ \varepsilon\ \Omega$ and $1 \leq j \leq n$. (Here $\partial_{ij}\,(u) = \dfrac{\partial u^2}{\partial x_i\ \partial x_i}$).

Results

Throughout this section Ω is a bounded domain in R^n.
Theorem 2.1. Let ε $L^p(\Omega), 1 \leq p \leq \infty$, then there is u ε $W^{2,p}(\Omega)$ such that $L(u) = f$
In the sense of distributions and

$$\|u\| w^{2,p}(\Omega) \leq K \| f \| L^p(\Omega) \tag{2.1}$$

where K is independent of f. (Here $W^{2,p}(\Omega), 1 \leq p \leq \infty$,are the usual Sobolev spaces.)

Theorem 2.2. Let f and each a_{ij} be continuous on Ω and f ε $L^1(\Omega)$. Then there is u ε $C^0(\Omega) \cap L^1(\Omega)$ such that $L(u) = f$.

Theorem 2.3. Let f and each a_{ij} be in $C^0(\overline{\Omega})$ and g ε $C^0(\Omega)$,then there is u defined on Ω,u ε $C^0(\Omega)$,such that

$$L(u) = f \text{ on } \Omega \tag{2.2}$$

and

$$u = g \text{ on } \partial\Omega \tag{2.3}$$

Solutions and estimates

In this section those parts of the theorems that need to be proved were proved. For Theorm 1,

let e be a fundamental solution of $\frac{\partial^2 e}{\partial x^2}$ in R, that $\frac{\partial^2 e}{\partial x^2} = \delta$, then the Diracdelta in R.

Define the distribution $E_j(\varphi) = e(\varphi(0,0,\ldots,j,\ldots,0,0))$, (3.1)

the action of e being in the jth coordinate;φ ε $D(R^n)$ – a test function.

Let f be zero outside Ω, $a_{ij} \equiv 1$ outside Ω and define v by

$$u_v = \frac{1}{n} \sum_{j=1}^{n} E_{j*}\left(\frac{f}{a_{jj}}\right) \tag{3.2}$$

where * is convolution. It is then clear that u, the restriction of v to Ω satistfied $L(u) = f$ and (2.1).

To prove Theorem 2, let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset$ ...with $\cup_{v=1}^{\infty} = \Omega_1$ be an exhaustion of Ω. Let $\{\varphi_v\}_{v=1}^{\infty}$ be a sequence of functions with φ_v ε $C_0^{\infty}(\Omega_{v+1}))$,$\varphi_v \equiv 1$ on Ω_v,and $0 \leq \varphi_v \leq 1$. Define u_v ε $C^0(R^n)$ by

$$u_v = \frac{1}{n} \sum_{j=1}^{n} E_j * \left(\frac{f}{a_{jj}}\right) \tag{3.3}$$

where again * is convolution.
Now, it is clear that $L(v_v) = f$ *in* Ω_v *and* $\{u_v\}$ trends locally uniformly to a continuous u ε $C^0(\Omega) \cap L^1(\Omega)$ such that

$$L(u) = f \tag{3.4}$$

To prove Theorem 3, let $\{\varphi_v\}$ be the sequence in $C_o^\infty(\Omega)$ constructed above and define

$$u_v := \left\{\frac{1}{n}\sum_{j=1}^{n} E_{j*}\left(\varphi_v \frac{f}{a_{jj}}\right)\right\}\varphi_v + (1-\varphi_v)g \tag{3.5}$$

Then $L(u_v)=f$ *in* Ω_v *and* $\{v_v\}$ converges locally uniformly in Ω to a function u in $C^o(\Omega)$ such that

$$L(u) = f \text{ in } \Omega \tag{3.6}$$

$$u = g \text{ on } \partial\Omega. \tag{3.7}$$

References

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Chen, Y-Z. & WV, L.-C. (1998) Second order elliptic equations and elliptic systems, *Translations of Mathematical Monographs* Vol.174, AMS, Providence, Rhode Island.

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