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THE GREEN'S FUNCTIONS OF SUPERCONDUCTIVITY- A REVIEW

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ABSTRACT

We present some basic Green's functions of superconductivity, making emphasis on their geneology and analytic properties. From calculations, we note that the temperature dependence of the Green's functions for fermionic (and bosonic) systems limits and defines the extent of their applications and results. Furthermore, the Gorkov interaction term of the four field fermion operators, is examined and interpreted in terms of the Landau condensate. Finally we show that the Gorkov interaction under a certain condition sustains superconductivity and spin density wave in the system.

KEYWORDS: Green's functions, Bethe-Salpeter equation, Landau condensate, superconductivity, spin density wave.

1. INTRODUCTION.

The theory of superconductivity received a tremendous boost in 1957 when Gorkov (Gorkov, 1958) formulated the Bardeen-Cooper-Schrieffer(BCS) theory (Bardeen, Cooper, Schrieffer, 1957) in the language of the Green's functions. The Green's functions were directly introduced into condensed matter physics from quantum field theory where they had been found to be particularly successful in solving many body problems (see e.g, Kushnirenko, 1971). The Green's function enables us to obtain the single-particle energy spectrum, the life-time of single-particle excitations, the ground state energy and the expectation value of any single-particle thermodynamic quantity in the ground state of the system.

A further development in the Green's function formalism was made through its diagrammatic representation of the electron-phonon interaction by Migdal (Migdal, 1958). Eliashberg later developed a perturbation method in which the Green's function calculated for the ground state of the superconductor was used as a zero approximation(Eliashberg,1960) in determining the exact Green's function of the electron through the Dyson equation. The metallic superconductors (e.g, tin and lead) and alloy superconductors (e.g, Nb_3Sn and V_3Ga) for which the BCS theory was formulated have serious cryogenic drawback- their critical temperatures (T_c) are in the range $0<T_c<25K$. This circumstance limits their application in technology. These low temperature superconductors (LTS)have now been superceeded by the ceramic high temperature superconductors (HTS) discovered in 1986(Bednorz and Muller,1986). The critical temperature of the HTS is greater than 40K, and room temperature superconductor (T_c=300K) is expected in the near future. The Green's function technique is still the main tool for describing and explaining the properties of these materials (Plakida, 2010).

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2. The single-particle causal Green's functions.

The single-particle Green's function for fermions is defined as (Lifshitz and Pitayevsky,1980)

$$G_{\alpha\beta}(X_1, X_2) = -i\langle T\Psi_\alpha(X_1)\Psi_\beta^\dagger(X_2)\rangle_0 \qquad (1)$$

where $X = X(\vec{r}, t), \vec{r}$ is the position vector and t the time. Ψ_α, Ψ_β are the annihilation and creation fermionic field operators respectively, and α, β are the spin indices.The angle bracket $\langle \dots \rangle_0$ denotes averaging with respect to the ground state of the system. The symbol T is the chronological operator which arranges the operators from right to left in the order of increasing times t_1, t_2, with t_1 being the reference or highest time on the left. Thus for fermions

$$G_{\alpha\beta}(X_1, X_2) = \begin{cases} -i\langle \Psi_\alpha(X_1)\Psi_\beta^+(X_2)\rangle, & t_1 > t_2 \\ i\langle \Psi_\beta^+(X_2)\Psi_\alpha(X_1)\rangle, & t_1 < t_2 \end{cases} \qquad (2)$$

If the system is not ferromagnetic and is not situated in an external field, then $G_{\alpha\beta} = \delta_{\alpha\beta}G$. In a similar manner the single-particle phonon Green's function is defined as (Schrieffer,1964)

$$D(X_1, X_2) = -i\langle T\varphi(X_1)\varphi^+(X_2)\rangle \qquad (3)$$

so that

$$D(X_1, X_2) = \begin{cases} -i\langle \varphi(X_1)\varphi^+(X_2)\rangle, & t_1 > t_2 \\ -i\langle \varphi^+(X_2)\varphi(X_1)\rangle, & t_1 < t_2 \end{cases} \qquad (4)$$

The operators $\Psi(X)$ and $\varphi(X)$ are Heisenberg operators, which, as an example for $\Psi(X)$ is

$$\Psi(r, t) = e^{iHt}\Psi(r)e^{-iHt} \qquad (5)$$

where H is the Hamiltonian operator. The corresponding Schrodinger representation of the wave function is

$$\Psi(r) = \frac{1}{\sqrt{V}}\sum_p a_p e^{ipr} \qquad (6)$$

Then (5) can be written in terms of the free particle energy $\varepsilon_0(p)$ as

$$\Psi(r, t) = \frac{1}{\sqrt{V}}\sum_p a_p e^{i[pr - \varepsilon_0(p)t]} \qquad (7)$$

The non-interacting single-particle Green's function(ignoring the spin indices) then becomes

$$G^{(0)}(r, t) = -\frac{i}{V}\sum_p e^{i[pr - \varepsilon_0 t]}\langle T a_p(t) a_p^+(0)\rangle \qquad (8)$$

Making use of the definition (2) we have

$$\langle T a_p(t) a_p^+(0)\rangle = \begin{cases} 1 - \langle a_p^+ a_p\rangle = 1 - n_p, & t > 0 \\ -n_p, & t < 0 \end{cases} \qquad (9)$$

In order to write $G^0(r, t)$ in terms of frequency and momentum we carry out a Fourier transformation of (8) as follows

$$G^{(0)}(p, \omega) = -i\frac{1}{V}\int_0^\infty G(r, t)e^{-i(pr - \omega t)}dt\,d^3r = -\frac{i}{V}\int_0^\infty e^{i(pr + \varepsilon_0 t)}e^{-i(pr - \omega t)}(1 - n_p)dt\,d^3r$$
$$+\frac{i}{V}\int_0^{-\infty} e^{i(pr + \varepsilon_0 t)}e^{-i(pr - \omega t)}n_p dt\,d^3r = -\frac{i}{V}\int_0^\infty (1 - n_p)e^{i(\omega - \varepsilon_0)t}dt\,d^3r + \frac{1}{V}\int_0^{-\infty} n_p e^{-i(\omega - \varepsilon_0)t}dt\,d^3r \qquad (10)$$

Noting that $n_p = a_p^+ a_p = 1,$ at $|p| < p_0$ and equal to zero at $|p| > 0$ for the occupied and unoccupied levels respectively, then (10) becomes

$$G^{(0)}(p, \omega) = -i\theta(|p| - p_0)\int_0^{\infty} e^{i(\omega - \varepsilon_0)t}dt + i\theta(p_0 - |p|)\int_0^{\infty} e^{-i(\omega - \varepsilon_0)t}dt$$
$$= \frac{\theta(|p| - p_0)}{\omega - \varepsilon_0(p) + i\delta} + \frac{\theta(p_0 - |p|)}{\omega - \varepsilon_0 - i\delta} = \frac{1}{\omega - \varepsilon_0(p) + i\delta sgn(|p| - p_0)} \qquad (11)$$

Here $sgn(x)$ stands for the sign of x and $\theta(x) = 1, for\ x > 0$ and $\theta(x) = 0\ for\ x < 0$. When $p > p_0,\ i\delta$ is positive, and when $p < p_0$, $i\delta$ is negative, thus the quantity $i\delta$ characterises the way the pole of the Green's function is by-passed in integration (Abrikosov et al, 1975). The phonon Green's function (4) contains the phonon field operator $\varphi(X)$ which can be written in terms of the annihilation b and creation b^+ operators of a phonon as

$$\varphi(X) = i\sqrt{\frac{\omega_q}{2}}[b_q e^{-i\omega_q t} + b_{-q}^+ e^{i\omega_q t}] \qquad (12)$$

The phonon Green's function is then

$$\theta(t_1 - t_2)e^{(-i\omega_q(t_1 - t_2) + \theta(t_2 - t_1)e^{i\omega_q(t_1 - t_2)}]} \qquad (13)$$

The Fourier transform of (13) is

$$D^{(0)}(q, \omega) = i\frac{\omega_q}{2}\int_0^\infty e^{i(\omega - \omega_q + i\delta)t}dt + i\frac{\omega_q}{2}\int_0^{\infty} e^{-i(\omega + \omega_q - i\delta)t}dt = \frac{\omega_q^2}{2(\omega - \omega_q + i\delta)}$$
$$-(\omega_q^2)/(2(\omega + \omega_q - i\delta) = (\omega_q^2)/(\omega^2 - \omega_q^2 + i\delta) \qquad (14)$$

3. The electron-phonon interaction

The electron-phonon interaction turns out to be a dominant process in condensed matter with its strength dependent on temperature. The Hamiltonian of the electron –phonon interaction is

$$H_{el-ph} = \sum_{pp'q} g_{pp'} a_p a_{p'}^{+} (b_q + b_{-q}^{+}) \qquad \text{...} \quad 15$$

where $g_{pp'}$ is the electron –phonon coupling constant and a_p, b_q are the electron and phonon annihilation operators respectively. The Hamiltonian (15) describes a scenario in which an electron interacts with phonons in two ways. In the first instance an electron absorbs a phonon from the field and in the second instance the electron emits a phonon into the field. Thus two separate vertices of electron-phonon interaction are seen to be present in the electronic system at $T \neq 0K$. The process takes a different turn when two electrons are within a phonon exchange range. The two vertices are now connected by the rapid exchange of phonons and the two electrons pair up. This mechanism of pairing was first proposed by Cooper(Cooper, 1956). Thus the electron-electron interaction is brought about by the mediation of phonons whereas the appearance of the electron-electron pairs due to an effective attraction is derived from the competition between the Coulomb repulsion and the electron- phonon exchange energy. Since the pairing temperature T_c is different for different metals or alloys,then there must be a renormalization of the electron-phonon vertices for each metal or alloy. We can extract only the single-electron Green's function from the expression (15), and this is not sufficient for the exact mathematical description of the pairing mechanism. A state with four operators has to be introduced. For that purpose we follow Schrieffer(Schrieffer,1964) to write the initial(I) and final states(F) of the phonons in the system as

$$|\vdash F\rangle = \{\blacksquare(b_1 q\,|\vdash I\rangle_{\perp} ab \dashv @ b_1(-q)^{\dagger} + |\vdash I\rangle_{\perp} em \dashv)\dashv \dashv \qquad \text{...} \quad 16$$

where the subscripts ab, em represent absorption and emission of phonons respectively. Since emission and absorption of phonons take place in equal probability, then

$$\langle F|H_{el-ph}|I\rangle = \langle F\left|\sum_{pp'q} g^2 a_p a_p^{+} a_{p'}^{+} a_{p'} (b_q + b_{-q}^{+})\right| I\rangle = \langle I\left|\sum_{pp'q} g^2 a_p a_p^{+} a_{p'}^{+} a_{p'} b_q^{+} (b_q + b_{-q}^{+})\right| I\rangle_{ab}$$
$$= \langle I\left|\sum_{pp'q} g^2 a_p a_p^{+} a_{p'}^{+} a_{p'} b_{-q} (b_q + b_{-q}^{+})\right| I\rangle_{em} \qquad \text{...} \quad 17$$

In the mean time if we ignore the phonon operators in (17) then we have in our hands the two-particles Green's function

$$G_{\alpha\beta,\gamma\delta}(P, P') = \langle T a_{p\alpha}(t) a_{p\beta}^{+} a_{p'\gamma}^{+} a_{p'\delta}(t')\rangle \qquad \text{.......................................} \quad 18$$

We may then apply Wick's theorem(Lifshitz and Pitayevsky,1980) to the eqn(18) to have

$$G_{\alpha\beta,\gamma\delta}(P, P') = \langle T a_{p\alpha}(t) a_{p\beta}^{+}\rangle \langle T a_{p'\gamma}^{+} a_{p'\delta}(t')\rangle + \langle T a_{p\alpha}(t) a_{p'\gamma}^{+}\rangle \langle T a_{p\beta}^{+} a_{p'\delta}(t')\rangle = G_{\alpha\beta}^{(0)} G_{\delta\gamma}^{(0)} + G_{\alpha\gamma}^{(0)} G_{\delta\beta}^{(0)} \qquad \text{...............} \quad 19$$

The expression (19) defines the two-electrons Green's function which in the zero order approximation reduces to the sum of the products of two single –electron Green's functions. Now let us consider (17), incorporating the phonon operators this time. For the absorption term we extract the expression

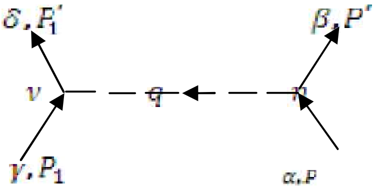
$$\langle T a_{p\alpha} a_{p\beta}^{+} a_{p'\gamma}^{+} a_{p'\delta} b_{q\eta}^{+} b_{q\nu}\rangle \qquad \text{..} \quad 20$$

For the emission term a similar expression is

$$\langle T a_{p\alpha} a_{p\beta}^{+} a_{p'\gamma}^{+} a_{p'\delta} b_{-q} b_{-q}^{+}\rangle \qquad \text{...} \quad 21$$

Wick's theorem may be carried out for the electron and phonon operators separately in (20) and (21). In the diagram technique for the Green functions the effect of including the phonon operators in (20) and (21) is that instead of having two non connecting parallel Green's lines, we now have diagrams of the type

$$\Gamma_{\alpha\gamma,\delta\beta}^{(0)}(p, q) =$$



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Here the dashed line represents the phonon Green's function, p, q are the momenta and $\Gamma_{\alpha\gamma,\delta\beta}^{(0)}$ is called the zero order vertex function. The expression (20) corresponds to the left vertex and (21), the right vertex of the diagram in (22). Furthermore the vertex function $\Gamma_{\alpha\gamma,\delta\beta}^{(0)}$ indicates attraction in the electron-phonon interaction(Sadovsky, 2006). Indeed

$$\Gamma_{\alpha\gamma,\delta\beta}^{(0)}(p, q) \equiv V(p, q) = g^2 D^{(0)}\left(\varepsilon_{p'} - \varepsilon_p; \vec{p}' - \vec{p}\right) = \frac{g^2 \omega^2(\vec{p}' - \vec{p})}{\left(\varepsilon_{p'} - \varepsilon_p\right)^2 - \omega^2(\vec{p}' - \vec{p})} \qquad \text{................} \quad 23$$

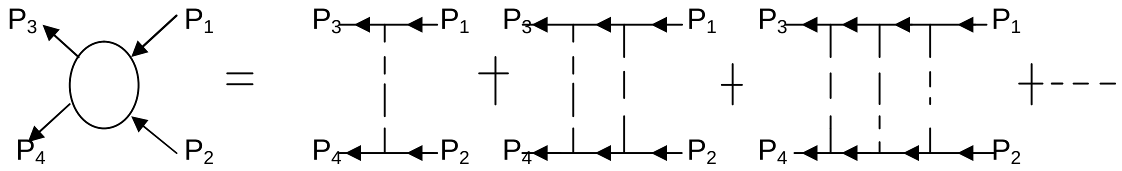
Here, $V(p, q)$ is another notation for the electron-phonon interaction energy and $D^{(0)}$ is the zero order phonon Green's function (14) which depends on the momenta p, p'. For electrons close to the Fermi surface $\varepsilon_{p'} \sim \varepsilon_p \sim 0$, with the result that for g positive

$$\Gamma^{(0)}_{\alpha\gamma,\delta\beta}(p,q) = -g^2 < 0 \qquad \ldots\ldots \qquad 24$$

The negative sign of g^2 shows the existence of attraction between electrons near the Fermi surface.

4. The vertex function and instability of the ground state (T=0)

In their classic paper Bardeen, Cooper and Schrieffer(Bardeen, Cooper, Schrieffer, 1957) showed that at sufficiently low temperature and in the presence of the interaction condition(24) the electrons in the metal will form pairs in momentum space near the Fermi surface. The two electrons must have equal and opposite momenta and spins. Multiple electron-electron scatterings are mediated by phonons as already noted. The scatterings can be represented by a series of ladder diagrams based on the diagram equation(22)(see, for example, Timm, 2012):



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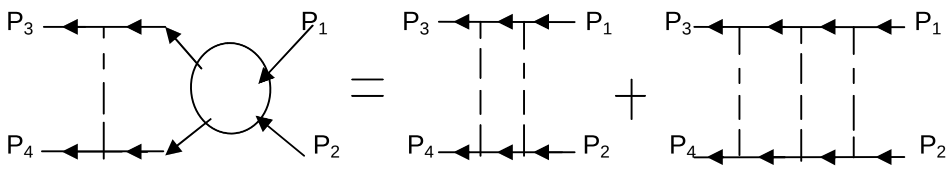
The empty circle with four momentum legs represents the repeated scattering of the electron with phonon,it can also be called the zero order vertex function. In order to sum the diagram (25) we find in the usual way(Lifshitz and Pitayevsky, 1980):



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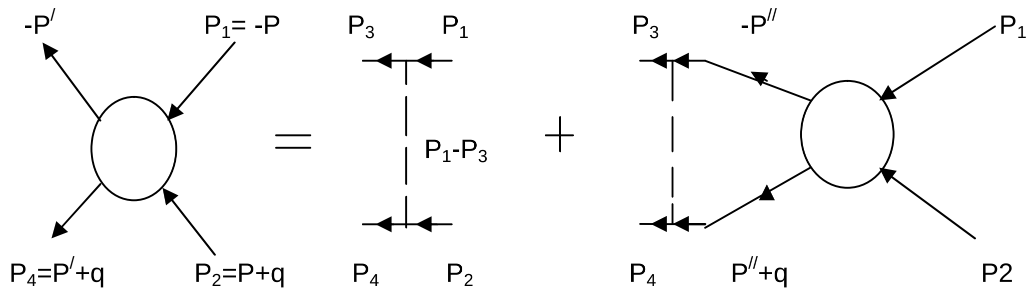
Solving (25),(26) simultaneously, we obtain the equation



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The corresponding integral equation is

$$\Gamma(P_3P_4,P_1P_2) = V(P'-P) + i\int V(P'-P'')G^{(0)}(-P'')G^{(0)}(P''+q)(-P'',P''+q|\Gamma|-P,P+q) \times \frac{d^4P''}{(2\pi)^4} \qquad \ldots\ldots \qquad 28$$

The interaction potential is

$$V(P'-P) = -g^2\omega_{p'}\omega_p = \lambda\omega_{p'}\omega_p \qquad \ldots\ldots \qquad 29$$

where $\lambda = -g^2$ is the electron-phonon coupling constant, and

$$\omega_p = \begin{cases} 1, \varepsilon_p < \omega_D \\ 0, \varepsilon_p > \omega_D \end{cases} \qquad \ldots\ldots \qquad 30$$

ω_D being the Debye frequency. The solution of the integral equation(28) is

$$\Gamma(P_3P_4,P_1P_2) = \frac{\lambda}{1 - i\lambda\int\frac{\omega_p^2 G^{(0)}(-P)G^{(0)}(P+q)(d^4p)}{(2\pi)^4}} \qquad \ldots\ldots \qquad 31$$

The problem now consists of evaluation of the integral and then nullifying the denominator in order to find the singularity of the vertex function. In view of this let us write

$$\int G^{(0)}(-P)G^{(0)}(P+q)\frac{d^4p}{(2\pi)^4} = \int\frac{G^{(0)}(q-P)G^{(0)}(P)(d^4p)}{(2\pi)^4}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d\varepsilon}{2\pi} \frac{1}{[\omega_0 - \varepsilon - \xi(q-p) + i\delta \, sgn\xi(q-p)][\varepsilon - \xi(p) + i\delta \, sgn\xi(p)]} \quad \text{..........} \quad 32$$

Here

$$\xi(p) = \frac{p^2}{2m} - \mu \quad \text{..........} \quad 33$$

is the quasiparticle energy and μ is the chemical potential. We may evaluate the integral by residue theorem, but first the domain of analyticity of ε must be noted. If both poles of the Green's function lie in one half plane of ε, then the integral(32) equals zero by Cauchy's theorem. This circumstance is avoided by observing two conditions of the quasiparticle energies

$$\xi(p) > 0, \xi(q-p) < 0; \qquad \xi(p) < 0, \xi(q-p) > 0 \quad \text{..........} \quad 34$$

These conditions place a pole in the upper half plane (uhp) and the other in the lower half plane (lwp). Let us denote the integral with respect to $d\varepsilon$ in the uhp by A_1, and that in the lhp by A_2, then

$$A_1 = \frac{1}{2\pi} \int \frac{d\varepsilon}{[\omega_0 - \varepsilon - \xi(q-p) + i\delta][\varepsilon - \xi(p) + i\delta]} = i \frac{1}{\omega_0 - \xi(p) - \xi(q-p) + i\delta \, sgn \, \xi(p)} \quad \text{..........} \quad 35$$

$$A_2 = -i \frac{1}{\omega_0 - \xi(p) - \xi(q-p) - i\delta \, sgn \, \xi(p)} \quad \text{..........} \quad 36$$

In (32) the integral with respect to momentum p can be evaluated by using the substitution

$$\int \frac{d^3p}{(2\pi)^3} \cong \frac{mp_F}{2\pi^2} \int d\xi \int d(\cos\theta) = \rho(E_F) \int_{-\infty}^{\infty} d\xi \int_{-1}^{1} d\xi \quad \text{..........} \quad 37$$

where $\rho(E_F)$ is the single-particle density of states of electrons on the Fermi surface. The result of evaluating the integrals in (32) is

$$-\lambda \frac{mp_F}{2\pi^2} \left\{ \frac{\frac{1 + \frac{1}{2}\ln(2\omega_D - i\delta)}{\omega_0 + v_F q - i\delta} + \frac{1}{2}\ln(2\omega_D - i\delta)}{-\omega_0 + v_F q - i\delta} + \frac{\omega_0}{2v_F q}\left(\frac{\ln(\omega_0 - i\delta)}{\omega_0 + v_F q - i\delta} + \frac{\ln(v_F q - \omega_0 - i\delta)}{-\omega_0 - i\delta} \right) \right\} \quad \text{..........} \quad 38$$

Putting $q = 0$ in (38) introduces instability in the system and allows the vertex function(31) to become

$$\Gamma(\omega_0) = \frac{\lambda}{1 + \frac{\lambda m P_F}{2\pi^2}\left\{ ln\left|\frac{2\omega_D}{\omega_0}\right| + i\frac{\pi}{2} \right\}} \quad \text{..........} \quad 39$$

Using $\omega_0 = |\omega_0| e^{i\varphi}$, then the denominator of (39) becomes

$$1 + \frac{\lambda m p_F}{2\pi^2} ln \left| \frac{2\omega_D}{\omega_0} \right| = 0 \quad \text{..........} \quad 40$$

Solving (40) gives the expression(Abrikosov,et al,1975)

$$\omega_0 = 2i\omega_D e^{-\frac{2\pi^2}{|\lambda| m p_F}} = i\tilde{\omega} \quad \text{..........} \quad 41$$

where

$$\tilde{\omega} = 2\omega_D e^{-\frac{2\pi^2}{|\lambda| m p_F}} \quad \text{..........} \quad 42$$

Therefore

$$\lambda = -\frac{2\pi^2}{m p_F} \frac{i\tilde{\omega}}{\omega_0 - i\omega} \quad \text{..........} \quad 43$$

Subtituting (43) in (39) yields

$$\Gamma(\omega_0) = -\frac{2\pi^2}{m p_F} \frac{i\tilde{\omega}}{\omega_0 - i\omega} \quad \text{..........} \quad 44$$

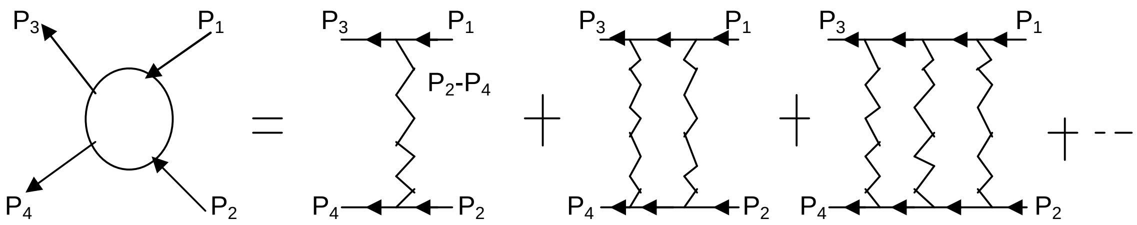
What we have here in(43) and (44) is that the point of instability of the electronic system towards the formation of Cooper pairs is given by the point of singularity of the vertex function, which in turn is the point at which electron-phonon interaction is switched on.

5. The Matsubara Green's function ($T \neq 0$)

In order to determine the temperature at which the pairing of the electrons occurred, the Matsubara temperature Green's function can be the most suitable technique. The cooperative phenomenon of interest is the pairing of electrons in such a way that at the Fermi surface the pair energy $\omega_{0n} = \varepsilon_{1n} + \varepsilon_{2n} = 0$, and the pair momentum is $q = p_1 + p_2 = 0$, are the singularities of the vertex function $\Gamma(\omega, q)$.These singularities are not possessed by the single-electron Green's function. The pairs now form a bosonic system and the Matsubara Green's function for the ideal Bose gas is

$$\mathcal{G}^{(0)}(i\omega_n', q) = \frac{1}{i\omega_n - \frac{q^2}{2m} + \mu} = \frac{1}{i\omega_n - \xi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 45$$

The frequency $\omega = i\omega_n = i2\pi nT$ coincide with those of the retarded Green's function $G^R(\omega, q)$ which is analytic in the upper half plane of the complex variable ω (Lifshitz and Pitayevsky, 1980) The pairing of electrons is again represented by a series of ladder diagrams whose rungs are made of wavy electron-electron interaction potential in contradistinction to electron-phonon interaction lines(25). The bare four legs interaction vertex function is given in terms of the ladder diagrams as



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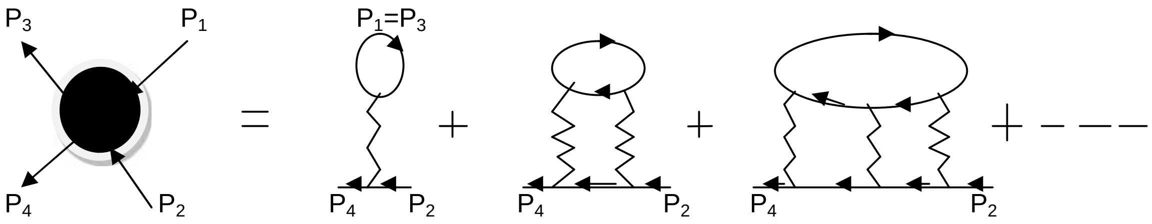
The renormalized vertex is the full four legs circle obtained by joining the end to the beginning of each upper electron line of the ladders in(46):



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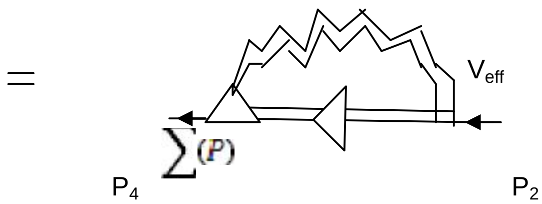
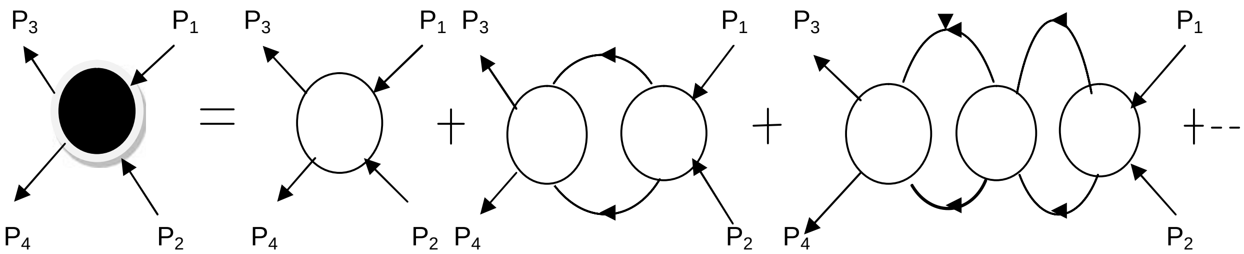


Each of the diagrams in (46) and (47) can be evaluated by using the rules outlined in Mattuck's book (Mattuck, 1967). In (47) the triangle labeled $\sum(P)$ is called the three legs self-energy insertion and it takes care of all the particle-hole(closed loops) contributions in the diagram equation(47),while the double wriggly lines represent the renormalized electron-electron interaction. The double straight line is the renormalized Green's function. If we cut off the external P_1 and P_2 lines (47) becomes an equation for the self-energy. Thus we may picture(25) as a process which aligns and brings two electrons together for pairing whereas (46) and (47) depict the actual pairs in the Cooper channel(cc) in which superconductivity occurs. From (46) and (47) the following diagram equation is obtained

The renormalized vertex is the full four legs circle obtained by joining the end to the beginning of each upper electron line of the ladders in(46):



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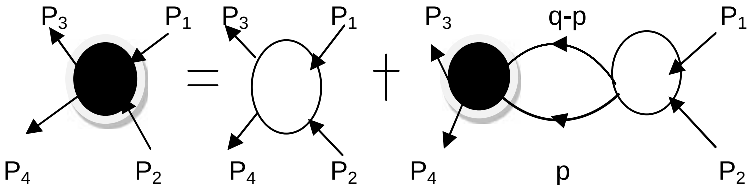
This series is summed to give the Bethe-Salpeter equation



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Here the full circle represents diagrams that cannot be cut into two parts joined together by two electrons lines such that one part contains only two ingoing external lines while the other part contains only two outgoing external lines. In short these are diagrams which represents the Peierls channel (PC). The intermediate lines, which are the Matsubara Green's functions represent the Cooper channel(CC). The corresponding algebraic equation is

$$\Gamma_{\alpha\gamma,\delta\beta}(P_3P_4,P_1P_2) = \Gamma^{(0)}_{\alpha\gamma,\delta\beta}(P_3P_4,P_1P_2) - \frac{\lambda^2}{(2\pi)^3}T\sum_{\omega_n}\int \mathcal{G}(p)\mathcal{G}(q-p)d^3p \quad \ldots\ldots\ldots\ldots \quad 50$$

The integral term in (50) gives

$$-\frac{\lambda^3}{(2\pi)^3}T\int\sum_n \frac{d^3p}{\omega^2+\left(\frac{p_F^2}{2m^*}\right)^2} = -\frac{\lambda^2 m^*}{2\pi^2}p_F ln\left(\frac{2\gamma\omega_D}{\pi T}\right) \quad \ldots\ldots\ldots\ldots \quad 51$$

Here m^* is the effective mass of the electron, ω_D is the Debye cut-off frequency and the critical temperature of superconductivity is obtained from (51) to be

$$T_c = 1.14\omega_D e^{-\frac{2\pi^2}{|\lambda|m^* p_F}} \quad \ldots\ldots\ldots\ldots \quad 52$$

6. The Gorkov interaction

The Hamiltonian of a system of interacting electrons is $H = H_0 + H_I$, where

$$H_0 = -\frac{1}{2m}\sum_\alpha\int \Psi^+_\alpha(r,t)\nabla^2 \Psi_\alpha(r,t)d^3X - \mu N \quad \ldots\ldots\ldots\ldots \quad 53$$

is the non-interacting Hamiltonian. The interaction Hamiltonian is given by

$$H_I = \frac{\lambda}{2}\sum_{\alpha\beta}\int \Psi^+_\alpha(r,t)\Psi^+_\beta(r,t)\Psi_\beta(r,t)\Psi_\alpha(r,t)\,dX \quad \ldots\ldots\ldots\ldots \quad 54$$

where λ is the interaction parameter. The Schrodinger equation for $\Psi_{\alpha'}$ is found to be

$$i\frac{\partial}{\partial t}\Psi_\alpha = -\left(\frac{\nabla^2}{2m}+\mu\right)\Psi_\alpha + \lambda\sum_\alpha \Psi^+_\delta\Psi_\delta\Psi_\alpha \quad \ldots\ldots\ldots\ldots \quad 55$$

Then the equation of motion for the causal Green's function is given by the expression

$$\left(\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - \mu\right)G_{\alpha\beta}(X_1,X_2) - i\delta_{\alpha\beta}(r_1 - r_2)\delta(t_1 - t_2) = \lambda\sum_{\alpha\beta}\langle T\Psi^+_\delta(X_1)\Psi_\delta\Psi_\alpha(X_1)(X_1)\Psi_\beta(X_2)\rangle \quad \ldots\ldots \quad 56$$

where the term on the right hand side of (56) is the Gorkov interaction(Gorkov,1958). If we apply Wick's theorem to the Gorkov interaction field operators,the result are two products of causal Green's functions and the anomalous Green's functions first obtained by Gorkov:

$$iF_{\alpha\beta}(X_1,X_2) = \langle N|T\Psi_\alpha(X_1)\Psi_\beta(X_2)|N+2\rangle \quad \ldots\ldots\ldots\ldots \quad 57$$

$$iF^+_{\alpha\beta}(X_1,X_2) = \langle N+2|\Psi^+_\alpha(X_1)\Psi^+_\beta(X_2)|N\rangle \quad \ldots\ldots\ldots\ldots \quad 58$$

An important notion that was introduced Lev Landau to explain superconductivity and superfluidity is that of condensate and the above-condensate states. As superconductivity sets in the Cooper pairs occupy the condensate and those electrons that have not formed pairs are in the above-condensate. The function $iF_{\alpha\beta}$ may be regarded as the wave function of the Cooper pairs in the condensate whereas $iF^+_{\alpha\beta}$ is the corresponding wave function of electrons in the above-condensate. In the absence of an external magnetic field contrary to Scherpelz's work (Scherpelz, et al,2012) we shall show that besides superconductivity, spin density wave (SDW) is also contained in the Gorkov interaction. For that let us put $X_1 = X_2$ or $t_1r_1 = t_2r_2$, then we have the altered Gorkov interaction in the form

$$H_{IG} = \lambda\langle\Psi^+_\delta\Psi_\delta\Psi_\alpha\Psi^+_\beta\rangle \quad \ldots\ldots\ldots\ldots \quad 59$$

According to the expansion (7), the expression (59) can be written as

$$\frac{\lambda}{V^2}\sum_{p,}\langle a^+_{p\sigma}a_{p\sigma}a_{-p,-\sigma}a^+_{-p,-\sigma}\rangle = \frac{\lambda}{2V^2}\sum_p\left(\langle a^+_{p\sigma}a_{p\sigma}a_{-p,-\sigma}a^+_{-p,-\sigma}\rangle + \langle a^+_{p\sigma}a_{p-q,-\sigma}a_{p'+q,\sigma}a^+_{p',-\sigma}\rangle\right) \quad \ldots\ldots \quad 60$$

Applying the mean field approximation to (60) we find

$$H_{IG} = -\frac{\lambda}{2V^2}\sum_p\left\{\langle a^+_{p\uparrow}a^+_{-p\downarrow}\rangle\left(a^+_{p\uparrow}a^+_{-p\downarrow} + a_{-p\downarrow}a_{p\uparrow}\right) + \langle a^+_\uparrow a_{p'-q\downarrow}\rangle\left(a^+_{p'\downarrow}a_{p+q\uparrow} + a^+_{p'+q\uparrow}a_{p\downarrow}\right)\right\}$$

$$= \Delta_{SC}\sum_p\left(a^+_{p\uparrow}a^+_{-p\downarrow} + a_{-p\downarrow}a_{p\uparrow}\right) + \Delta_{SDW}\sum_p\left(a^+_{p\downarrow}a_{p+q\uparrow} + a^+_{p+q\uparrow}a_{p\downarrow}\right) \quad \ldots\ldots\ldots\ldots \quad 61$$

where the superconducting order parameter is

$$\Delta_{SC} = -\frac{\lambda}{2V^2}\sum_p \langle a^+_{p\uparrow} a^+_{-p\downarrow}\rangle \qquad \text{………………………………………………………..} \qquad 62$$

and spin density wave order parameter is given as

$$\Delta_{SDW} = -\frac{\lambda}{2V^2}\sum_p \langle a^+_{p\uparrow} a_{p-q\downarrow}\rangle \qquad \text{……………………………………………………….} \qquad 63$$

CONCLUSION

The Green's functions studied in this paper are the single-particle causal Green's function for electrons and phonons at zero temperature, and the Matsubara and anomalous Green's functions suitable for systems at non-zero temperature. We have only discussed Green's functions as applied to fermionic systems,for which the commutator of two operators is a constant. But for spin systems (such as the high temperature superconductors) a good candidate for Green's function is the double-time Green's function first introduced by Zubarev (Zubarev, 1960). The double-time Green's function can be constructed with spin and Hubbard operators (see for example, Ovchinikov and Valkov, 2011). The BCS theory operates on the electron-phonon exchange mechanism as is well known, but the notion of the condensate and Gorkov interaction as worked out in this paper admits antiferromagnetic (AF) ordering of spins and its modulation as a precursor to electron pairing. Thus the condensate and above-condensate described by the Gorkov interaction (Gorkov, 2011) can sustain both the SDW and superconductivity although the same condensate when described by the anomalous Green's functions masks the SDW.

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