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ON THE BARDEEN-COOPER-SCHRIEFFER(BCS) HAMILTONIAN

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ABSTRACT

We present a detailed diagonalization calculation of the Bardeen-Cooper-Schrieffer (BCS) Hamiltonian via the unitary transformation method which allows for the Bogolyubov-Valatin transformation to be arrived at naturally. The energy gap equation is derived and solved, and averages of some representative thermodynamic quantities are obtained.

KEYWORDS: Hamiltonian, unitary matrix, superconductivity, interaction, ground state.

1. INTRODUCTION

Superconductivity is the flow of electric current without resistance. The phenomenon was discovered by Kammerling Onnes in Leiden in 1911 and was given a microscopic quantum explanation by Bardeen, Cooper and Schrieffer (Bardeen, Cooper, Schrieffer,1957). The BCS theory was soon found to apply to superconductors such as tin, lead, titanium and zinc besides many other elements in the periodic table. The critical temperatures(Tc) of these elemental superconductors fall below 10.0K. Superconductivity was later discovered in compounds such as Nb_3Sn,V_3Si,Nb_3Al and Nb_3Ge. However elemental and compound superconductors all have a serious drawback: their transition temperatures are well below 30K. This circumstance has proved to be a constraint in the exploitation of these superconductors in technology. Therefore there has been a great interest in finding materials that would superconduct at higher temperatures.

Theoretical microscopic explanation of superconductivity was not forthcoming before 1957, however, in 1950 Frohlich conjectured that electron-phonon interaction was the main process involved in superconductivity (Frohlich,1950). Frohlich's work generated optimism about superconductivity, enough to make Kittel write in 1953(Kittel, 1953) that a period of progress was at hand.

In 1986, IBM researchers Bednorz and Muller discovered high temperature superconductivity (HTSC) in ceramics (Bednorz, Muller, 1986). The Bednorz-Muller superconductor has Tc≅ 40K.

Soon after a still higher temperature Tc≅ 90K was found in Yttrium compound (Wu et al,1987).In 2009 a Tc=135K was seen in mercury barium copper oxide by Chu (Chu, et al,2009); the transition temperatures of recent cuprates are expected to increase further. Thus it is now possible to anticipate room temperature superconductors in the near future(Maksimov,2008).As for the theory behind the Hi-Tc, the situation is that even twenty five years after the discovery of superconductivity in the cuprates, there is still no consensus about the mechanism of electron pairing in these substances(Sadovskii,2004;Ashkanazi and Johnson,2012;Sica,et al,2012). However many reseachers favour the interpretation of experimental data(especially, Nuclear Magnetic Resonance(NMR) and Angle Resolved Photoemission Spectroscopy(ARPES)) in terms of the electron-phonon interactions(EPI) mechanism(Kuli, 2004;Wysokinski,1998).One main reason for the revival of interest in EPI is the discovery of superconductivity in MgB_2 in 2001 with an unexpectedly high Tc≅ 40K(Muranaka,Akimitsu,2011). All experimental data on MgB_2 satisfy the EPI mechanism (Kotegawa, et al, 2001). In this paper we shall study the BCS Hamiltonian, which is concretely based on the EPI.

A technically convenient way to derive the equations of superconductivity theory is by using a unitary matrix to diagonalize the BCS Hamiltonian. The unitary matrix method(UMM) allows us to obtain the Bogolyubov-Valatin transfomations as a matter of course, making the gap equation to follow naturally. By making use of the grand potential difference with respect to the normal and superconducting states, themodynamic quantities can be found.

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2. The BCS Hamiltonian

Let us write the BCS Hamiltonian in terms of the chemical potential μ and particle number N as

$$H' = H_0 - \mu N + H_I = H_0 - \mu N + \frac{1}{V}\sum_{pp'} U(p-p')a^+_{p\uparrow}a^+_{-p\downarrow}a_{-p'\downarrow}a_{p'\uparrow} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

where U(p-p′) is the pair potential, $a^+_{p\sigma}, a_{p\sigma}$ are the spin ($\sigma = \uparrow,\downarrow$) creation and destruction operators of electrons respectively and V is the volume. The kinetic part of the Hamiltonian is

$$H_0 - \mu N = \sum_{p\sigma}(\varepsilon_p - \mu)a^+_{p\sigma}a_{p\sigma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

Combining the two expressions, we have

$$H' = \frac{1}{V}\sum_{p\sigma}\xi(p)a^+_{p\sigma}a_{p\sigma} + \frac{1}{V}\sum_{pp'}U(p-p')a^+_{p\uparrow}a^+_{-p\downarrow}a_{-p'\downarrow}a_{p'\uparrow} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3$$

where $\xi(p) = \varepsilon_p - \mu$ is the quasiparticle energy. Let us transform the interaction part of the Hamiltonian (3) in the mean field approximation:

$$H_I = \frac{1}{V}\sum_{pp'}U(p-p')\langle a^+_{p\uparrow}a^+_{-p\downarrow}\rangle a_{-p'\downarrow}a_{p'\uparrow} + \frac{1}{V}\sum_{pp'}U(p-p')a^+_{p\uparrow}a^+_{-p\downarrow}\langle a_{-p'\downarrow}a_{p'\uparrow}\rangle \quad \ldots\ldots\ldots\ldots\ldots \quad 4$$

Making the notation

$$\Delta_p = \frac{1}{V}\sum_{pp'}U(p-p')\langle a_{-p\downarrow}a_{p\uparrow}\rangle \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 5$$

then

$$H_I = \sum_{pp'}\left(\Delta^*_p a_{-p'\downarrow}a_{p'\uparrow} + a^+_{p\uparrow}a^+_{-p\downarrow}\Delta_p\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 6$$

The Hamiltonian (1) takes the form,

$$H' = \sum_p \xi(p)a^+_{p\sigma}a_{p\sigma} + \sum_p \Delta^*_p a_{-p\downarrow}a_{p\uparrow} + \sum_p \Delta_p a^+_{p\uparrow}a^+_{-p\downarrow} = \sum_p \begin{pmatrix}a^+_{p\uparrow} & a_{-p\downarrow}\end{pmatrix}\begin{pmatrix}\xi_p & \Delta_p \\ \Delta^*_p & -\xi_p\end{pmatrix}\begin{pmatrix}a_{p\uparrow} \\ a^+_{-p\downarrow}\end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad 7$$

3. Diagonalization of H'

We seek a canonical transformation that will diagonalize the Hamiltonian(7);the unitary matrix method (UMM) is best suited for this programme. Consider the unitary matrix U_p and its adjoint U^+_p in the form of 2x2 matrix.

$$U_p = \begin{pmatrix}u_p & -v^*_p \\ v_p & u^*_p\end{pmatrix}, \qquad U^+_p = \tilde{U}^*_p = \begin{pmatrix}u^*_p & v^*_p \\ -v_p & u_p\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 8$$

Now for the 2x2 matrix in(7),we have(ignoring the subscripts for convenience),

$$H'_m = U^+_p\begin{pmatrix}\xi_p & \Delta_p \\ \Delta^*_p & -\xi_p\end{pmatrix}U_p = \begin{pmatrix}u^* & v^* \\ -v & u\end{pmatrix}\begin{pmatrix}\xi & \Delta \\ \Delta^* & -\xi\end{pmatrix}\begin{pmatrix}u & -v^* \\ v & u^*\end{pmatrix}$$
$$= \begin{pmatrix}(u^2 - v^2)\xi + 2uv\Delta & (u^2 - v^2)\Delta - 2uv\xi \\ (u^2 - v^2)\Delta - uv\xi & (v^2 - u^2)\xi - 2uv\Delta\end{pmatrix} = \begin{pmatrix}E & 0 \\ 0 & -E\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 9$$

In forming (9) we used the reality of u^*, v^* and Δ^*. The two equations obtained from (9)are

$$(u^2 - v^2)\xi + 2uv\Delta = E\ , \qquad (u^2 - v^2)\Delta - 2uv\xi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 10$$

These equations are solved to give: $E\xi = (\xi^2 + \Delta^2)a, \qquad E\Delta = (\xi^2 + \Delta^2)b,$ where $a = u^2 - v^2, b = 2uv.$

Therefore we have the eigenvalue $E_p = \sqrt{\xi_p^2 + \Delta_p^2}$ and as a result

$$diag\begin{pmatrix}\xi_p & \Delta_p \\ \Delta^*_p & -\xi_p\end{pmatrix} = \begin{pmatrix}\sqrt{\xi_p^2 + \Delta_p^2} & 0 \\ 0 & -\sqrt{\xi_p^2 + \Delta_p^2}\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 11$$

Now for the sake of clarity we write the diagonalization of the entire Hamiltonian(7) in the form

$$H'_{diag} = \sum_p \left(a^+_{p\downarrow} a_{-p\downarrow}\right) UU^+ \begin{pmatrix} \xi_p & \Delta_p \\ \Delta^*_p & -\xi_p \end{pmatrix} UU^+ \begin{pmatrix} a_{p\uparrow} \\ a^+_{-p\downarrow} \end{pmatrix} \quad (12)$$

Here

$$U^+ \begin{pmatrix} a_{p\uparrow} \\ a^+_{-p\downarrow} \end{pmatrix} = \begin{pmatrix} \gamma_{p\uparrow} \\ \gamma^+_{-p\downarrow} \end{pmatrix}, \qquad \left(a^+_{p\uparrow} a_{p\downarrow}\right) U = \left(\gamma^+_{p\uparrow} \gamma_{-p\downarrow}\right) \quad (13)$$

where γ^+_p, γ_p are the new creation and destruction operators that have replaced a^+_p, a_p respectively. The new operators also satisfy the commutation rule $\gamma_p \gamma^+_p + \gamma^+_p \gamma_p = 1.$ Then

$$H'_{diag} = \sum_p \left(\gamma^+_{p\uparrow} \gamma_{-p\downarrow}\right) \begin{pmatrix} E_p & 0 \\ 0 & -E_p \end{pmatrix} \begin{pmatrix} \gamma_{p\uparrow} \\ \gamma^+_{-p\downarrow} \end{pmatrix} = \sum_p E_p \gamma^+_p \gamma_p \quad (14)$$

The operators a, a^+ describe the state of the non interacting individual particles of the Fermi gas and can no more be used when the gas particles are interacting. Thus when the interaction is switched on the Fermi gas will now be described in terms of γ_p and γ^+_p (Lifshitz, Pitayevskii,1978).

4. Bogolyubov-Valatin transformation

The 'old' operators a^+ and a can be written in terms of the γ^+_p and γ_p as follows:

$$\begin{pmatrix} a_{p\uparrow} \\ a^+_{-p\downarrow} \end{pmatrix} = inv \begin{pmatrix} u^*_p & v^*_p \\ -v_p & u_p \end{pmatrix} \begin{pmatrix} \gamma_{p\uparrow} \\ \gamma^+_{-p\downarrow} \end{pmatrix} = \begin{pmatrix} u_p \gamma_{p\uparrow} + v^*_p \gamma^+_{-p\downarrow} \\ -v_p \gamma_{p\uparrow} + u^*_p \gamma^+_{-p\downarrow} \end{pmatrix} \quad (15)$$

where inv idicates the taking of the inverse of the 2x2 matrix. Therefore

$$a_{p\uparrow} = u_p \gamma_{p\uparrow} + v_p \gamma^+_{-p\downarrow}, \qquad a^+_{-p\downarrow} = u_p \gamma^+_{-p\downarrow} - v_p \gamma_{p\uparrow} \quad (16)$$

Equations (16) are the Bogolyubov- Valatin transformations. The row matrix of a^+, a can also be written in terms of γ^+, γ as follows

$$\left(a^+_{p\uparrow} \;\; a_{-p\downarrow}\right) = \left(\gamma^+_{p\uparrow} \gamma_{-p\downarrow}\right) U_p^{-1} = \left(u\gamma^+_{p\uparrow} + v\gamma_{-p\downarrow} \qquad u\gamma_{-p\downarrow} - v\gamma^+_{p\uparrow}\right) \quad (17)$$

This last expression can also be obtained directly from the Bogolyubov-Valatin transformation(16). We may now be able to calculate the mean of a pair of operators such as

$$\langle a_{p'\uparrow} a_{-p'\downarrow} \rangle = u_{p'} v_{p'} \left(-1 + 2\langle \gamma^+_{p'\uparrow} \gamma_{p'\uparrow} \rangle\right) = -\frac{u_{p'} v_{p'} \tanh E_{p'}}{2T} \quad (18)$$

From equation(10) and the eigenvalue E_p, we find

$$u_p v_p = \frac{1}{2}\left(1 - \frac{\xi^2_p}{E^2_p}\right)^{1/2}, \qquad u^2_p + v^2_p = 1 \quad (19)$$

where

$$u^2_p = \frac{1}{2}\left(1 + \frac{\xi_p}{\sqrt{\xi^2_p + \Delta^2_p}}\right), \qquad v^2_p = \frac{1}{2}\left(1 - \frac{\xi_p}{\sqrt{\xi^2_p + \Delta^2_p}}\right) \quad (20)$$

are the Bogolyubov coefficients. Substituting (18) and (19) in (5) we obtain

$$\Delta_p = \frac{1}{2V} \sum_{pp'} U(p - p') \frac{\Delta_{p'}}{E_{p'}} \tanh E_{p'} \quad (21)$$

The expression (21) is the BCS gap equation.

5. Solution of the gap equation

In eqn.(21) the summation can be replaced by an integration at the thermodynamic limit N/V as $N \to \infty, V \to \infty$. That is

$$\frac{1}{2V}\sum_{p} \to \int \frac{d^3p}{(2\pi)^3} = \int \frac{p^2 dp}{2\pi^2}\frac{d\Omega_p}{4\pi} = N(0)\int d\xi \int \frac{d\Omega_p}{4\pi} \qquad 22$$

where N(0) is the density of states at the Fermi surface. In the BCS Fermi system the summary spin of the pair is zero, therefore the gap function Δ_p is isotropic. The solid angle integration helps to write $\int U(p-p')^{d\Omega_p}/_{4\pi} = V_{eff} = g$, and the gap equation becomes

$$1 = N(0)g\int_0^{\omega_D} \frac{\frac{d\xi}{\sqrt{\xi^2+\Delta^2}}\tanh\left(\beta\sqrt{\xi^2+\Delta^2}\right)}{2} \qquad 23$$

where ω_D is the Debye cut-off frequency,and $\beta = 1/kT$, at zero temperature, $\frac{\tanh\left(\beta\sqrt{\xi^2+\Delta^2}\right)}{2} = 1$, then

$$\int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2+\Delta^2}} = sinh^{-1}\frac{\omega_D}{\Delta_0} \cong In\left(\frac{2\omega_D}{\Delta_0}\right) \qquad 24$$

Eqn.(23) will now take the following form

$$1 = N(0)g\, In\ ((2\omega_1 D)/\Delta_1 0\ ). \qquad 25$$

This yields the BCS gap function

$$\Delta_0 = 2\omega_D exp\left(-\frac{1}{N(0)g}\right), \quad \Delta_0 = \Delta(T=0) \qquad 26$$

From expression (23) the equation for the transition temperature Tc can also be determined if the integrand is represented as

$$\frac{\int_0^{\omega_D}\frac{dE}{E}\tanh E}{2kT} = \int_0^{x=\omega_D/kT}\frac{dx}{x}tanhx = In\left(\frac{\omega_D\gamma}{\pi T}\right) \qquad 27$$

here $In\gamma \approx 0.577$ is the Euler constant. Equation (23) can now be written as

$$1 = N(0)gIn\left(\frac{2\omega_D\gamma}{\pi T_c}\right) \qquad 28$$

Solving equation (28) for Tc yields the BCS formula for transition temperature (Bardeen,Cooper,Schrieffer,1957)

$$T_c = 1.14\omega_D \exp\left(-\frac{1}{N(0)g}\right) \qquad 29$$

6. The mean of some thermodynamic quantities

The analyses given above can enable averages of pertinent themodynamic quantities to be calculated(Fetter,Walecka,1971). From the eqn. (2), we write the mean of the kinetic energy as

$$u_v\gamma_{-v\downarrow})$$ 30

Let us calculate the mean potential or pairing energy by defining it as follows

$$\langle E_\downarrow pot\rangle = 1/V\,\Sigma_\downarrow(pp^\top)\ [U(pp^{\top\prime}]\ )\langle a^+_{p'_\uparrow}a^+_{-p\downarrow}\rangle\langle a_{-p\downarrow}a_{p\uparrow}\rangle$$ 31

then using the Bogolyubov-Valatin transformations(16) we get

$$\langle E_{pot}\rangle = -\frac{\sum_p \frac{\Delta_p^2}{2E_p}\tanh E_p}{2T}$$ 32

The total average energy of the Fermi system is thus equal to

$$\langle E-\mu N\rangle = \sum_p\left(\frac{\xi_p - \frac{\xi_p^2}{E_p}\tanh E_p}{2T} - \frac{\Delta_p^2}{2E}\tanh E_p\right)}{2T}$$ 33

The difference in energy between the normal and superconducting states at zero temperature is known as the condensation energy; this can be determined if it is noticed that for the normal state at T=0K, $\Delta_p = 0, E_p = \xi_p$. Thus

$$\langle E-\mu N\rangle_s - \langle E-\mu N\rangle_n = \sum_p\left(\xi_p - \frac{\xi_p^2}{E_p} - \frac{\Delta_p^2}{2E_p}\right) = \frac{VN(0)}{4\pi}\int_0^{\omega_D}\left(\xi_p - \frac{\xi_p^2}{\sqrt{\xi_p^2+\Delta_0^2}} - \frac{\Delta_p^2}{2\sqrt{\xi_p^2+\Delta_0^2}}\right)d\xi$$

$$= \frac{VN(0)}{4\pi}\frac{\Delta_0^2}{2}$$ 34

7. Calculations in terms of the grand potential

The mean value of the interaction energy(4) can be represented in the form

$$\langle H_I\rangle = \frac{1}{V}\sum_p U(p,p')\langle a^+_{p\uparrow}a^+_{-p\downarrow}\rangle\langle a_{-p'\downarrow}a_{p'\uparrow}\rangle = -\frac{\Delta_p^2}{g}$$ 35

where $\Delta_p = g\langle a_{-p'\downarrow}a_{p'\uparrow}\rangle = -g\langle a^+_{p\uparrow}a^+_{-p\downarrow}\rangle$. From statistical physics, we have for the grand potential (Abrikosov et al,1975) $\frac{d\Omega}{dg} = \frac{1}{g}\langle H_I\rangle$, and integrating this, with respect to the superconducting and normal states, we get the result (Abrikosov et al,1975)

$$\Omega_s - \Omega_n = V\int\frac{dg}{g}\langle H_I\rangle = -V\int_0^g\frac{\Delta^2}{g^2}dg = V\int_0^\Delta\frac{d\left(1/g\right)}{d\Delta}\Delta^2 d\Delta$$ 36

Taking 1/g from (23) and substituting it in (36), one obtains the expression

$$\Omega_s - \Omega_n = VN(0)\int_0^{\omega_D}d\xi\int_0^\Delta 2\frac{\Delta'}{E}\frac{\tanh E}{2T}d\Delta'$$ 37

which is integrated by parts and after a little algebra yields

$$\Omega_s - \Omega_n = -\frac{VN(0)\Delta^2}{2} - VN(0)\Delta^2 In\frac{\Delta_0}{\Delta} - \frac{4VN(0)}{\beta}\int_0^{\omega_D} In\left(1+e^{-\beta E}\right)d\xi + \frac{1}{\beta^2}\frac{VN(0)\pi^2}{3}$$ 38

According to the general principles of statistical physics(see for example, Lifshitz and Pitayevskii,1978), the expression (38) is true for all thermodynamic potentials. As a result we can write (37) for the internal energy as

$$E_s - E_n = -VN(0)\frac{\Delta^2}{2} - VN(0)\Delta^2 In\frac{\Delta_0}{\Delta} + \frac{1}{3}VN(0)\pi^2(kT)^2 - 4VN(0)kT\int_0^{\omega_D} In\left(1+e^{-\beta E}\right)d\xi \qquad 39$$

On the strength of (39) for the normal state of the electronic system, the energy and heat capacity are

$$E_n = \frac{1}{3}VN(0)\pi^2k^2T^2, \qquad \frac{C_n}{V} = \frac{2}{3}N(0)\pi^2k^2T \qquad 40$$

The integral in the last term of eqn.(39) yields $e^{-\beta\Delta}\left(\frac{2\pi\Delta}{\beta}\right)^{1/2}$, therefore when all temperature independent terms are dropped in that equation, the electronic energy and specific heat in the superconducting state are respectively

$$E_s = 2VN(0)(kT)^{1/2}(2\pi\Delta_0)^{1/2}e^{-\beta\Delta_0}, \qquad \frac{C_s}{V} = (2\pi)^{1/2}\,N(0)\left(\frac{\Delta_0}{kT}\right)^{1/2}e^{-\Delta_0/kT} \qquad 41$$

The calculations presented here are valid at low temperatures where $\Delta \gg T$, but for T close to Tc the size of the gap Δ is small and an approximate formula for the gap and Tc can be found.For this let us rewrite the gap equation (21)in the form

$$\Delta_p = g\int_0^{\omega_D}\Delta_{p'}T\sum_n\frac{1}{\omega_n^2+E_p^2}\frac{d^3p}{(2\pi)^3} = \frac{2VgN(0)}{4\pi\beta}\int_0^{\omega_D}d\xi\sum_n\frac{\Delta_{p'}}{\omega_n^2+\xi^2+\Delta^2} \qquad 42$$

Using the expansion

$$\frac{1}{(\omega_n^2+\xi^2)+\Delta^2} = \frac{1}{\omega_n^2+\xi^2} - \frac{\Delta^2}{(\omega_n^2+\xi^2)^2} + \frac{\Delta^4}{(\omega_n^2+\xi^2)^3} - \cdots \qquad 43$$

we have from (42) after some integration:

$$\frac{1}{g} = \frac{2VN(0)}{4\pi\beta}\int_0^{\omega_D}\frac{1}{2\xi}\tanh\frac{\xi}{2T}d\xi - \frac{2VN(0)}{4\pi\beta}\left(\frac{\beta}{\pi}\right)^2\Delta^2\sum_{n=0}\frac{1}{(2n+1)^2} + \frac{3}{16\pi\beta}VN(0)\Delta^4\left(\frac{\beta}{\pi}\right)^5\sum_{n=0}\frac{1}{(2n+1)^5} \qquad 44$$

In the last expression, we have used for fermions $\omega_n = \pi(2n+1)/\beta$, and now applying the prescription $\sum_{p=0}\frac{1}{(2n+1)^p} = (1-2^{-p})\zeta(p)$ to eqn.(43),and replacing its first term by eqn.(27)we obtain the result

$$In\left(\frac{2\omega_D\gamma}{\pi T_c}\right) = In\left(\frac{2\omega_D\gamma}{\pi T}\right) - \frac{7}{8}\zeta(3)\frac{\Delta^2}{(\pi T)^2} + \frac{93}{128}\zeta(5)\frac{\Delta^4}{(\pi T)^4} - \cdots \qquad 45$$

where $\zeta(p)$ is the zeta function. Then,

$$In\frac{T}{T_c} = -\frac{7}{8}\zeta(3)\frac{\Delta^2}{(\pi T)^2} + \frac{93}{128}\zeta(5)\frac{\Delta^4}{(\pi T)^4} - \cdots \qquad 46$$

Retaining the first term on the right hand side and using the expansion $In(T/T_c) = 1 - T/T_c$, we get

$$\left(1-\frac{T}{T_c}\right) = \frac{7}{8}\zeta(3)\frac{\Delta^2}{(\pi T)^2} \qquad 47$$

Therefore the equation of the gap at a temperature Tc-T<< Tc is

$$\Delta \cong \pi T_c \sqrt{\frac{8}{7\zeta(3)}} \sqrt{1-\frac{T}{T_c}} \cong 3.063 T_c \sqrt{1-\frac{T}{T_c}} \qquad 48$$

8. Relationship between $\Delta(T=0)$ and $\Delta(T)$.

Consider the general gap equation (21) which can be written in the form

$$\frac{g}{2}\int \frac{1-2n_p}{\sqrt{\xi^2+\Delta^2}} \frac{d^3p}{(2\pi)^3} = 1 \qquad 49$$

where $n_p = \left(1+e^{-E/T}\right)^{-1}$.

Equation (48) can be represented as (Lifshitz and Pitayevskii,1978)

$$-1+\frac{g}{2}\int \frac{1}{\sqrt{\xi^2+\Delta^2}} \frac{d^3p}{(2\pi)^3} = g\int \frac{n_p}{\sqrt{\xi^2+\Delta^2}} \frac{d^3p}{(2\pi)^3} \qquad 50$$

At T=0K,

$$\frac{1}{(2\pi)^3}\int \frac{d^3p}{\sqrt{\xi^2+\Delta_0^2}} = N(0)\int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2+\Delta_0^2}} = N(0)\,In\left(\frac{2\omega_D}{\Delta_0}\right) \qquad 51$$

and at $T \neq 0$,

$$\frac{1}{(2\pi)^3}\int \frac{d^3p}{\sqrt{\xi^2+\Delta^2}} = N(0)In\left(\frac{2\omega_D}{\Delta}\right) \qquad 52$$

Now substituting eqn.(25),(51),and (52) in (50) yields for the left hand side of that equation

$$-\frac{N(0)g}{2}In\frac{2\omega_D}{\Delta_0} + \frac{N(0)g}{2}In\frac{2\omega_D}{\Delta} = N(0)gIn\frac{\Delta_0}{\Delta} \qquad 53$$

The integral on the right hand side of eqn. (50) is evaluated (Lifshitz and Pitayevskii, 1978) to give $N(0)g\exp\left(-\Delta_0/T\right)\sqrt{2\pi T/\Delta_0}$. Finally eqn.(50) can be written as

$$In\frac{\Delta_0}{\Delta} = e^{-\frac{\Delta_0}{T}}\sqrt{\frac{2\pi T}{\Delta_0}} \qquad 54$$

In the usual way the left hand side of (54) is expanded to give $(\Delta_0-\Delta)/\Delta_0$ as a result of which eqn (54) becomes

$$\Delta = \Delta_0\left(1-e^{-\frac{\Delta_0}{T}}\sqrt{\frac{2\pi T}{\Delta_0}}\right) \qquad 55$$

We may write also

$$In\left(\frac{2\omega_D}{\Delta}\right) = \frac{1}{N(0)g} + In\left(\frac{\Delta_0}{\Delta}\right) \qquad 56$$

then

$$In\left(\frac{\Delta_0}{\Delta}\right) = In\left(\frac{2\omega_D}{\Delta}\right) - In\left(\frac{2\omega_D\gamma}{\pi T}\right) + \frac{\Delta^2}{(\pi T)^2}\frac{7\zeta(3)}{8} - \frac{\Delta^4}{(\pi T)^4}\frac{93}{128}\zeta(5)\ldots \qquad 57$$

From here in the first order approximation the expression for Tc in terms of Δ_0 is obtained

$$T_c = \frac{\gamma\Delta_0}{\pi} \qquad 58$$

A special property of superconductors is the $\lambda-$ shape jump in the specific heat capacity at the critical point. The jump is sensed by the thermodynamic quantities such as the grand potential, the Helmholtz free energy and the entropy. Substituting 1/g (eqn (44)) in eqn (36) yields the Helmholtz free energy difference at $T \to T_c$ as

$$F_s - F_n = -\frac{4VN(0)(T_c^2)}{7\zeta(3)}\left(1-\frac{T}{T_c}\right)^2 \quad 59$$

If note is taken that $\Omega_s - \Omega_n = \Delta\Omega$, then the entropy change of the electron gas at T_c is given as

$$S_s - S_n = -\left(\frac{\partial(\Delta\Omega)}{\partial T}\right)_{\mu=0} = -\frac{8VN(0)T_c}{7\zeta(3)}\left(1-\frac{T}{T_c}\right) \quad 60$$

The jump in specific heat capacity at $T \to T_c$ is given by

$$C_s - C_n = \Delta C_v = \frac{T}{V}\frac{\partial^2}{\partial T^2}(\Delta\Omega) = \frac{8VN(0)}{7\zeta(3)}T_c \quad 61$$

Everywhere in this work the density of states is taken at the Fermi surface, its explicit value being $N(0) = m^* p_F / 2\pi^2$, where m^* is the electron effective mass and p_F is the Fermi momentum.

CONCLUSION

The relevance of the BCS theory consists of its methodology,its physical, and rich mathematical foundation. It serves as a testing ground for statistical physics and thermodynamics of many-body systems and provides a variety of techniques for handling new physical problems(Barlas and Varma, 2012). In recent times one of such problems is finding a correct mechanism for high temperature superconductivity in the cuprates and pnictides(Choi, 2012). It is our opinion that such a mechanism when found will receive an explanation that will not radically deviate from the BCS formalism.

REFERENCES

Abrikosov, A. A., Gorkov, L. P and Dzyaloshinski, I. E., 1975. Methods of quantum field theory in statistical physics. Dover, New York.

Ashkenazi, J and Johnson, N. F., 2012. Quantum criticality stabilizes high Tc superconductivity against competing symmetry-breaking instabilities. Cond-mat/12076789.

Bardeen, J., Cooper, L. N and Schrieffer, J. R., 1957. Theory of superconductivity. Phys.Rev. 108, pp 1175-105.

Barlas, Y and Varma, C. M., 2012. Higgs bosons in d-wave superconductors.Cond-mat/12060400.

Bednorz, J. G and Muller, K. A., 1986. Possible high temperature superconductivity in the Ba-La-Cu-O system. Zeitz fur Phy.B, Condensed matter 64, 189-193.

Choi, H., 2012. Comments on d-wave pairing mechanism for cuprates high Tc superconductors: higher is different? Journ. Korean Phy.Soc. 60, (7): 978-986.

Chu, C. W., Gao, L., Chen, F., Huang, Z. G., Meng, R. L and Xue, Y .Y., 1993. Superconductivity above Above 150K in $HgBa_2Ca_2Cu_3O_{8+x}$ at high pressures. Nature 365, 323.

Fetter, A. L and Walecka, J. D., 1971. Quantum theory of many- particle systems. McGraw-Hill.

Frohlich, H., 1950. Theory of superconducting state I. The ground state at absolute zero of temperature. Phy.Rev.79, 845-856.

Kittel, C., 1953. Introduction to Solid State Physics, p 209. John Wiley.

Kotegawa, H., Ishida, K., Kitaoka, Y., Muraraka, T and Akimutsu, J., 2001. Evidence for strong coupling s-wave superconductivity in MgB_2: ^{11}B NMR study.Cond-mat/0102334.

Kuli$c^{'}$, M. L., 2004. Electron phonon interaction and strong correlations in high temperature superconductors: one cannot avoid unavoidable.Cond-mat/0404287.

Lifshitz, E. M and Pitayevskii, L. P., 1978. Statistical physics, part 2. Nauka, Moscow.

Maksimov, E. G., 2008. Room temperature superconductivity-myth or reality?Usp.Fiz.Nauk 178, 175-179.

Muranaka, T and Akimitsu, J., 2011. Superconductivity in MgB_2 . Kristallogr. 226, 385-394.

Sadovskii, M. V., 2004. Models of the pseudogap state in high-temperature superconductors Cond-mat/0408489.

Sica, G., Samson, G. H and Alexandrov, A. S., 2012. Pseudogap in high-temperature

superconductors from realistic Frohlich and Coulomb interactions. Cond-mat/12080240.

Wu, M. R., Ashburn, J. R., Torng, C. J., Hor, P. H., Meng, R. L., Gao, L., Huang, J., Wang, J. Q and Chu, C. W., 1987. Superconductivity at 93K in a new mixed phase Y-Ba-Cu-O compound system at ambient pressure. Phy.Rev.Lett 58, (9): 905-910.

Wysokinski, K. I., 1998. Strong correlations and electron- phonon interaction in superconductors. New developments in high temperature superconductivity. Proceedings of the 2nd Polish-US conference, Wroclaw and Karpacz, Poland. Eds. Klamut J, Vaal B, Kazimiersky M.