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FOURIER SERIES MODELS THROUGH TRANSFORMATION

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ABSTRACT

This study considers the application of Fourier series analysis (FSA) to seasonal time series data. The ultimate objective of the study is to construct an FSA model that can lead to reliable forecast. Specifically, the study evaluates data for the assumptions of time series analysis; applies the necessary transformation to the data and fits multiplicative and additive FSA models. In order to meet the aforementioned objectives of the study, the average monthly temperature data (1996 – 2005) collected from the National Root Crops Research Institute, Umudike are subjected to statistical analysis. The preliminary analysis of the data makes transformation necessary. As a result, the square transformation which outperforms the others is adopted. Consequently, each of the multiplicative and additive FSA models fitted to the transformed data are then subjected to a test for white noise based on spectral analysis. The result of this test shows that only the multiplicative model is adequate. Hence, it used to make forecast of the future values of the original data.

KEY WORDS: Fourier series, square transformation, multiplicative model, additive model, white noise and spectral analysis.

INTRODUCTION

Sometimes, a given series shows regular fluctuations in addition to the presence of the secular trend. These fluctuations or movements are often periodic(Ekpeyong,2005). A function is said to be periodic if it is repeats itself after a given period. Thus, if a function of t *say* $g(t)$ has a period θ, then $g(t+\theta) = g(t)$. If θ is an integer, $g(1),\ g(2),\ ...,\ g(n)$ are possible. However, when θ is not an integer, the periodic function $g(t)$ can be approximated by a number of trigonometric functions (Priestly, 1981). Periodic time series are said to be seasonal. Several time series tools have been used by analysts in modelling seasonality in a given time series data. In this regard, Omekara and Ekpeyong (2006) employed the Fourier series model in the analysis of temperature data of Uyo metropolis. The procedure for using experimental smoothing for analysing seasonal time series data is given by Delurgio (1998). Analysis of seasonal time series data can also be done using the Box and Jekins Method. This may be as a result of the availability of the statistical package (MINITAB) and model identification tools namely autocorrelation function (ACF) and partial autocorrelation (PACF) that facilitate model building(Chatfield,2004). Fourier series analysis is usually preferred to the other methods of modelling seasonal time series data described above. This is because, it involves Orthogonal Coefficient. As a result, if a coefficient is found not to be significantly different from zero, the concerned term is dropped without necessarily refitting be model(Delurgio, 1998).

In spite of the interesting feature of Fourier series analysis, not much has been done or said about its use in modelling climatic data (Dyer, 1977). Again, some analysts fit time series models without actually checking if they are adequate or not. Inadequate models may lead to unreliable forecast. Since one of the goals of time series analysis is to make good predication with a fitted model; it is often expedient to check the adequacy of any fitted model before making forecast. If a model is inadequate, it should be refitted. Model inadequacy may be as a result of violation of one or more of the assumptions necessary for the use of a model. Transformation can be applied to improve the quality of a time series data in terms of normality and constant variance (Draper etal ,1981) .

Iwueze and Akpanta (2009) outlined the procedure for applying an appropriate transformation to time series data without fitting the time series model first.

Jenkins and Watts (1968) pointed out that situations arise in practice when it is necessary to detect departures from the theoretical white noise process caused by periodic effect and in this case, a test based on spectrum is more appropriate. This calls for the use of a spectral based test for white noise while working on Fourier series analysis of seasonal time series data. Therefore, the ultimate objective of this work is to fit a Fourier series analysis model that can lead to reliable forecast. Specifically, the study:

- Evaluates the Umudike temperature (seasonal) data for the assumptions of time series models.
- Transforms the given data if necessary
- Fit time series (multiplicative and additive) models to the seasonal data.
- Check the adequacy of the fitted models.
- Forecast the future values of the Umudike temperature data using the adequate model(s).

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2.0 METHOD OF ANALYSIS FOURIER SERIES MODELS

The basic concept of Fourier series analysis used in analyzing the seasonal time series data is discussed in this section. The common form of this Fourier series model for time series decomposition is given by (Delurigio 1998) as:

$$\hat{X}_t = a + bt + \sum_{j=1}^{k}\left(a_j \cos jwt + b_j \sin wt\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where

$\hat{X}_t =$ fitted value of the series at time t

$a =$ constant used to set the level of the series

$b =$ trend estimate of the series

$a_j, b_j (j = 1,2,3,\ldots,k) =$ Fourier coefficients

$\omega = \dfrac{2\pi f}{n}$ is the Fourier frequency

$k =$ highest harmonic of ω

The coefficients $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are obtained by regressing the detrended series on $\cos wt, \sin wt, \cos 2wt, \sin 2wt, \ldots, \cos kwt, \sin kwt.$

It shall be noted that a test for each of the parameters in the trend equation is required in order to find out if there is a significant trend or not, if the trend is not significant, the grand mean may be used as an estimate of trend. Equation (1) may be referred to as the additive Fourier series analysis model.

Furthermore, Chatfield (1975) points out that the components of time series can be combined using the additive, multiplicative or mixed model. Hence, we introduce the multiplicative form of Fourier time series models. This model is of the form:

$$\hat{X}_t = (a + bt)\sum_{j=1}^{k}\left(a_j \cos jwt + b_j \sin wt\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where the meanings of $a, b, a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are as stated under equation (1).

2.1 CHOICE OF APPROPRIATE TRANSFORMATION

The need for transformation arises when one or more of the assumptions of a model is not met (Osburne, 2000). Among the common transformation are the logarithmic transformation, square transformation, square root transformation and inverse transformation.

Studies have shown that it is possible to determine if the given data require transformation before the main statistical modelling. Iwueze and Akpanta (2009) suggested that to determine the appropriate transformation to apply to the given time series data, the regression analysis of the natural logarithms of the standard deviations on the natural logarithms of their corresponding means is required. Hence, the estimated regression equation is given by:

$$\log_e \hat{\sigma}_j = \hat{\alpha} + \hat{\beta}\log_e \overline{X}_j, \; j = 1,2,3, \ldots m \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where

$m =$ the total number of years considered in the given time series data

$\hat{\sigma}_j =$ the estimated standard deviation for the jth year data

$\hat{\alpha} =$ the estimated regression parameter independent of $\overline{X}_j$

$\hat{\beta} =$ the estimated regression coeffient.

$\overline{X}_j =$ the estimated mean for the jth year data

We now consider the power transformation:

$$W_t = \left.\begin{array}{l} \log_e X_1, \text{ if } \hat{\beta} = 1 \\ X^{\left[1 - \hat{\beta}\right]}, \text{ if } \hat{\beta} \neq 1 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The transformed data become normally distributed if the relationship between the annual standard deviations and means is not significant.

The Bartlett's test for the constant variance assumption in the transformed data is also to be considered. Here, the test statistic is given by:

$$Q = \frac{2.3026\left\{ v\log\left[\sum_{j=1}^{m} \frac{v_j \hat{\sigma}_j^2}{v} \right] - \sum_{j=1}^{m} v_j \log \hat{\sigma}_j^2 \right\}}{1 + \frac{1}{3(m-1)}\left[\sum_{j=1}^{m} \frac{1}{v_j} - \frac{1}{v} \right]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

where $\hat{\sigma}_j^2$ is the estimate of the variance for the jth year data

and

$v = \sum_{j=1}^{m} v_j$ is the degree of freedom.

It shall be noted that Q approximately follows a χ^2 Distribution with m – 1 degrees of freedom.

2.2 MODEL ADEQUACY

After fitting a time series model to data, it is imperative to check if the model is adequate or not. This can be done by ascertaining if the residual is a white noise or not.

Alvarez et al (2006) considered a simple test for white noise based on spectral analysis. The test statistics is given by:

$$\delta = T \sum_{j=1}^{T/2} \left(R_j^2 - 2/T \right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

where

T = total number of observations in a series

$$R_j^2 = a_j^2 + b_j^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$a_j = 2 \sum_{t=1}^{T/2} x_t \cos\left[\frac{2\pi ft}{T} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$b_j = 2 \sum_{t=1}^{T/2} x_t \sin\left[\frac{2\pi ft}{T} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

and

x_t = a stationary (detrended) series

To find out if a series is a white noise or not, the critical valve of δ at a given significant level should be known. These values for $\alpha = 0.1,\ 0.05,\ and\ 0.01$ are contained in table 1 of Alvarez et al (2006). If δ is less than its corresponding critical value at a given level of significance α, the residual series is a white noise. Otherwise, the null hypothesis of white noise is rejected.

2.3 F ORECAST ACCURACY MEASURE

Thiel's coefficient is used to measure the forecast accuracy of the fitted model. This coefficient is given by Friedhelm(1973) as

$$U_2 = \frac{\left[\sum_{i=1}^{n} (P_i - A_i)^2 \right]^{\frac{1}{2}}}{\left[\sum_{i=1}^{n} A_i^2 \right]^{\frac{1}{2}}}$$

Where A_i and P_i are the ith actual and predicted values of the series respectively. U_2 lies between 0 and 1 inclusive i.e $0 \le U_2 \le 1$. If $U_2 = 0,$ there is a perfect prediction. When $U_2 = 1,$ there is a perfect inequality or negative proportionality between actual and predicted values (Raymond, 1975).

RESULTS AND DISCUSSION

The statistical procedure discussed in section II for Fourier analysis of seasonal time series data are applied to average monthly temperature data of Umudike (1996 – 2005). The following table contains natural logarithms of the annual standard deviations and their corresponding natural logarithms of annual means of Yt.

Table 1: Natural Logarithms of Annual Standard Deviation and Means

Ln(std)	0.42	-0.33	0.39	-0.22	-0.22	0.11	-0.99	0.11	0.22	0.12
Ln(mean)	3.28	3.29	3.31	3.29	3.32	3.39	3.29	3.30	3.31	3.31

The regression analysis of the natural logarithms of the annual standard deviations on the natural logarithms of the annual means is given by

$$\ln \hat{\sigma}_j = 4.1 - 1.21 \ln \overline{Y}_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

From equation (10), $\hat{\beta} = -1.21$

Considering the approximate value of β in the power transformation, we have:

$$X_t = Y_t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Again, when the actual value of $\hat{\beta}$ (i.e – 1.21) is considered in the transformation, we have

$$W_t = Y_t^{2.21} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

The transformed values generated by Y_t, X_t *and* W_t are subjected to regression analysis. Result of the regression analysis of $\ln\hat{\sigma}_j$ *on* $\ln\overline{Y}_j, \ln\overline{X}_j$ *and* $\ln\overline{W}_j$ are shown in the following table.

Table 2: Regression Analysis of Log_e (Standard Deviation) on Log_e (Mean) of Various Transformations

Transform	Regression	R^2	T-test for slope		Bartltlett's test	
			Df	t-value	Q	Df
Y_t	$\ln\hat{\sigma}_j = 4.1 - 1.21\ln\overline{Y}_j$	0.004	8	-0.18	19.5781	9
X_t	$\ln\hat{\sigma}_j = -2.2 + 0.27\ln\overline{X}_j$	0.001	8	0.8	20.1291	9
W_t	$\ln\hat{\sigma}_j = 1.8 + 0.42\ln\overline{W}_j$	0.002	8	0.14	20.2641	9

The transformation X_t and W_t appear to remove the relationship between In (standard deviation) and In (mean) at $\alpha = 0.05$. The relationship exists in the original data (Y_t). Again, Y_t, X_t, *and* W_t exhibit some sense of constant variance assumption. Our choice of X_t as the appropriate transformation is based on the facts that it is easier to compute $X_t = Y_t^2$ *than* $W_t = Y_t^{2.21}$. R^2 is also minimum in the transformation X_t as shown in table **2.** The least square estimates of the trend parameters are $\hat{a} = 728.27$ *and* $\hat{b} = 0.120$. We shall now consider table 4.3 for the test for significance of the parameters.

Table 3: Anova Table for Test of Significance of the Trend Parameters of X_t.

Source of Variation	Degree of freedom	Sum of squares	Mean sum Of squares	F	P
Regression	1	0.777	0.777	0.60	0.438
Error	118	151.772	1.286		
Total	119	152.549			

From table 4.3, there is no significant trend in the transformed data (X_t). Hence, the trend is estimated using the grand mean $\overline{X}_t = 735.546$. For the additive Fourier series model, the detrended series is obtained by subtracting 735.546 from the transformed data (X_t). The corresponding terms of the seasonal component of the model obtained by regressing the detrended series on the trigonometric terms $\cos wt, \sin wt, \cos 2wt, \sin 2wt, ..., \cos 6wt$ and their significance test are contained in the table below.

Table 4: Test for Significance of the Parameters of Seasonal Component of the Additive FSA Models

Predictor	Coefficient	P
Constant	-0.000	1.000
$\cos \omega t$	9.384	0.049
$\sin \omega t$	65.189	0.000
$\cos 2\omega t$	-17.280	0.000
$\sin 2\omega t$	-13.198	0.006
$\cos 3\omega t$	-8.543	0.073
$\sin 3\omega t$	-13.205	0.006
$\cos 4\omega t$	-30.14	0.524
$\sin 4\omega t$	-9.609	0.044
$\cos 5\omega t$	-0.643	0.892
$\sin 5\omega t$	-3.279	0.488
$\cos 6\omega t$	0.774	0.817

Based on table 4, terms associated with $\cos \omega t, \sin \omega t, \cos 2\omega t, \sin 2\omega t, \sin 3\omega t$ *and* $\sin 4\omega t$ are significant. Hence, the estimated Fourier series model (additive) is given by

$$\hat{X}_{1t} = 735.546 + 9.384\cos\omega t + 65.189 \sin\omega t - 13.205\sin 3\omega t - 9.60 \sin 4\omega t \quad .... \quad (13)$$

On the other hand, the detrended series corresponding to the multiplicative model is obtained by dividing X_t by the grand mean $\overline{X}_t$=735.546. A test for significance of the parameters of the seasonal component of the multiplicative FSA model is summarized in the following table.

Table 5: Test for Significance of the Parameters of the Seasonal Component of the Multiplicative FSA Model.

Predictor	Coefficient	P
Constant	1	0.000
$\cos \omega t$	0.128	0.049
$\sin \omega t$	0.089	0.000
$\cos 2\omega t$	-0.023	0.000
$\sin 2\omega t$	-0.018	0.006
$\cos 3\omega t$	0.012	0.073
$\sin 3\omega t$	-0.018	0.006
$\cos 4\omega t$	-0.004	0.524
$\sin 4\omega t$	-0.013	0.044
$\cos 5\omega t$	-0.001	0.892
$\sin 5\omega t$	-0.004	0.488
$\cos 6\omega t$	0.001	0.817

In this case, only the constant term and terms associated $\cos \omega t, \sin \omega t, \cos 2\omega t, \sin 2\omega t, \sin 3\omega t$ and $\sin 4\omega t$ are significant at $\alpha = 0.05$. Hence, the multiplicative FSA model is given by

$$\hat{X}_{2t} = 735.546(1 + 0.128\cos\omega t + 0.089\sin\omega t - 0.023\cos 2\omega t - 0.018\sin 2\omega t - 0.018\sin 3\omega t - 0.013\sin 4\omega t)...(14)$$

Furthermore, the calculated value of δ for the additive model is 2345821067. At $\alpha = 0.05$, the calculated value of δ is greater than the critical value of 292. Therefore, the residual series of the additive model is not white noise. Similarly the value of δ for the multiplicative model is 1.95 which is less than 2.9, hence the residual series of the multiplicative model is a white noise. This implies that the multiplicative model is adequate and can be used to make forecast.

The graphical representations of the actual data and their estimates for both additive and multiplicative models are shown in figures 1 and 2 respectively. It can be seen that the graph of the estimates appears to be generally closer to that of the actual data in figure 2 than in figure 1. Thus, the multiplicative model fits better to the data than its additive counterpart.

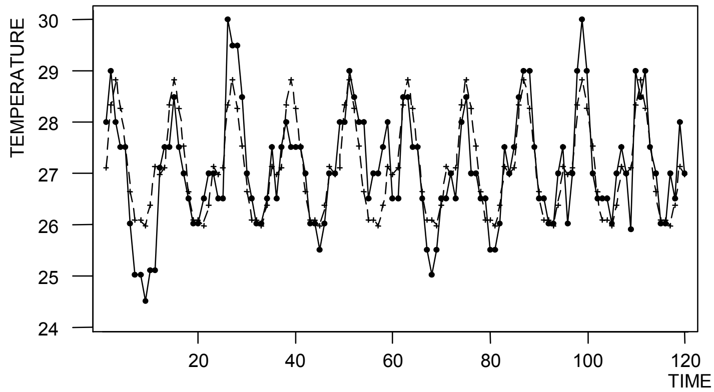

Fig. 1: Plot of the Actual Data and their Estimates for Additive Model
In figure 1 **above, the act**ual data are in solid circles while the estimated values are in pluses.

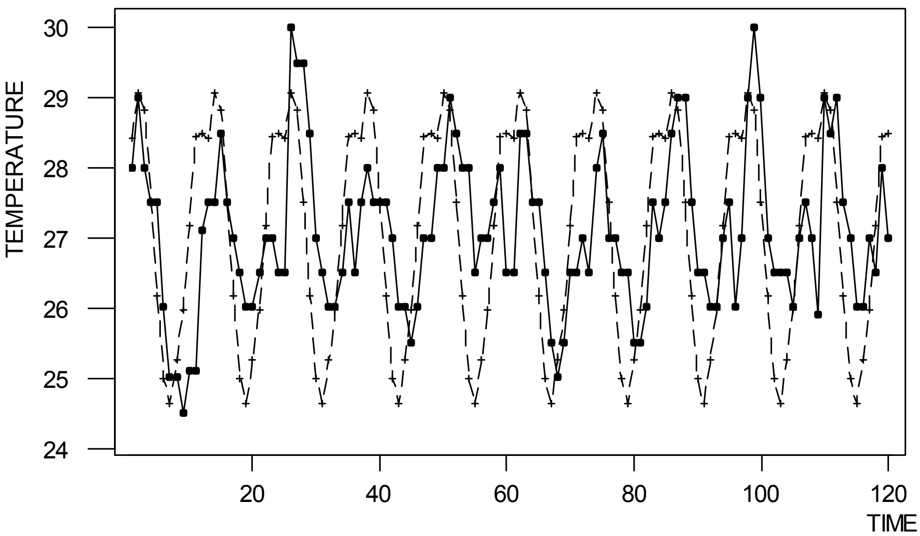

Fig. 2: Plot of Actual Data and their Estimates for Multiplicative Model
In figure 2 above, the actual data are in solid circles while the estimated values are in pluses.

3.1 FORECASTING

The forecast values of Umudike temperature for the year 2006 obtained using the model in equation 14 are contained in table 6.

Table 6: Actual and Forecast Values of Umudike Temperature for 2006

Mean Temperature	28.5	28.5	29.0	28.5	27.0	27.0	26.5	25.5	25.5	26.5	27.5	26.0
Forecast Value	28.4	29.1	28.8	27.5	26.2	25.0	24.6	25.2	26.0	27.2	28.4	28.5

Here, the Thiel's coefficient $U_2 = 0.04$ which is approximately equal to zero. Thus, the model is suitable enough to describe the series.

CONCLUSION

We have considered the basic concept of fitting Fouier series models to time series data. The used data are evaluated in order to find out if there is any need for transformation using the procedure outlined by Iwueze and Akpanta (2009). Two appears to be computationally easier than the latter. It also leads to the minimum coefficient of determination and therefore is preferred to the latter.

The conventional additive Fourier series model is fitted to the transformed series (X_t). The analysis of the residual series of the model using the Alvarez's proposed test for white noise shows that the model is inadequate.

The introduction of the multiplicative form of FSA model is in line with what is obtainable in other methods of analyzing time series data. On evaluating the residual of the series this model performs better than the additive model in this case. We therefore recommend that analysts working on Fourier series analysis of time series data should consider the nature of the given data before making a choice of model.

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APPENDIX I: Mean Temperature of Umudike Data from 1996 – 2006

	MONTH											
YEAR	JAN	FEB	MAR	APRIL	MAY	JUNE	JULY	AUG	SEPT	OCT	NOV	DEC
1996	28.0	29.0	28.0	27.5	27.5	26.0	25.0	25.0	24.5	25.1	25.1	27.1
1997	27.5	27.5	28.5	27.5	27.0	26.5	26.0	26.0	26.5	27.0	27.0	26.5
1998	26.5	30.0	29.5	29.5	28.5	27.0	36.5	26.0	26.0	26.5	27.5	26.5
1999	27.5	28.0	27.5	27.5	27.5	27.0	26.0	26.0	25.5	26.0	27.0	27.0
2000	28.0	28.0	29.0	28.5	28.0	28.0	26.5	27.0	27.0	27.5	28.0	26.5
2001	26.5	28.5	28.5	27.5	27.5	26.5	25.5	25.0	25.5	26.5	26.5	27.0
2002	26.5	28.0	28.5	27.0	27.0	26.5	26.5	25.5	25.5	26.0	27.5	27.0
2003	27.5	28.5	29.0	29.0	27.5	26.5	26.5	26.0	26.0	27.0	27.5	26.0
2004	27.0	29.0	30.0	29.0	27.0	26.5	26.5	26.5	26.0	27.0	27.5	27.0
2005	25.9	29.0	28.5	29.0	27.5	27.0	26.0	26.0	27.0	26.5	28.0	27.0
2006	28.5	28.5	29.0	28.5	27.0	27.0	26.5	25.5	25.5	26.5	27.5	26.0