GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL. 18, NO. 1&2, 2012: 75-85
COPYRIGHT© BACHUDO SCIENCE CO. LTD PRINTED IN NIGERIA ISSN 1118-0579
www.globaljournalseries.com, Email: info@globaljournalseries.com

TORSION OF BARS WITH REGULAR POLYGONAL SECTIONS

OUIGOU MICHEL ZONGO AND SIÉ KAM, ALIOUNE OUEDRAOGO

(Received 8 June 2010; Revision Accepted 14 March 2011)

ABSTRACT

In this article, the study of the torsion of cylindrical bars using large singular finite elements method leads to the resolution of the system of linear equations using MATLAB software. Particularly, the numerical solution of the problem of beams with regular section shows clearly the precision of the method depending upon the choice of different collocation points and gives in many cases, the exact solution with a relatively short computation time. The case of bars with regular polygonal section treated numerically, illustrates the precision of the method. If the number of sides is more than five, we always observe an exponential decrease in the total error with the number of coefficients preserved under field, this one reaching a minimal value for each polygon and starts increasing beyond this value. When the number of sides becomes larger, the solution tends towards the one of the circle that is known.

KEYWORDS: Torsion, collocation, singularities, large elements.

INTRODUCTION

The weak torsion of cylindrical bars leads to the resolution of an elliptic partial differential equation with homogeneous Dirichlet boundary conditions. It is supposed that the torsion occurs without any change in volume, i.e. a deformation of pure slip.

The case of polygonal bars is extremely delicate to treat numerically; the border of the studied domain constitutes a broken line with tops where the external normal is discontinuous. The solution of such a problem remains necessarily singular.

When singularities arise, the usual methods of finite elements or the finite differences give unsatisfactory results if they are used in their traditional form. But in improving these methods slightly, this allows obtaining very good results by taking account, when it is possible, of the analytical form of the solution (Fix,1969, p.645-658); (Wait, et al., 1971; p.45-52); (Emery, 1973, p.344-351); (Strang, et al, 1973); (Whiteman,1975, p.101). Since this produced good results, it can be replaced by a more efficient process, i.e. large singular finite elements method (Tolley, 1977); (Tolley, et al, 1977, p.26) used in solving equations of torsion of bars with regular polygonal using MATLAB software and going from the equilateral triangle to the regular polygon with 100,000 sides.

Method

The equations of the torsion of a thin bar of cross section Ω are written (Landau, et al, 1967):

$$\Delta u(x,y) = -1 \ (x,y) \in \Omega \quad \text{.....................} \quad (1)$$

$$u(x,y) = 0 \ (x,y) \in \partial\Omega \quad \text{.....................} \quad (2)$$

Theses above equations related are two dimensional problem and the function of constraint ***u*** thus depends only on two variables. While placing a system of axes of coordinates in the plan of the cross-sectionΩ, the only components of the tensor of the constraints different from zero are:

$$\tau_{xz} = 2G\alpha \frac{\partial u}{\partial y} \quad \text{..................................} \quad (3)$$

$$\tau_{yz} = -2G\alpha \frac{\partial u}{\partial x} \quad \text{..................................} \quad (4)$$

Where G is the modulus of rigidity, α the unit torsion angle, x and y are the Cartesian coordinates of a point of Ω and z the axis forming with x and y a direct orthogonal reference mark.

The resolution of the problem (equations (1) and (2)) does not have any difficulty as long as the contour S of the domain does not have any tops. If the domain is a polygon, the contour is a broken line and it is advisable to be extremely careful in the treatment of the singularities. The choice of the calculation algorithm is then fundamental. Large singular finite elements method is particularly appropriate in studying torsion of polygonal bars, this method comprises three steps:

Step 1: Division of the domain.

The polygonal domain Ω is decomposed into subdomains Ω_i each containing one (only one) singular point. When the domain of the cross-section is a regular polygon with N sides, there are N subdomains (Fig. 1). The subdomains are called 'large singular finite elements'. The aperture at a top is $2\alpha = (N-2)\pi / N$ in the case of a regular polygon with N sides.

Ouigou Michel Zongo, Department of Physics, UFR-SEA, University of Ouagadougou, B.P 7021, Ouagadougou 03, Burkina Faso
Sié Kam, Alioune Ouedraogo, Department of Physics, UFR-SEA, University of Ouagadougou, B.P 7021, Ouagadougou 03, Burkina Faso

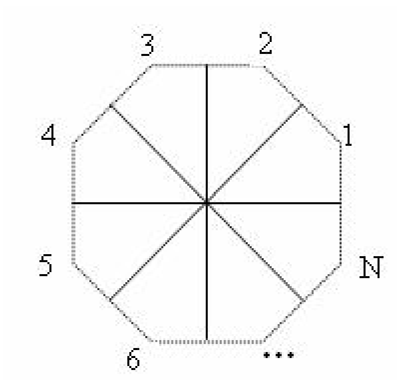

Fig. 1 Division of the domain

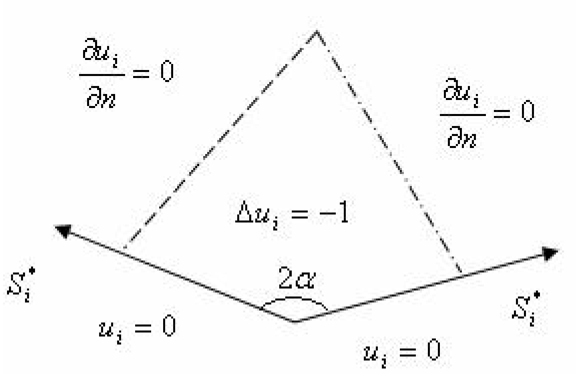

Fig. 2 Domain of the auxiliary problem

Step 2: Resolution of auxiliary problems

The auxiliary problems are identical; thus this leads to think of identical solutions in the identical subdomains. If this is the case, the calculation of a single auxiliary solution would be enough to determine the solution of the initial problem. Then, there is a total symmetry of the physical problem.

For every subdomain Ω_i^* solve:

$\Delta u_i(r_i, \theta_i) = -1$ in Ω_i^* with the boundary conditions (5)

$u_i(r_i, \theta_i) = 0$ on $S_i^* = \partial\Omega_i^*$ (6)

Where domain Ω_i^* contains Ω_i completely and where the boundary S_i^* of Ω_i^* contains completely S_i^* which is made by the half-right hand sides limiting support on the sides as figure 2 indicates it.

The solution of the auxiliary problem (5) and (6) is not fully given. Indeed, this problem is particular, because no constraint is put on u_i solution to infinite. It is thus possible to find an infinity of functions which solve the equations (5) and (6).

That is to say the point angle i of the polygon to which is linked a local polar coordinates (r_i, θ_i), (Fig. 3)

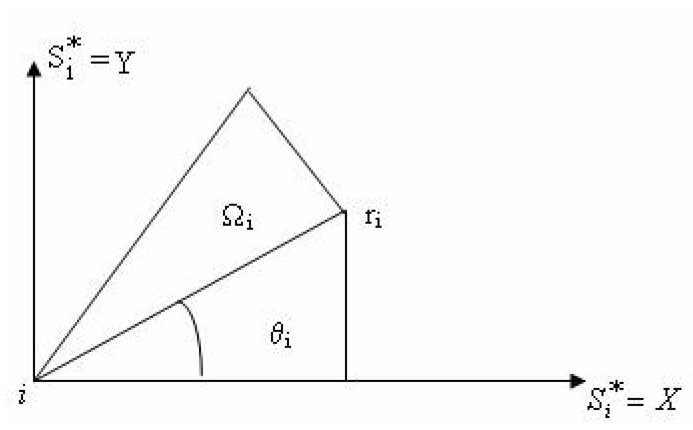

Fig. 3: Local system of polar coordinates linked to the field Ω_i

An unspecified solution of the problem (5) and (6) is written as the sum of a particular solution of the equation with second member and the solution of the homogeneous equation:

$$u_i(r_i, \theta_i) = \sum_{n=1}^{\infty} a_{in} r_i^{\frac{n\pi}{2\alpha}} \sin\left(\frac{n\pi\theta_i}{2\alpha}\right) + \frac{r_i^2}{4}[\cos(2\theta_i) + \sin(2\theta_i)\tan(2\alpha) - 1] \quad (7)$$

where $2\alpha = (N-2)\pi / N$ and N indicates the number of sides of the regular polygon and a_{in} are constants and are the unknown factors of the problem (5) and (6). In practice, it is generally impossible to find the exact analytical solution (i.e. to solve an infinite system).

This solution is valid for all regular polygons except the square where the particular solution is as follow:

$$u_p(r_i,\theta_i) = r_i^2[\lambda_{1i}\log r_i + \lambda_{2i})\sin 2\theta_i + (\lambda_{3i} + \lambda_{4i}\theta_i)\cos 2\theta_i + \lambda_{5i}] \qquad (8)$$

Step 3: Connection of auxiliary solutions

To obtain the solution of the initial problem (1) and (2), one must just make a "good choice" of constants a_{in} involved in various auxiliary problems. It is possible to show (Tolley, 1977) that the relevant choice is made by expressing the continuity of functions u_i and u_j and that of their normal derivative all along each segment of the curve Γ_{ij} (under the line separating two adjacent elements Ω_i and. Ω_j). In practice, one cannot obviously make the connection u_i and u_j but only in a limited number of points of Γ_{ij} , and generally approximate solutions are found. This procedure provides a linear algebraic system, non homogeneous for constants a_{in}. Continuity is imposed, for example, within the meaning of collocation or least squares. The use of collocation consists in imposing the continuity of the function and its normal derivative in a certain number of points located along sub-borders separating two adjacent subdomains Ω_i and Ω_j.

With regard to the method of least squares, it consists in minimizing the sum I of the following integrals defined on each sub-border Γ_{ij} separating two adjacent subdomains.

$$I_{ij} = \int_{\Gamma_{ij}} \left[(u_i - u_j)^2 + [\frac{\partial u_i}{\partial n_i} + \frac{\partial u_j}{\partial n_j})^2 \right] ds \qquad (9)$$

In this expression, s indicates the curvilinear coordinate on the sub-border Γ_{ij} , n_i and n_j respectively indicate the unit outward normal along Γ_{ij} .

The method provides the exact solution when connection is perfect in all points of sub-borders Γ_{ij} , and it is thus appropriate to assess the precision of the method by calculating an estimate of connection errors on each one of the sub-borders Γ_{ij} . An error on a sub-border can be defined as follows:

$$\eta = \sqrt{\frac{1}{L_{ij}} \int_{\Gamma_{ij}} [(u_i - u_j)^2 + (\frac{\partial u_i}{\partial n_i} + \frac{\partial u_j}{\partial n_j})] ds}$$ where L_{ji} is the length of the segment Γ_{ij} . (10)

The total error is defined as being the sum of the errors of all sub-borders Γ_{ij} balanced by the number K of sub-borders:

$$\varepsilon = \frac{1}{K}\sum_{i<j} \sqrt{\frac{1}{L_{ij}} \int_{\Gamma_{ij}} [(u_i - u_j)^2 + (\frac{\partial u_i}{\partial n_i} + \frac{\partial u_j}{\partial n_j})^2] ds} \qquad (11)$$

Application to some regular polygons

If the cross-section Ω is an unspecified polygon, one obtains the different subdomains, by lowering from the centre of the polygon, the perpendiculars to its sides. If Ω is a regular polygon with N sides, the various subdomains are identical. The auxiliary problems are then identical and the auxiliary solutions are also the same.

a) Case of the equilateral triangle

The first step of large singular finite elements method leads to three identical subdomains: the quadrilaterals Ω_1, Ω_2, Ω_3, obtained by lowering the perpendiculars to the sides, starting from the centre of the triangle.

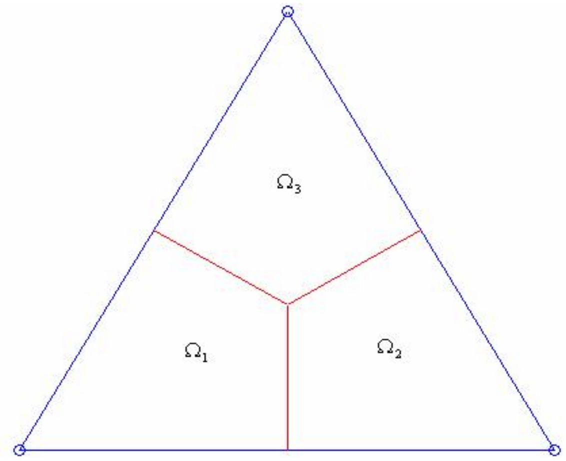

Fig. 4: Division of the triangular domain

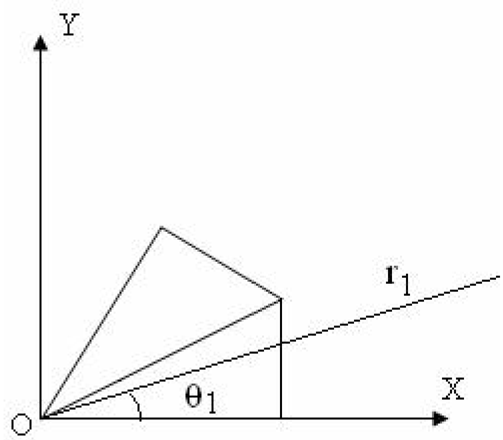

Fig 5: Local frames of reference

The three auxiliary problems (step 2) are similar. Equations for the one on subdomain Ω_1 are:

$$\Delta u_1(r_1,\theta_1) = -1 \text{ in } \Omega_1 \quad (12)$$

$$u_1(r_1,0) = 0 \quad (13\text{-a})$$

$$u_1(r_1,\pi/3 = 0 \quad (13\text{-b})$$

This auxiliary problem with Ω_1 admits solutions of type (7) and by taking properties of symmetry into account, for a given index N, coefficients a_{in} must be equal and coefficients a_{in} with odd index are different from zero and: $n = 2p-1 \quad (p = 1,\ 2,\ ...)$

$$u_1(r_1,\theta_1) = \sum_{p=1}^{N} r_1^{(6p-3)} \sin[(6p-3)\theta_1] + \frac{r^2}{4}[\cos(2\theta_1) + \sin(2\theta_1)\tan(2\alpha) - 1] \quad (14)$$

The connection of auxiliary solutions (step 3) is done by requiring that:

$$u_1 = u_2 \text{ and } \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \quad (15)$$

$$u_2 = u_3 \text{ and } \frac{\partial u_2}{\partial y} = \frac{\partial u_3}{\partial y} \quad (16)$$

$$u_1 = u_3 \text{ and } \frac{\partial u_1}{\partial y} = \frac{\partial u_3}{\partial y} \quad (17)$$

In practice, an approximate solution is obtained if the relations (15) to (17) are true in n points of each sub-border. As there are two equations to solve at each point of collocation, this allows getting a system of 6n equations, which makes it possible to find the 6n coefficients a_{in}. Since the order symmetry of the problem is 3, the study can therefore be limited to the solving of a subsidiary problem in a subdomain Ω_1. Equations (15) to (17) are reduced then to:

$$\frac{\partial u_1}{\partial x} = 0 \quad (18)$$

out of n points of collocation of the sub-border Γ_{12}. This gives a system of n equations for the unknown coefficients a_{1p} factor of approximate N order of the solution:

$$\sum_{p=1}^{N}(6p-3)a_{1p}r_1^{(6p-3)}\sin[(6p-3)\theta_1]+\frac{r\sin\theta_1\sqrt{3}}{6}=0 \qquad (19)$$

To get coefficients a_{1p}, this needs just to minimize the integral $I=\int_{\Gamma_{12}}\left(\frac{\partial u_1}{\partial x}\right)^2 dx$ defined all along the sub-border Γ_{12}.

In an equilateral triangle with unit side, there is the following relation between r_1 and θ_1 $r_1 = 1/2\cos\theta_1$ with $0<\theta_1<\pi/6$.

The method of least squares gives: $\partial I/\partial a_{1n}=0$ or

$$\sum_{p=1}^{N}\int_0^{\pi/6}[[(6p-3)r_1^{6p-3}\sin(6p-3)\theta_1+(\sqrt{3}\sin\theta_1/6)](6p-3)r_1^{(6p-3)}\sin(6n-3)\theta_1]d\theta_1 \qquad (20)$$

If: $\beta_{np}=\int_0^{\pi/2}(6p-3)(6n-3)\frac{\sin[(6p-3)\theta_1]\ \sin[(6n-3)\theta_1]}{[2\cos\theta_1]^{6(p+n)-8}}d\theta_1$

$$\gamma_n=-(\sqrt{3}/6)\int_0^{\pi/6}[6n-3]\frac{\sin[(6n-4)\theta_1]}{(2\cos\theta_1)^{6n-3}}d\theta_1$$

When varying n and p from 1 to N, there is then the linear system to solve to obtain the coefficients a_{1p}

$$\sum_{p=1}^{N}\beta_{np}a_{1p}=\gamma_n \qquad (21)$$

The solving of the system (18) gives the analytical solution; only the first term is different from zero. Therefore, the exact analytical solution of the torsion of a bar with triangular right cross-section on unit side is as the following one:

$$u_1(r_1,\theta_1)=-\frac{\sqrt{3}}{6}r_1^3\sin(3\theta_1)+\frac{r^2}{4}\left[\cos(2\theta_1)+\sqrt{3}\sin(2\theta_1)-1\right] \qquad (22)$$

b) Case of a square with unit side.

In case of a square domain, this one has symmetry of revolution of order. It is then advisable to divide Ω into four identical sub-domains: Ω_1, Ω_2, Ω_3, Ω_4 first step of the method.

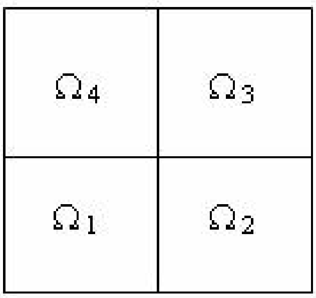

Fig. 6: Division of the square field

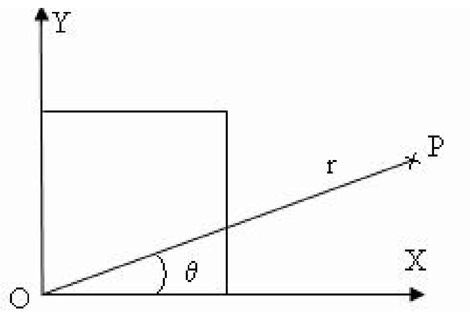

Fig. 7: auxiliary problem with Ω_1

The four auxiliary problems (step 2) are of the same type and equations concerning the subdomain Ω_1 are as follow:

$$\Delta u_1(r_1,\theta_1)=-1 \text{ in } \Omega_1 \qquad (23)$$

$$u_1(r_1,0)=0 \qquad (24\text{-a})$$

$$u_1(r_1,\pi/2)=0 \qquad (24\text{-b})$$

Such a problem admits the following solution: (Tolley, 1977, p.902-912):

$$u_1(r_1,\theta_1) = \sum_{n=1}^{\infty} a_{1n} r_1^{2n} \sin 2n\theta_1 + \frac{r_1^2}{4\pi}\left[(\pi - 4\theta_1)\cos(2\theta_1) - \pi - 4\log r_1 \sin(2\theta_1)\right] \quad (25)$$

The connection of subsidiary solutions (step 3) is done while requiring:

$$u_1 = u_2 \text{ and } \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \text{ along } \Gamma_{12} \quad (26)$$

$$u_2 = u_3 \text{ and } \frac{\partial u_2}{\partial y} = \frac{\partial u_3}{\partial y} \text{ along } \Gamma_{23} \quad (27)$$

$$u_3 = u_4 \text{ and } \frac{\partial u_3}{\partial x} = \frac{\partial u_4}{\partial x} \text{ along } \Gamma_{34} \quad (28)$$

$$u_1 = u_4 \text{ and } \frac{\partial u_1}{\partial y} = \frac{\partial u_4}{\partial y} \text{ along } \Gamma_{14} \quad (29)$$

In practice, an approached solution is obtained if relations (26) to (29) are checked in N points of each sub-border Γ_{ij}. Since there are two equations to satisfy for each point chosen, then a system of equations is obtained, making it possible to find the $8N$ coefficients a_{in}.

$(i = 1,\ 2,\ 3,\ 4\ ;\ n = 1,\ 2,\ .....,\ 2N)$.

Taking account of the properties of symmetry, one may note that coefficients a_{in} must be equal for a given index n and only coefficients a_{in} which are odd n = 2p-1 (p = 1, 2, 3,. .) are different from zero. Then, relations (26) to (29) are simply reduced to: $\frac{\partial u_1}{\partial x} = 0$ in N points of sub-border Γ_{12}. This gives a system of N equations for N unknown coefficients a_{1p} of N-like approximation:

$$u_1(r_1,\theta_1) = u_1^N(r_1,\theta_1) = \sum_{p=1}^{N} a_{1p} r_1^{4p-2} \sin[(4p-2)\theta_1] + \frac{r_1^2}{4\pi}[[(\pi - 4\theta_1)\cos(2\theta_1) - \pi - 4\log r_1 \sin(2\theta_1)] \quad (30)$$

Similar relations can be obtained for solutions in others sub-fields and the problem is entirely solved.
If the relation (30) is true all over the sub-border Γ_{12}, then the exact solution of the problem (1) and (2) could be obtained in the sub-field Ω_1, namely $u_1 = \lim u_1^N$.
In solve (30) directly i and p, varying from 1 to N, coefficients obtained present very slight errors, even negligible as for N>4 and a relatively very short time of calculation.

c) Case of polygonal bars having N sides with N superior or equal to N>5.

Step 1: of the method gives N identical subdomains Ω_i (Fig. 8)

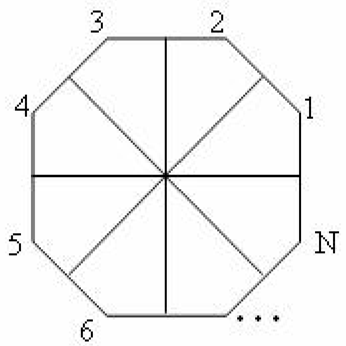

Fig. 8: Division of the polygonal domain

Auxiliary problems have been identical (step 2) and must be solved:

$$\Delta u_i(r_i,\theta_i) = -1 \text{ in the sub-field } \Omega_i \quad (31)$$

With boundary conditions

$$u_i = 0 \text{ if } \theta = 0 \ \forall r_i \quad (32)$$

$$u_i = 0 \text{ if } \theta = (N-2)\pi / N \ \forall r_i \quad (33)$$

For reasons of symmetry, the study is brought back to the half domain Ω_i

The solution of such an equation is always like in (7) i.e.

$$u_i(r_i,\theta_i) = \sum_{p=1}^{\infty} a_{in} r_i^{\frac{p\pi}{2\alpha}} \sin(\frac{p\pi\theta_i}{2\alpha}) + \frac{r_i^2}{4}[\cos(2\theta_i) + \sin(2\theta_i)\tan(2\alpha) - 1] \quad (34)$$

where $2\alpha = (N-2)\pi / N$ and N indicates the number of sides of the regular polygon.

The various coefficients a_{in} must be equal between them for a given odd index n for same the reasons as above.

The connection of auxiliary solutions (step 3) is done by requiring equality of the functions and like in their normal derivative along under borders Γ_{ij} between two contiguous fields Ω_i and Ω_j, that results in the following relations:

$$u_1 = u_2 \text{ and } \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ along } \Gamma_{12}$$

$$u_2 = u_3 \text{ and } \frac{\partial u_2}{\partial n} = \frac{\partial u_3}{\partial n} \text{ along } \Gamma_{23}$$

$$u_3 = u_4 \text{ and } \frac{\partial u_3}{\partial n} = \frac{\partial u_4}{\partial n} \text{ along } \Gamma_{34}$$

- -

$$u_1 = u_N \text{ and } \frac{\partial u_1}{\partial n} = \frac{\partial u_N}{\partial n} \text{ along } \Gamma_{1N}$$

For similar reasons as above mentioned and for N identical fields; the relations of continuity are reduced as follows (15) to (17): $\frac{\partial u_i}{\partial x} = 0$

This relation is valid in N points of the common sub-border between the first sub-fields, getting therefore back to a system of N equations for N unknown coefficients a_{in} .

The expression (14) is therefore as follows: $\sum_{p=1}^{N} \lambda_p a_{1p} r_1^{\lambda_p - 1} \sin[(\lambda_p - 1)\theta_1] = \frac{r_1 \sin\theta_1 \tan(2\alpha)}{2}$

RESULTS AND DISCUSSION

a) For a bar having an equilateral triangle as cross-section, there is only one coefficient different from zero which gives the exact solution of the problem. The layout of the curve of the total error according to the number of points of collocation on the sub-border Γ_{12} shows that it grows beyond the two points of collocation. It is therefore useless to choose a number of coefficients superior or equal to two; all the other coefficients being zero. This solution was found with a very low total error of calculation (Fig. 8) and a relatively short calculation time.

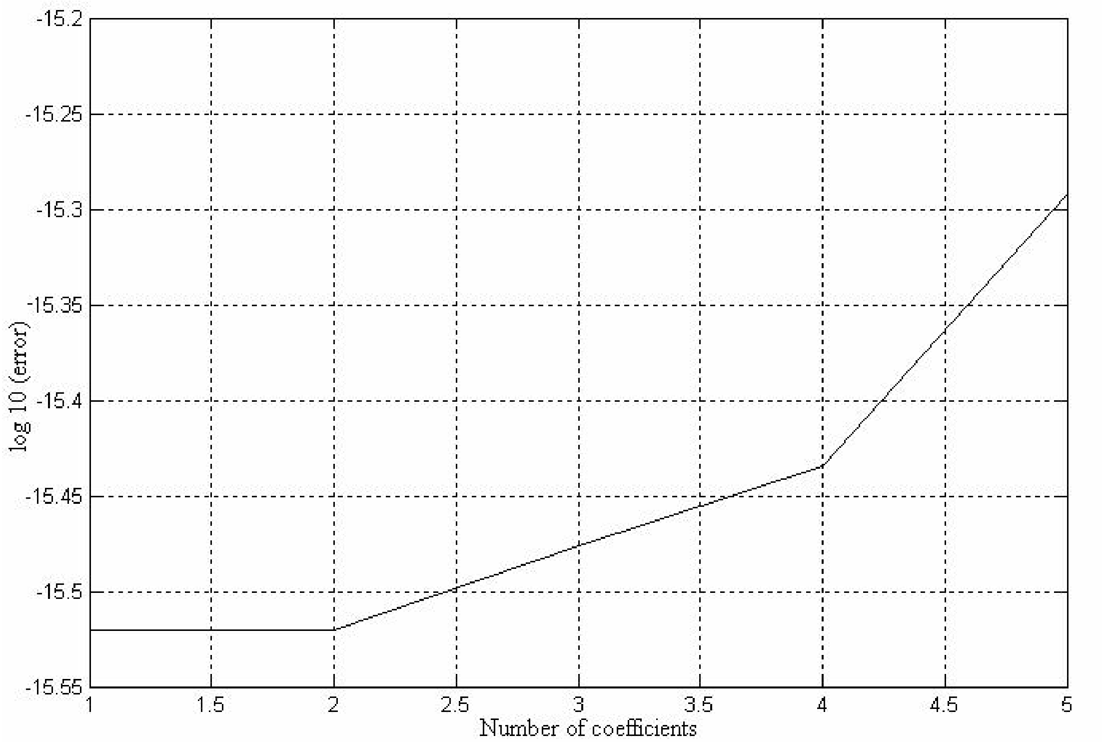

Fig. 8: Curve of total error in case of equilateral triangle

b) In the case of the square, one determined the solution by various modes of collocation. Best results are obtained in equiangular and equidistant collocations, followed by Gauss's collation and poor results are obtained using Chebyshev's collocation. The total error determined by the method of least squares decreases exponentially when the number of preserved coefficients increases, reaches a minimal value with forty parameters preserved and starts increasing beyond this value (Fig. 9).

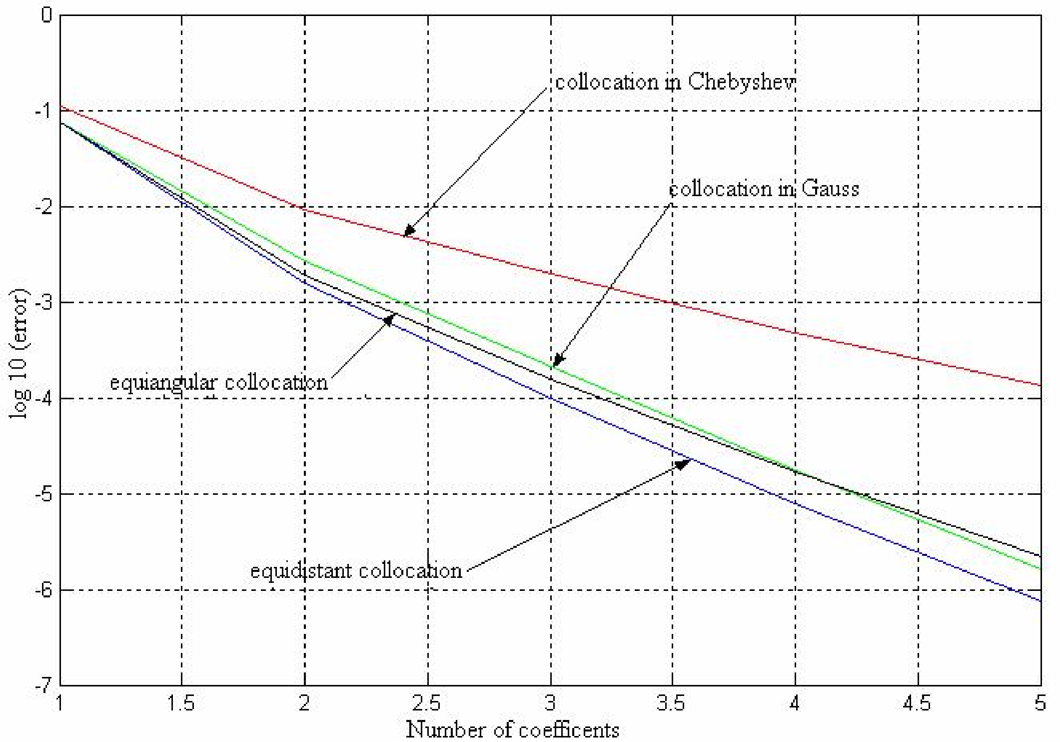

Fig. 9: Curve of total error in case of the square

c) If the number of sides of the polygonal field is more than four, equidistant collocation was used and results obtained are less good than in the previous cases. Nevertheless, it can be noted the decrease of the total error with the number of points of collocation. This error grows when the number of polygon sides increases. It decreases up to eight sides before starting increasing for a given number of points of collocation.

The continuity of the function and its normal derivatives in the meaning of least squares enabled us to note then: when the number of the polygonal section sides becomes larger, slopes of various exponentially decreasing curves of various total errors become almost parallel. (Fig.10)

- when the number of sides increases, the total error increases
- the total error decreases exponentially when the number of coefficients preserved per sub-field increases, reaches a minimal value for a number of coefficients ranging between fifty and eighty, going from pentagon to decagon (Fig. 11)
- for polygons whose sides range between twenty and one hundred thousand, the minimal total error is reached for a number of coefficients less than in the preceding cases.

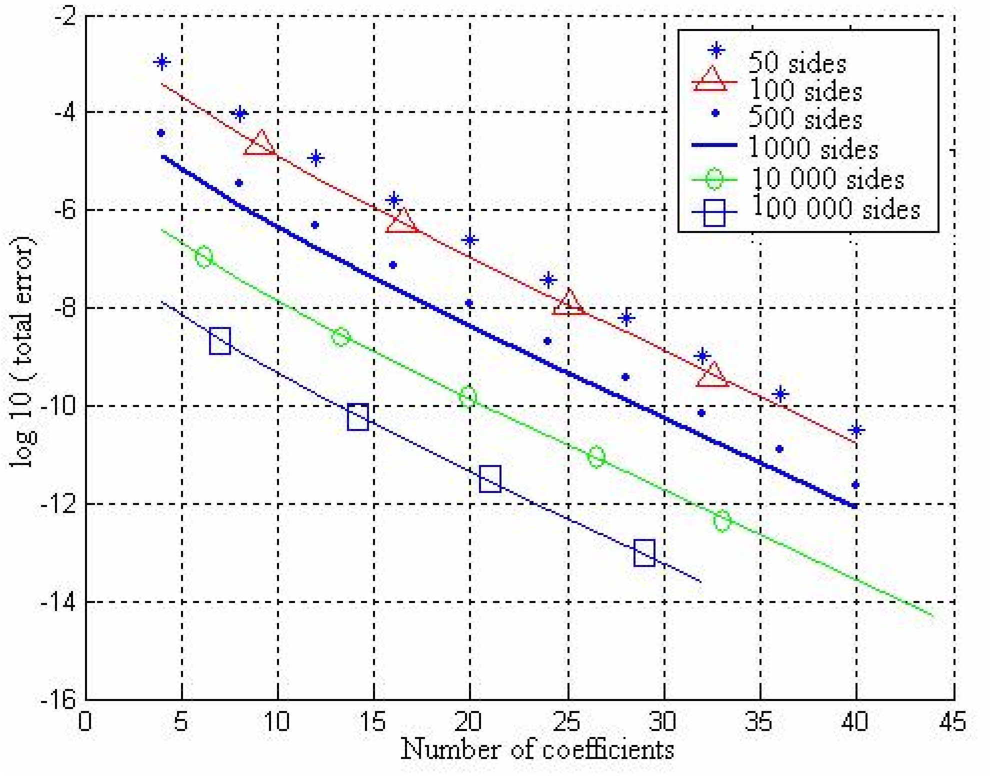

Fig.10: Curves of total error of some regular polygons

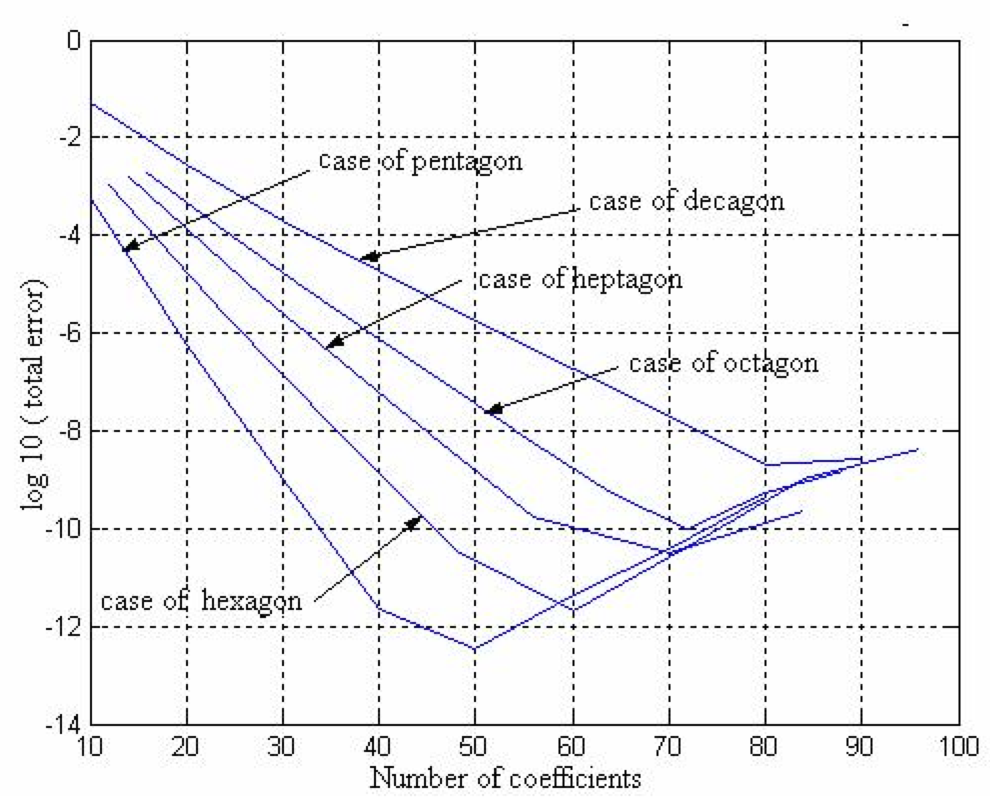

Fig.11: Curves of total error in cases of pentagon to decagon

CONCLUSION

We applied the method of large singular finite elements to the resolution of the problems of torsion of cylindrical bars of regular polygonal cross-section. This study gives satisfactory results with a very low total error.

In the case of a bar with equilateral triangular section, the exact solution is found with only one coefficient different from zero in auxiliary solutions.

For the bar with square section, one compared the solutions obtained by various modes of collocation. The best result is obtained in equiangular collocation and the result is the least using Chebishev's collocation.

If the cross-section is a regular polygon with a number of sides more than five, we always observe an exponential decrease in the total error with the number of coefficients preserved under field, this one reaching a minimal value for each polygon and starts increasing beyond this value. When the number of sides becomes important, the polygon tends towards the circle with unit radius and the solution therefore tends towards that of the torsion of a circular bar which is known.

ACKNOWLEDGEMENTS

The authors would like to thank CUD, Belgium, for the grant (CIUF, P2, Physics) given to Mr. Zongo and for the provision of MATLAB Software.

REFERENCES

Emery, A. F., 1973. The use of singularity Programming in finite-difference and finite-element computations of temperature, Trans. ASME(c), J. Heat Transfer, (95): 344-351.

Fix, G., 1969. Higher order Rayleigh-Ritz approximations, J. Math. Mech., (18): 645-658, 1969.

Landau, L and Lifshitz, E., 1967. Theory of Elasticity. Edn. Mir, Moscow.

Strang, G and Fix, G., 1973. An Analysis of the finite elements method, Prentice Hall.

Tolley, M. D., 1977. Large singular finite elements, Ph.D Thesis Faculty of Applied Sciences, ULB.

Tolley, M. D., 1977. Torsion des barres polygonales. Bull. Cl. Sc. 5ème série, LXIII (11): 902-912.

Tolley, M. D and Wajc, S. J., 1977. Advances in Computer Methods for Partial Differential Equations, II, Vichnevetsky, Edn. Rugers University. p.26

Wait, R and Mitchell, A. R., 1971. Corner singularities in elliptic problems by finite element methods, J. Comp. Phys. (8): 45-52.

Whiteman, J. R., 1975. In finite elements in fluids, 2, 101.