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MODELING OF THE CONSTRUCTION OF SEWER SYSTEMS

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ABSTRACT

The construction of sewer systems has proven to be challenging. It has been noted that the margin between safety and optimality is small and as such it is not so easy to predict the efficiency of sewer design. This paper tries to look for a model to describe the motion of water in a channel. A coupled surface runoff-water flow approach is applied to model the water transport from the catchment areas. The Saint-Venant's equations have been employed and the Lax-Friedrichs scheme has been used to numerically solve the nonlinear system of equations.

KEYWORDS: Sewer systems, surface runoff-water, The Saint-Venant's equations, Lax-Friedrichs scheme, nonlinear system of equations

1. INTRODUCTION

In many developing towns like Kumasi and Accra in Ghana, after a strong rainfall event, streets and houses get flooded (Andreini, et. al. 2000, Glowa, 2006, Ofori-Sarpong, 1985). This is due to the fact that the sewer systems in the town are not able to transport the whole amount of rainfall water. Many damages occur and a great amount of time, work and money is needed to repair all damages and clean up the resulting chaos.

The purpose of the study is to evolve the means of testing the feasibility of a sewer design, thereby helping to construct an optimal sewer system in townships. It is obvious that when designing a sewer system, a compromise between building costs and effectiveness of the sewer system must be made. The most effective sewer system will be highly expensive, whereas an ineffective system will cause additional costs (or secondary costs) to its original cost from overflow damage repair. The general goal (in economy) is to minimize overall costs. It is clear that effectiveness dictates the secondary costs and in order to judge the effectiveness of a given design a computer simulation must be performed. The purpose of this paper is therefore to provide a suitable model for sewer system designs.

2. MATHEMATICAL MODEL

Hydrology

The main source of water is rain as we consider water flow into a sewer system. Thus a closer look at rainfall characteristics is needed. Hydrology is a multidisciplinary subject that deals with the occurrences, circulation and distribution of earth's water. A *hydrological cycle* is a continuous process in which water is evaporated from the oceans, moves inland as moist air masses, and produces precipitation (rainfall). The precipitation that falls on the land surface is dispersed via several pathways. A portion of the precipitation or rainfall is returned to the atmosphere by evapotranspiration, the conversion of water to water vapour, and transpiration, the loss of water vapour through plant tissue (Andreini, et. al. 2000, Bedient and Huber, 1988, Glowa, 2006, Ayibotele, 1984). Another portion becomes overland flow or direct runoff (in forms of local streams and rivers). Finally, some water enters the soil as infiltration and may re-enter channels later as inter-flow or may go to the deeper ground water. For further explanation, see Hall, 1986, Hanh, 1996 and Hilden, 1996.

Surface Runoff

Surface runoff is that part of rainfall that is not absorbed by soil infiltration and evaporation (Bedient and Huber, 1988, Chow et. al. 1988). If the intensity of rain is less than the soil infiltration, the rain is all absorbed and there is no surface runoff. On the other hand, if the rain intensity is greater than the soil infiltration, surface runoff will occur. Drainage area usually consists of subareas or catchments of different surface characteristics, like surface slopes and roughness. Steep slopes for instance reduce time of concentration and detention volume. Time of concentration is the time for runoff from the most remote point of a catchment area to reach its outlet. Roughness increase surface storage and promotes large infiltration, thus decreasing runoff.

Method to Calculate Runoff

If rainfall of constant intensity *I* begins instantaneously and continues indefinitely, the rate of runoff will increase until the time of concentration t_c, when all areas of the watershed are contributing to the flow at the outlet. The rate of peak discharge *Q* (which occurs at time t_c) is

$$\mathrm{Q} = \mathrm{CIA}\ , \quad 0 \le C \le 1\ ,$$

where *C* is runoff coefficient and *A* is watershed area. The runoff coefficient is the least precise variable and can be thought of as the percentage of rainfall that becomes runoff. Proper selection of this coefficient requires judgment and experience on the part of the hydrologist. The proportion of the total rainfall that will reach the drains depends on the percentage

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imperviousness, slope, and ponding character of the surface. Impervious surface such as asphalt, for example, will produce nearly 100 percent runoff after the surface has become thoroughly wet, regardless of the slope.

Model for Catchments

A composite analysis is required that must account for the various surface characteristics of a drainage area consisting of subcatchments. Figure 1 shows a section of drainage area

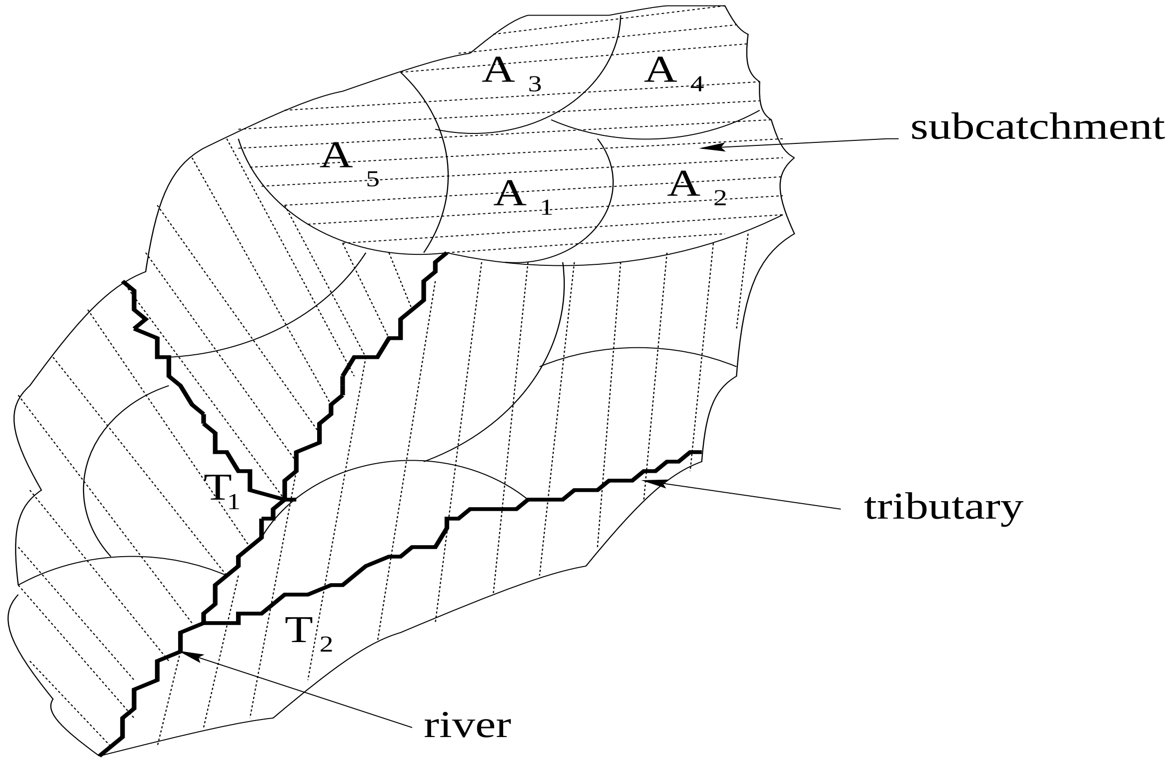

Figure 1: Example of a section of a river

with inlets T_1 and T_2. The areas of the subcatchments are denoted by A_j and the runoff coefficients of each subcatchment are denoted by C_j. The peak runoff is then computed using the well-known Darcy's Law:

$$Q(t) = \sum_{j=1}^{M} C_j A_j I_j (t - t_j),$$

where *M* = number of subcatchments drained. It is assumed that the rainfall intensity I_j vary for each subcatchment. Furthermore, a time shift t_j has been assumed for each subcatchment since the time of concentration employed for the runoff to become established and flow from the remote part of the drainage to the inflow point of the channel vary for the subcatchments. With this division of the catchment into subcatchments, it is possible to estimate the runoff within the same model framework in a realistic way.

Model for Water Flow

The flow of water in the channel is modeled via the *Saint-Venant's* equations (Hanh, 1996, Hilden, 1996, MacDonald, 1996), which are based on the conservation of mass and momentum. These equations are nonlinear hyperbolic partial differential equations (P.D.E.) and they are coupled with nonlinear algebraic cross structure equations. The equations are:

Mass: $$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q$$

Momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial z}{\partial x} = -gA(S_f + S_o)$$

Where

t = time [s]
x = longitudinal abscissa in the direction of flow [m]
z = water surface absolute elevation [m]
A = cross-sectional area [m^2]
v = average velocity in the cross-section [ms^{-1}]
Q = Av = flow discharge [$m^3 s^{-1}$]
q = lateral inflow or outflow [$m^2 s^{-1}$]
g = acceleration due to gravity [9.81 ms^{-2}]
S_o = slope of the channel
S_f = friction slope = $\eta^2 Q^2 / A^2 R^{4/3}$
η = Manning coefficient [3]
R = hydraulic radius [m] = *A/P*
P = wetted perimeter [m]

The Saint-Venant's equations are based on the following assumptions:

1. The flow is one-dimensional; depth and velocity vary only in the longitudinal direction of the channel. This implies that the velocity in a cross-section is constant and the water surface is horizontal across any section perpendicular to the longitudinal axis.
2. Flow is assumed to vary gradually along the channel so that hydrostatic pressure prevails and vertical accelerations can be neglected.
3. The longitudinal axis of the channel is approximated as a straight line.
4. The bottom slope of the channel is small and the channel bed is fixed; that is, the effects of scour and deposition are negligible.
5. Resistance coefficients for steady uniform turbulent flow are applicable so that relationships such as Manning's equation can be used to describe resistance effects.
6. The fluid is incompressible and of constant density throughout the flow.

Since there was no data available, the researcher has considered only an example. A simple structure like the rectangular channel cross-section, as in Figure 2 has been used.

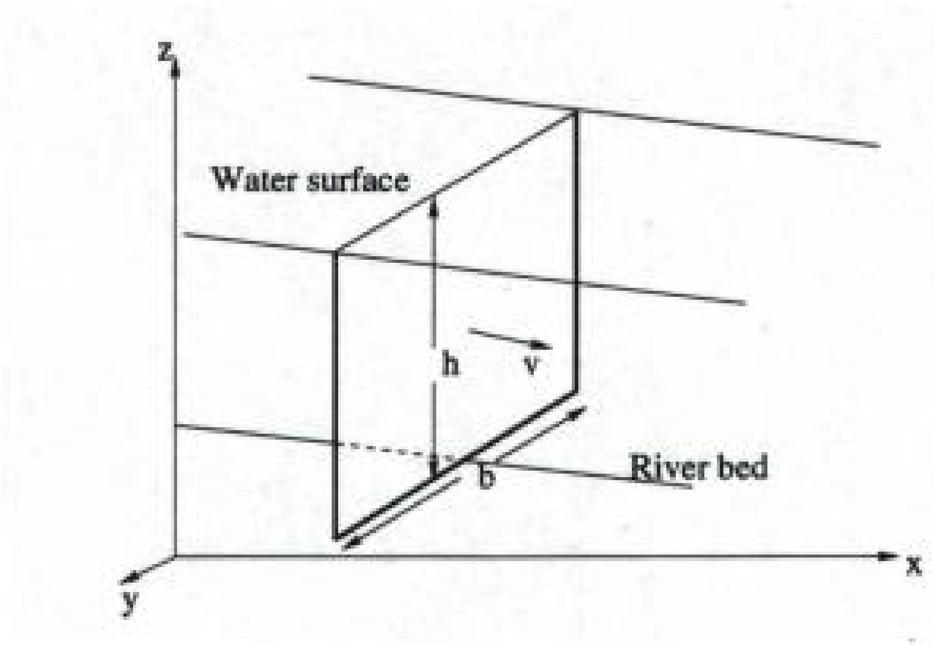

Figure 2: The one dimensional flow of water along a river of a rectangular cross-section

Now, $A = bh$, so that $Q = bhv$ and $Q^2/A = bhv^2$. In this case, the Saint-Venant's equations become

$$b\frac{\partial h}{\partial t} + b\frac{\partial}{\partial x}(hv) = q\,,$$

$$b\frac{\partial}{\partial t}(hv) + b\frac{\partial}{\partial x}\left(hv^2\right) + gbh\frac{\partial h}{\partial x} = -gbh\left(S_f + S_o\right),$$

where h is the height of the stream.

$$\Rightarrow \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv) = \frac{q}{b}\,,$$

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}\left(hv^2 + g\tfrac{h^2}{2}\right) = -gh\left(S_f + S_o\right),$$

From *Manning's* equation

$$S_f = \frac{\eta^2 Q^2}{A^2 R^{4/3}} = \frac{\eta^2 A^2 v^2}{A^2 R^{4/3}} = \frac{\eta^2 v^2}{R^{4/3}}$$

$$\Rightarrow \qquad v = \sqrt{\frac{S_f R^{4/3}}{\eta^2}}\,.$$

But $R = A/P$ with $P = b + 2h$.

Let $u = \begin{pmatrix} h \\ hv \end{pmatrix}$

and $f(u) = \begin{pmatrix} hv \\ \dfrac{(hv)^2}{h} + g\dfrac{h^2}{2} \end{pmatrix}$.

Then the system can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \begin{pmatrix} q/B \\ -gh\left(S_f + S_o\right) \end{pmatrix}.$$

The Jacobian of f is

$$J_f = \begin{pmatrix} 0 & 1 \\ gh - v^2 & 2\mathrm{v} \end{pmatrix}.$$

It's eigenvalues are

$$\lambda_1 = v - \sqrt{gh}$$
$$\lambda_2 = v + \sqrt{gh}$$

and the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ v - \sqrt{gh} \end{pmatrix}$$
$$r_2 = \begin{pmatrix} 1 \\ v + \sqrt{gh} \end{pmatrix}.$$

The diagonalised matrix is therefore of the form

$$\Lambda = \begin{pmatrix} v - \sqrt{gh} & 0 \\ 0 & \mathrm{v} + \sqrt{\mathrm{gh}} \end{pmatrix}.$$

To solve the partial differential equations, boundary conditions are needed and these are presented in Section 3.
Saint-Venant's equations have no known analytical solution in real geometry. In some simple cases, the hydraulic behavior of such system can be studied through the method of characteristics. But, for further tests on real systems, these equations have to be solved numerically. In this research, the *Lax Friedrichs scheme* (LeVeque, 1992) has been used.

3. NUMERICS

Simulation of Runoff for Catchments

To estimate the runoff for a given catchment area, the catchment area was divided into regions of subcatchments. Since there was no data available, simulations were done for the rain intensities over time and space. This was done by considering rainfall as an exponential function of the form

$$I(t) = \alpha \exp(-\beta(t-\gamma))^2,$$

where α, β and γ, are some constants. The area of each subcatchment was multiplied by the corresponding runoff coefficient. The total runoff, *Q(t)* for the catchment area was then computed using the model:

$$Q(t) = \sum_{j=1}^{M} C_j A_j I_j (t - t_j),$$

which has already been described in Subsection 2. The values for C_j and t_j were guessed since there was no data available.

The One-Dimensional Water Flow

The following is a description of the methods for solving the system of Saint-Venant's equations. The finite difference method has been chosen since the system is partial differential equations of the hyperbolic type. As already mentioned in the previous Section, the *Lax-Friedrichs scheme* (LeVeque, 1992) has been used in this work. The generalization of the Lax-Friedrichs method to nonlinear systems takes the form:

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right) + \Delta t \cdot g_j^n$$

The calculations are performed on a uniform grid placed over the *x-t* plane. The *x-t grid* is a network of points defined by taking distance increments of length Δx and time increments of duration Δt. The distance points have been denoted by the index *j* and the time points by index *n*. A *time line* is a line parallel to the *x*-axis through all the distance points at a given value of time. The finite difference equations represent the spatial and temporal derivatives in terms of the unknown variables on the current time line, *n+1*, and the preceding time line, *n*, where all the values are known from previous computations. The stencil for the Lax-Friedrichs scheme is shown in Figure 3.

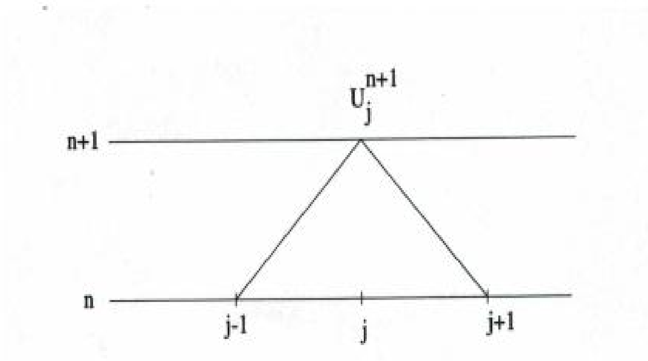

Figure 3: Stencil of Lax-Friedrichs scheme

Since the system of partial differential equations has been decoupled by the diagonalisation of the system, one approximates the spatial derivatives by the method of lines and gets a system of ordinary differential equations (O.D.E.) in time. The discretization then turns out to be an explicit Euler one, which can be solved by the use of any well-known discretization schemes for ODEs. Such ODE solvers are packages in MATLAB.

Boundary Conditions and Coupling of the Models

As mentioned in the previous Section, one needs boundary conditions to solve the system. On the left side of the channel (inflow boundary) for every time step the discharge *Q(t)* is given. Thus, the water flows with a corresponding height and speed, calculated from the Manning's formula (Chow et. al. 1988, Rentrop and Steinebach, 1997). As initial condition, the discharge *Q(0)* is assumed for every point of the river discretization. Again Manning's formula is used to calculate water levels and velocities. On the right side of the river, the water is allowed to flow out with the Manning's speed corresponding to the height at that point.

The surface runoff model and the river flow model are coupled via the boundary conditions. The computed total runoff was fed into the left boundary of the channel section. Here, the speed in Manning's equation was expressed in terms of the runoff and height. That is, from

$$v = \sqrt{\frac{S_f R^{4/3}}{\eta^2}}, \qquad \text{R = A/P}$$

and also from $A = bh$ and $Q = Av$,

$$v = \left(\frac{Q^{2/3} \sqrt{S_f}}{\eta P^{2/3}} \right)^{3/5}.$$

It was assumed that *Q* did not fall under a certain limit; hence problems with a dry channel bed are avoided.

For a given segment of the channel, the slope was taken to be negative since it was expected that the water transport is from the left to the right side. It is also possible to consider different slopes and widths. Runoffs for other catchment areas along the channel were computed and these were fed into the system at some specified points of the channel. Such inputs represent the *q/b* term in the Saint-Venant's mass equation. It should be noted here that a rectangular cross-section has been assumed throughout the channel and the direction of this lateral inflow, is orthogonal to the flow in the main channel. The height at this point of the channel needs to be computed since the *q* has now been divided by the width, *b*. This quantity is now of dimension $[ms^{-1}]$ and it is thus responsible for height changes and not for a constant cross-sectional area. This amount was divided by Δx for the section of length Δx.

4. RESULTS

It should be noted that the sewer will have an overflow when the maximum height is greater than the height of the sewer channel. As already mentioned, Saint-Venant equations and Manning equation are valid for the open channel flow. As soon as the sewer channel is full with water, these equations cannot be used anymore. There exists a method called the Preisman Slot (Hanh, 1996, Liebscher, 1993, MacDonald, 1996), which takes into account the phase of pressure flow. In this theory the sewer system will fail when the pressure becomes too high. However, this has not been applied to this system.

Several calculations and tests for different choices of parameters were carried out. Thus, it was possible to examine the effects of lateral inflows, slope, friction, cross-section area and also of initial and boundary conditions. In this section, some results of the coupled model approach are presented.

The results of the simulation of the catchment area, which was used as the inflow for the channel, can be found in Figure 5 and Figure 6. Figure 5 shows the simulation of runoff for five subcatchments of areas $2m^2$, $4m^2$, $5m^2$, $6m^2$ and $3m^2$ with runoff coefficients 0.4, 0.5, 0.4, 0.3 and 0.2 respectively. The simulation was done for a period of one hour. Figure 6 shows the total runoff for the whole catchment. The rain intensity used for this area is the one shown on Figure 4.

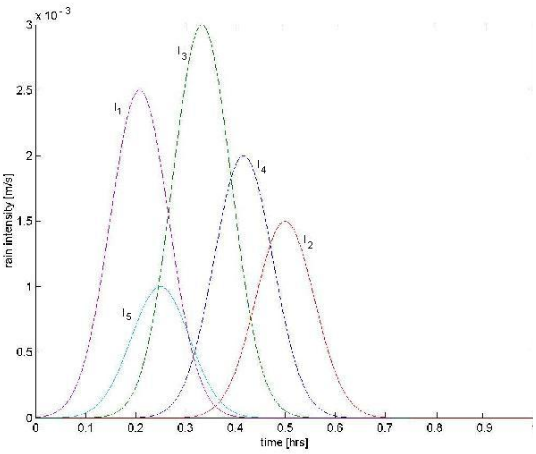

Figure 4: The rain intensity of the area which was used as the inflow for the channel

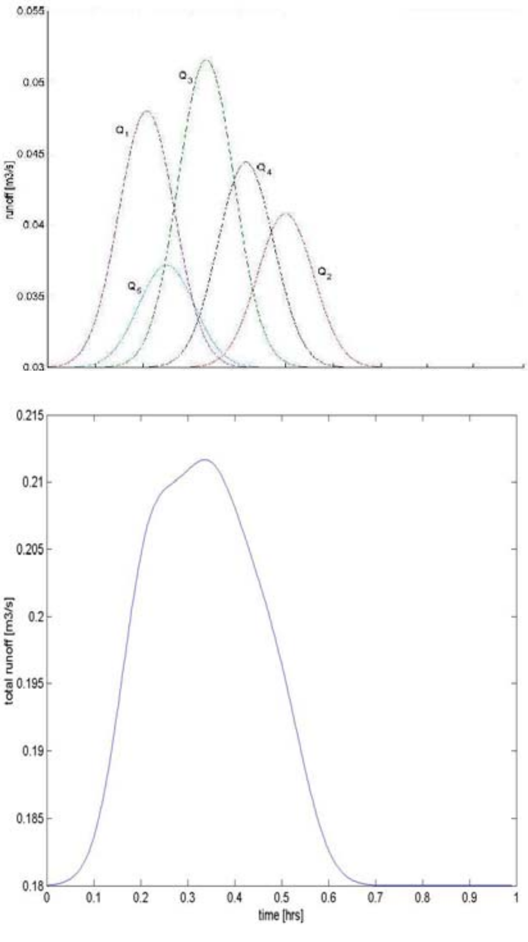

Figure 6: Result of the simulation of total runoff for the catchments as an inflow for the channel

A channel of length 300*m*, width of 0.4*m*, a slope of -0.003 and Manning's coefficient of 0.012 have been considered. The total runoff hydrograph (Figure 6) was then used as the inflow for the channel. Similar simulations but with different variables were carried out for two other areas, which were considered as lateral inflows at points x = 80*m* and x = 150*m* of the channel. Figure 7 shows the total runoff hydrograph used as lateral inflow for the first lateral inflow while Figure 8 shows the one used for the second lateral inflow.

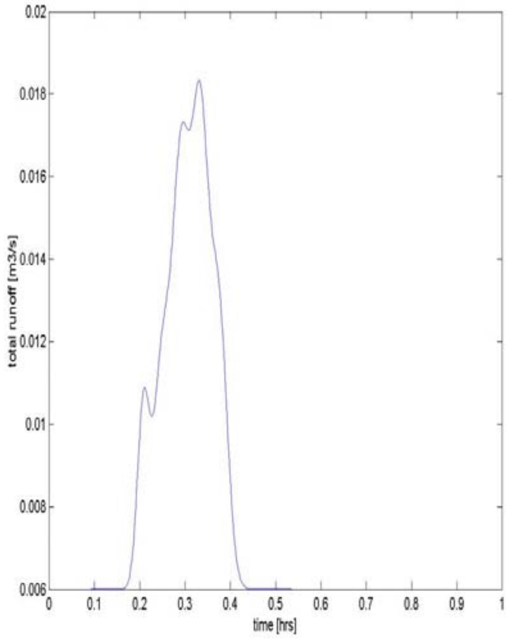

Figure 7: Result of the simulation of total runoff for the first lateral inflow

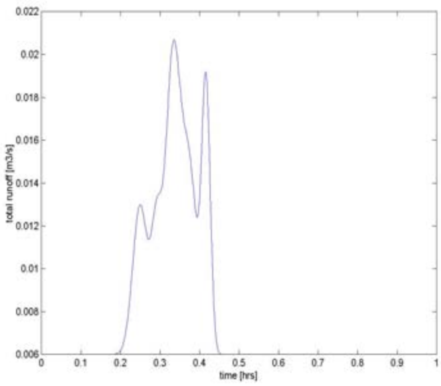

Figure 8: Result of the simulation of total runoff for the second lateral inflow

The solution gives the water height, the velocity and (by their multiplication) the discharge at every point for every time step. Some of these results are presented in the following graphs. The height and velocity as against the length of the channel after a period of two hours are shown in Figure 9 and Figure 10 respectively. The rate of water discharged, Q at the end of the channel is shown in Figure 11. Three-dimensional pictures of the height of water and the velocity of water along the channel can be found in Figure 12 and Figure 13 respectively. This shows that the model is suitable for sewer system design, since at any point on the three-dimensional pictures, one can predict the height or velocity of the water for a given length of the channel with respect to time, t. The effects of the lateral inflows at points $x = 80m$ and $x = 150m$ of the length of the channel can also be seen on the pictures and this shows that the numerical scheme used is efficient.

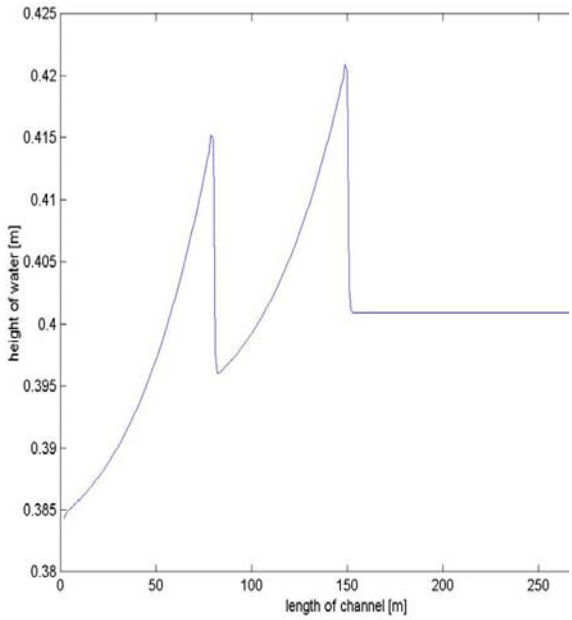

Figure 9: Height of water along the channel after a period of two hours

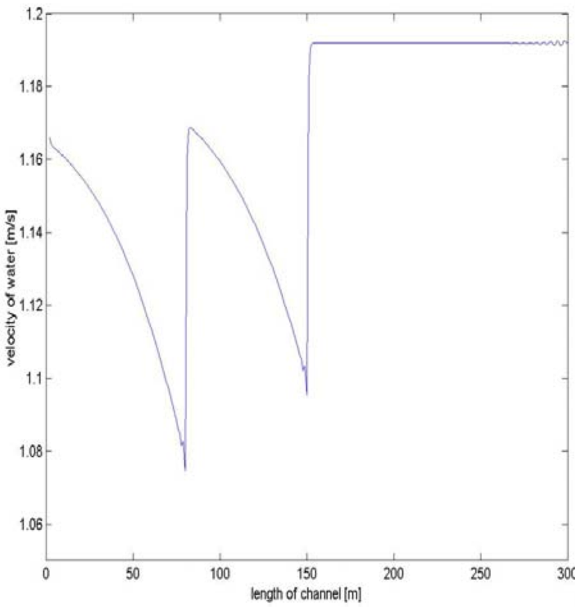

Figure 10: Velocity of water along the channel after a period of two hours

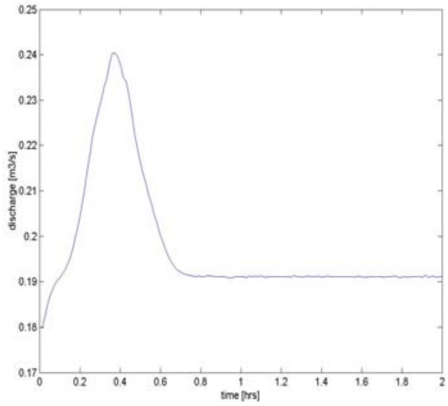

Figure 11: Rate of discharge at the end of the 300m channel

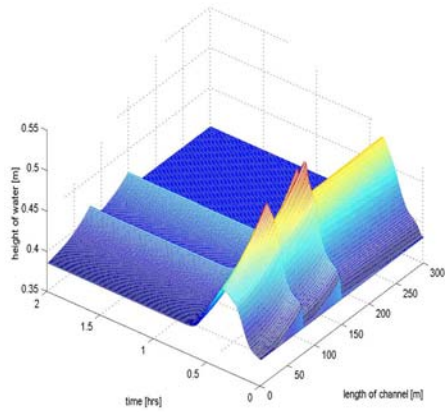

Figure 12: A 3-D picture of the height of water along the channel

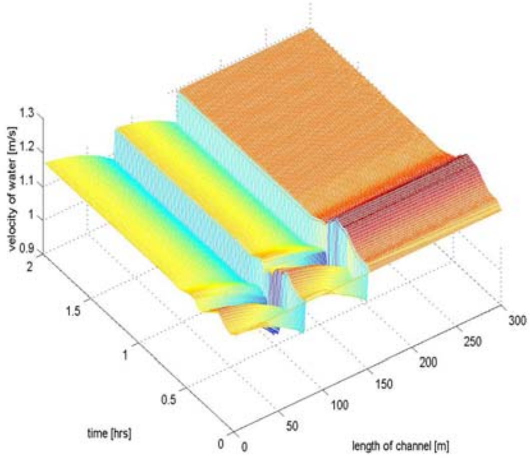

Figure 13: A 3-D picture of the velocity of water along the channel

5. CONCLUSION

The calculations from the simulation show that it is possible to simulate the water transport from big catchment areas within this coupled model approach. The assumptions, equations and the coupled process were presented and results for a fictive catchment area were given, thus, to show the performance of the approach. The programme was run for a small area but it can be used for that of a larger area. It would be interesting to have more precise data of rainfall in order to judge the system behavior in situations of really strong rain over very short time. One can also change the form of the channel from rectangular to arbitrary form of channel. From the results, one can calculate the probability of failure of the sewer system based on the probability of rainfall. In this light, the cost of failure and that of construction can be compared to get the optimal point where both of the costs can be balanced. When all these are done, then a comprehensive computational model for the sewer system will be obtained.

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