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ADDITION OF SMALLER UNIT'S EFFICIENCY IN NETWORK DATA ENVELOPMENT ANALYSIS

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ABSTRACT

In this work, we present data envelopment analysis modeling approach for network data envelopment analysis where addition of efficiency is assumed for sub units or stages. The approach is applied under both constant return to scale and variable return to scale assumptions. We looked at the general multi stage processes classified as serial, parallel and non-immediate flow processes. The overall efficiency is expressed as a weighted sum of the efficiencies of the individual stages. We observe the more general problem of an open multistage process where some output from a given stage may leave the system while others become inputs to the next stage. We finally apply our approach to the numerical example.

KEYWORDS AND PHRASES: Efficiency; Network; smaller Units.

INTRODUCTION

Data Envelopment Analysis (DEA) is a methodology that is based on the application of linear programming approach for evaluating the performance of a set of peer entities called Decision-Making Units (DMUs), which convert multiple inputs into multiple outputs. Data envelopment analysis have seen a great variety of applications in evaluating the performances of many different kinds of entities engaged in many different activities in many different contexts in many countries. From beginning DEA technique, this approach has been widely used in many real-world problems and applications [Emrouznejad and Yang, (2018); Liu et al., (2013); Peykani, Farzipoor Saen, et al., (2021)]. Network DEA models consider systems that have a network structure in which system inputs, after passing several intermediate interactions, are transformed into intermediate productions and finally leave the system as output products. However, many realworld cases do not necessarily conform to this network structure, which is related to the system outputs during multiple time periods or the same dynamic impacts. These structures cannot handle dynamic impacts. Therefore, this paper presents a novel structure that can consider the dynamic impacts and influences of sub-units on each other at various time periods.

Besides, two models based on slack variables are proposed which can consider dynamic effects and calculate the efficiency of such networks. Salehzadeh et al. (2024). According to the work of Chen et al. (2009) and Cook et al. (2010), the derivation of a radial measure of efficiency can be decomposed into a convex combination of radial measures for the individual components that make up the DMU. We note that in these two work, the weights used for individual stage's efficiency aggregation are variables, and not imposed exogenously. Chen et al. (2009) presented a methodology for representing overall radial efficiency of a DMU as an additive weighted average of the radial efficiencies of the individual stages or components that make up the DMU. While the approach of Chen et al. (2009) can be extended to DMUs that have more than two stages, such an extension requires that the multi-stage processes share the unique feature that all outputs from any stage represent the only inputs to the next stage. In other words, except for the first stage, all other stages do not have their own independent inputs (and/or outputs), that enter (exit) the process at that point. While these closed systems do exist, the more prevalent case is where each stage is open, that is it has its own inputs (and/or outputs) in addition to the intermediate measures (that exist in-between two stages).

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Such open multistage structures are relatively common, particularly in processing industries. This study starts with the approach of Chen et al (2009) where a simple two-stage network process is studied. We then present the work of Cook et al (2010) where additive efficiency decomposition approach is applied to general network structures. Examining open serial systems, we then present a model for measuring the overall radial efficiency of the general serial multi-stage process, and show that this measure can be decomposed into radial measures of efficiency for the components or stages making up the overall process. Farhadet al. (2024) used additive efficiencey decomposition forkidney allocation problem undermedical andlogistical uncertainty. Motivated by the work of Atul, et al. (2024) on paralel network envelopment analysis and Wade and Joe (2014) who presented a model for measuring the overall radial efficiency of the general serial multistage process, and show that this measure can be decomposed into radial measures of efficiency for the components or stages making up the overall process, the model is extended to structures with more complex multistage processes. Ming-miin and 1. Li-Hsueh (2014) illustrated that in order to conform to real operational situations, the construction of a DEA model should consider and match the internal operational characteristics of decision-making units. Some DEA application models based on the multi-activity frame work or network structure are provided to describe the internal structure of financial institutions and hotels. These models provide managerial insight into the sources of inefficiency within an organization.

2. preliminaries

Definition 2.1. DEA Efficiency: the performance of DMU_0 is fully (100%) efficient if and only if both

(1) $\theta^* = 1$ and

(2) all slack $S_i^{-*} = S_r^{+*} = 0$

where θ^* is the optimal efficiency score

Definition 2.2. Relative Efficiency: a DMU is said to be rated as fully (100%) efficient on the basis of available evidence if and only if the performance of other DMUs does not show that some of it input or output can be improved without worsening some of its other inputs or outputs. let $y_{rj}(r = 1,...,s)$ be the output levels secured by DMU_j and x_{ij} the levels of input (i = 1, ..., m) it uses, ∀i.

Definition 2.3. Decision Making Units (DMU) refer to any entity that is to be evaluated in terms of its ability to convert inputs to outputs.

Definition 2.4. Efficiency (Extended Pareto - Koopmans Definition): full (100%) efficiency is attained by any decision making units if and only if non of it input or output can be improved without worsening some of its other input or output.

Slack Based Measure of Efficiency: this is formulated as

$$(SBM_{in}) \min\ p_{in} = 1 - \frac{1}{n}\sum_{i=1}^{n} S_i^- / x_{io} \qquad (2.1)$$

subject to

$x_o = X\lambda + S^- \; y_o = Y\,\lambda - S^+ \lambda \geq 0, S^- \geq 0, S^+ \geq o.$

let an optimal solution of (SBM_{in}) be $(P_{in}, \lambda^*, S^{-*}, S^{+*})$ then we have the relationship $P_{in} \leq \theta^*_{CCR.}$

where, x, y are input and output respectively, s is a slack variable and λ is the weight.

Theorem 2.1. The equality $P_{in^*} = \theta_{CCR^*}$ holds if and only if the input-oriented CCR model has zero inputslacks for every optimal solution.

Remark: the strict inequality $P_{in^*} < \theta_{CCR^*}$ if and only if the CCR solution reveals an input mix efficiency.

From A Fractional To A Linear Program. We now replace the fractional program formulated as:

$$\max_{\nu,\mu} \theta = \frac{\mu_1 y_{1o} + \mu_2 y_{2o} + \cdots + \mu_s y_{so}}{\nu_1 x_{1o} + \nu_2 x_{2o} + \cdots + \nu_m x_{mo}} \qquad (2.2)$$

subject to

$$\frac{\mu_1 y_{1j} + \mu_2 y_{2j} + \cdots + \mu_s y_{sj}}{\nu_1 x_{1j} + \nu_2 x_{2j} + \cdots + \nu_m x_{mj}} \leq 1 \quad (j = 1, 2, \cdots, n)$$

$$\nu_1, \nu_2, \cdots, \nu_m \geq 0$$

$$\mu_1, \mu_2, \cdots, \mu_s \geq 0$$

where μ is the weight associated with output and v is the weight associated with input.

Model (2.2) is equivalent to the following linear programming after the CharnesCooper(1962) transformation.

$\max\theta = \mu_1 y_{1o} + \cdots + \mu_s y_{so}$ (2.3)

μ,v

subject to

$v_1 x_{1o} + \cdots + v_m x_{mo} = 1$

$\mu_1 y_{1j} + \mu_2 y_{2j} + \cdots + \mu_s y_{sj} \leq v_1 x_{1j} + v_2 x_{2j} + \cdots + v_m x_{mj}$ $(j = 1,2,\cdots,n)$ $v_1, v_2, \cdots, v_m \geq 0$

$\mu_1, \mu_2, \cdots, \mu_m \geq 0$

Theorem 2.2. The fractional program (2.2) is equivalent to linear program (2.3).

Proof. Under the nonzero assumption of v and x > 0, the denominator of the constraint of (2.2) is positive for every j, and hence we obtain (2.2) by multiplying both sides of the first constraint by the denominator. Next we note that a fractional number is invariant under multiplication of both numerator and denominator by the same nonzero number. After making this multiplication, we set the denominator of (2.2) equal to 1, more of it to a constraint , as is done on the first constraint of (2.3)and maximize the numerator, resulting in (LP). Let an optimal solution of LP be ($v = v^*, \mu = \mu^*$) and the optimal objective value θ^*. The solution ($v = v^*, \mu = \mu^*$) is also optimal for FP, since the above transformation is reversible under the assumption above. FP and LP therefor have the same optimal objective value θ^*. Wade and Joe (2014)

Also, the measures of efficiencies presented are unit invariant, i. e., they are independent of the unit of measurement used in the sense that multiplication of each output by a constant $\alpha_i > 0$, i = 1,2,⋯ ,m and each input by a constant $\beta_r > 0$,r = 1,2,⋯ ,s does not change the result. Precisely, we have

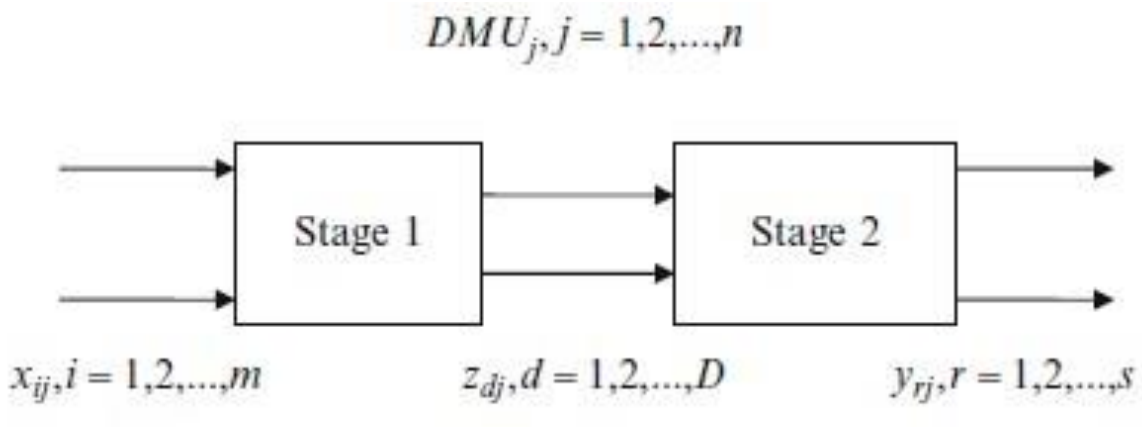
(units Invariant) Max $\theta = \theta^*$ in equation (2.2) and (2.3) are independent of the unit in which the input and output are measured provided these units are the same for every decision making units.

MAIN RESULTS

ADDITION OF SMALLER PART'S EFFICIENCY IN NETWORK PROCESSES

In data envelopment analysis original settings, only the inputs supplied to the system and the outputs produced from it are considered, neglecting the operations and interrelations of the processes within the system. The system is thus called a black-box system, and the associated model a black-box one.

Usually a production system is composed of several interrelated processes. When the internal structure is considered, one faces a network system. Comparing the black-box system with the network system, it is noted that, for each DMU j, the sum of the exogenous inputs of all processes is equal to the inputs of the system, and the sum of the exogenous outputs of all processes is equal to the outputs of the system. Moreover, the sum of the intermediate products used by all processes is equal to the sum of the intermediate products produced by all processes; that is, all intermediate products are produced and consumed within the system. Several models for measuring the efficiency of a network system have been developed.



ADDITION OF TWO STAGE UNDER CONSTANT RETURN TO SCALE. Suppose we have n DMUs, and that each DMU_j(j = 1,2,...,n) has m inputs to the first stage, x_{ij}(i = 1,2,...,m), and D outputs from this stage, z_{dj},(d = 1,2,...,D). These D outputs then become the inputs to the second stage, and are referred to as intermediate measures. The outputs from the second stage are denoted y_{rj},(r = 1,2,...,s) and γ is the weight associated with intermediate output. Based upon the Constant return to scale model (Charnes et al. 1978), the (Constant return to scale efficiency scores for DMU_{jo} in the first and second stages can be calculated in the following two Constant return to scale equations (3.1) and (3.2), respectively:

$$\theta_{jo}^1 = max \frac{\sum_{d=1}^{D} \gamma_d^A z_{dj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o}} \quad (3.1)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d^A z_{dj}}{\sum_{i=1}^{m} \nu_i x_{ij_o}} \leq 1 \quad j = 1, 2, ..., n$$

Figure 1. Two stage process

$\gamma dA, vi \geq 0$

$$\theta_j^2 = max \frac{\sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{d=1}^{D} \gamma_d^B z_{dj_o}} \qquad (3.2)$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} \gamma_i z_{dj}} \leq 1 \quad j = 1, 2, ..., n$$

$$\gamma_d^B, u_i \geq 0$$

The overall CRS efficiency score can be calculated from the following CRS equation below

$$max \frac{\sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o}} \qquad (3.3)$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1 \quad j = 1, 2, ..., n$$

$$u_r, \nu_i \geq 0$$

D

Given the inputs to the first stage x_{ij}, that stage yields the optimal intermediate measure $^{P}\gamma_i z_{dj}$ which is d=1 then used as the (aggregated) input in the second stage. Thus, it is assumed that $\gamma_d{}^A = \gamma_d{}^B = \gamma_d$ and the overall efficiency of a DMU is given by:

$$\theta_{jo} = max \frac{\sum_{d=1}^{D} \gamma_d z_{dj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o}} . \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} \gamma_d z_{dj}} = max \frac{\sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o}} \qquad (3.4)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj}}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1 \quad j = 1, 2, ..., n$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} \gamma_d z_{dj}} \leq 1, \quad j = 1, 2, ..., n$$

$$\gamma_d, \nu_i, u_r \geq 0$$

The objective function of equation (3.4) is that the overall efficiency is the product of the efficiencies of the two stages, i.e.,

$$\theta_{j_o}^1 . \theta_{j_o}^2 = \frac{\sum_{r=1}^{s} u_o^* y_{rjO}}{\sum_{i=1}^{m} \nu_o^* x_{ijO}} = \theta_{jO}$$

The key and rational assumption is that the value of the outputs from the first stage is reasonably the value when they assume the additional role as inputs to the second stage, that is $\gamma_d{}^A = \gamma_d{}^B$. Without this assumption,

D D

(1) equation (3.4) becomes a non linear program, as the terms $^{P}\gamma_d{}^A z_{do}$ and $^{P}\gamma_d{}^B z_{do}$ cannot be canceled

d=1 d=1

in the objective function.

(2) solving equation (3.4) is equivalent to applying the constant return to scale model to stages 1 and 2 independently, and then taking the geometric mean of the two CCR efficiency scores.

For the purpose of general two stage modeling processes, and specifically to allow for variables return to scale settings, rather than combine the stages in a multiplicative (geometric) manner as in Kao and hwang (2008) and liang et al (2008), we use a weighted additive (arithmetic mean) approach.

The multiplicative and additive models are two different but equally valid ways of aggregating the components of a two stage process. Thus, overall efficiency of two stage process is

$$w_1.\frac{\sum_{d=1}^{D}\gamma_d z_{dj_o}}{\sum_{i=1}^{m}\nu_i x_{ij_o}} + w_2.\frac{\sum_{r=1}^{s}u_r y_{rj_o}}{\sum_{d=1}^{D}\gamma_d z_{dj_o}}, \quad (3.5)$$

where w_1 and w_2 are user-specified weights such that $w_1 + w_2 = 1$. These weights are not optimization variables, but rather are functions of the optimization variables. Thus, overall efficiency of the process is by solving the following problem:

$$max \left(w_1.\frac{\sum_{d=1}^{D}\gamma_d z_{dj_o}}{\sum_{i=1}^{m}\nu_i x_{ij_o}} + w_2.\frac{\sum_{r=1}^{s}u_r y_{rj_o}}{\sum_{d=1}^{D}\gamma_d z_{dj_o}} \right) \quad (3.6)$$

subject to

$$\frac{\sum_{d=1}^{D}\gamma_d z_{dj}}{\sum_{i=1}^{m}\nu_i x_{ij}} \leq 1$$

$$\frac{\sum_{r=1}^{s}u_r y_{rj}}{\sum_{d=1}^{D}\gamma_d z_{dj}} \leq 1$$

$$\gamma_d, u_r, \nu_i \geq 0, \quad j = (1, 2, ..., n).$$

It is obvious that equation (3.6) cannot be turned into a linear program using the usual Charnes and Cooper (1962) transformation. For instance, if we let

$t_1 = \frac{1}{\sum_{r=1}^{s}\nu_i y_{ij_o}}, t_2 = \frac{1}{\sum_{d=1}^{D}\gamma_d z_{dj_o}}$, and set $\pi_d^1 = t_1.\gamma_d, w_1 = t_1.v_i, \mu_r = t_2.u_r, \pi_d^2 = t_2.\gamma_d$ then the transformation

$\pi_d^1 = t_1.\gamma_d$, and $\pi_d^2 = t_2.\gamma_d$ imply a linear relationship between π_d^1 and π_d^2, namely,

$\pi_d^1 = \frac{\sum_i w_i xij_o}{\sum_k \pi_k^1 z_{kj_o}}.\pi_d^2$. Then, equation (3.6) becomes

$$max \left[w_1.\sum_{d=1}^{D}\pi_d^1 z_{dj_o} + w_2.\sum_{r=1}^{s}\mu_r y_{rj_o} \right] \quad (3.7)$$

subject to

$$\sum_{i=1}^{m} w_i x_{ij} - \sum_{d=1}^{D}\pi_d^1 z_{dj} \geq 0$$

$$\sum_{d=1}^{D}\pi_d^2 z_{dj} - \sum_{r=1}^{s}\mu_r y_{rj} \geq 0$$

$$\sum_{i=1}^{m} w_i x_{ij_o} = 1$$

$$\sum_{d=1}^{D}\pi_d^2 z_{dj_o} = 1$$

$$\pi_d^1 = \frac{\sum_i w_i.x_{i,j_o}}{\sum_k \pi_k^1.z_{k,j_o}}.\pi_d^2$$

$$\pi_d^1, \pi_d^2, \mu_r, w_i \geq 0 \quad j = 1, 2, ..., n$$

m D

which is a non linear program. Letting ^{P} $v_i x_{ijo}$ + ^{P} $\gamma_d z_{djo}$ the total size of (amount of resources consumed
i=1 d=1 m D
by) the two stage process, and ^{P} $v_i x_{ijo}$ and ^{P} $\gamma_d z_{djo}$, the sizes of the stages 1 and 2 respectively, we define
=1 d=1 i

$$w_1 = \frac{\sum_{i=1}^{m} \nu_i x_{ij_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}} \quad and \quad w_2 = \frac{\sum_{d=1}^{D} \gamma_d z_{dj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}}$$

. (3.8)

Then, the objective function of equation (3.6) becomes

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj_o} + \sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}}. \quad (3.9)$$

Under constant return to scale case, equation (3.6) becomes

$$max \frac{\sum_{d=1}^{D} \gamma_d z_{dj_o} + \sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}} \quad (3.10)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj}}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1$$

s D
X X
μryrj$_o$ + πdzdj$_o$
r=1 d=1 $\gamma_d, u_r, v_i \geq 0$, j = 1,2,...,n.

Using the Charnes- Cooper transformation, equation (3.10) is equivalent to

s D
maxXμryrjo + Xπdzdjo (3.11)
r=1 d=1 subject to
D m
X X
πdzij − wixij ≤ 0
d=1 i=1 s D
X X
μryrj − πdzdj ≤ 0
r=1 d=1 m D
Xwixijo + X πdzdjo = 1
i=1 d=1
$\pi_d, \mu_r, w_i \geq 0$, j = 1,2,...,n

When the optimal solution of equation (3.11) is obtained, we can calculate the efficiency of the two individual stages. However, equation (3.11) can have alternative optimal solutions. As a result, the decomposition of the overall efficiency defined in (3.5) may not be unique. Therefore we follow Kao and Hwangs (2008) approach to find a set of multipliers which produces the largest first (or second) stage efficiency score while maintaining the overall efficiency score. The following procedure is therefore proposed. Given the overall efficiency obtained from (3.11) (denoted as θ_o), we calculate either the first stage's efficiency (θ_j^1) or the second stage's efficiency (θ_j^2) first, and then derive from that the efficiency of the other stage. In case the first stage is to be given pre-emptive priority, the following model determines its efficiency (θ_o^{1*}), while maintaining the overall efficiency score at θ_o calculated from equation (3.11).

$$\theta_{jo}^1 = max \frac{\sum_{d=1}^{D} \gamma_d^A z_{dj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o}} \quad (3.12)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d^A z_{dj}}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} \gamma_d z_{dj}} \leq 1$$

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj_o} + \sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}} = \theta_o$$

$$\gamma_d^A, \nu_i, u_r, \geq 0 \quad j = 1, 2, ..., n$$

or equivalently,

D

θo1∗ = maxXπdzdjo (3.13)

d=1

subject to

$$\sum_{d=1}^{D} \pi_d z_{ij} - \sum_{i=1}^{m} w_i x_{ij} \leq 0$$

$$\sum_{r=1}^{s} \mu_r y_{rj} - \sum_{d=1}^{D} \pi_d z_{dj} \leq 0$$

$$(1 - \theta_o) \sum_{d=1}^{D} \pi_d z_{ij} + \sum_{i=1}^{s} \mu_r y_{rj} = \theta_o$$

$$\sum_{i=1}^{m} w_i x_{ij_o} = 1$$

$$\pi_d, \mu_r, w_i \geq 0, \quad j = 1, 2, ..., n$$

The efficiency for the second stage is then calculated as

$$\theta_o^2 = \frac{\theta_o - w_1^*.\theta_o^{1*}}{w_2^*}$$

where w_1^* and w_2^* represent optimal weights obtained from model (3.11) by way of (3.8). Note that here we use $(*) \text{ in } \theta_o^{1*}$ to indicate that the efficiency of the first stage is given the pre-emptive priority and is optimized first. In this case, the resulting second stage efficiency score is denoted as θ_o^2. In case the second stage is to be given pre-emptive priority, the following model determines the second stage's efficiency (θ_o^{2*}) while maintaining the overall efficiency score at θ_o calculated from equation (3.11).

$$\theta_o^{2*} = max \frac{\sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{d=1}^{D} \gamma_d z_{dj_o}} \qquad (3.14)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj}}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} \gamma_d z_{dj}} \leq 1$$

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj_o} + \sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{i=1}^{m} \nu_i x_{ij_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}} = \theta_o$$

$$\gamma_d^A, \nu_i, u_r, \geq 0 \quad j = 1, 2, ..., n$$

or equivalently,

s

θo2∗ = maxXμdyrjo (3.15)

r=1

subject to

D m

X X

πdzij − wixij ≤ 0

d=1 i=1 s D

X X

μryrj − πdzdj ≤ 0

r=1 d=1

D s m

X X X

πdzijo + μryrjo − θo wixijo = θo

d=1 i=1 i=1

D

X

πdzdjo = 1 d=1 $\pi_d, \mu_r, w_i \geq 0$, j = 1,2,...,n

and the efficiency for the first stage is calculated as $\theta_o^1 = \frac{\theta_o - w_2^*.\theta_o^{2*}}{w_1^*}$

similarly, we use (*)θ_o^{2*}in to indicate that second stage is given pre-emptive priority in terms of its efficiency being optimized first. In this case, the resulting first stage efficiency score is denoted as $\theta_o{}^1$. Finally, note that if $\theta_o^{1*} = \theta_o^1$ or $\theta_o^{2*} = \theta_o^2$, then this indicates that we have a unique efficiency decomposition.

3.2. **ADDITION OF TWO STAGE UNDER VARIABLES RETURN TO SCALE.** The variables return to Scale efficiency scores for the two stages can be determined by the following variables return to Scale efficiency models Banker et al. (1984)

$$max\ Q_{jo}^1 = \frac{\sum_{d=1}^{D} \gamma_d^A z_{dj_o} + U^A}{\sum_{i=1}^{m} \nu_i x_{ij_o}} \qquad (3.16)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d^A z_{dj} + U^A}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1 \quad j = 1, 2, ..., n$$

$$\gamma_d^A, \nu_i \geq 0 \quad U^A free\ in\ sign$$

and

$$max\ Q_{j_o}^2 = \frac{\sum_{r=1}^{s} u_r y_{rj_o} + U^B}{\sum_{d=1}^{D} \gamma_d^B z_{dj_o}} \qquad (3.17)$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + U^B}{\sum_{d=1}^{D} \gamma_i z_{dj}} \leq 1 \quad j = 1, 2, ..., n$$

$$\gamma_d^B, u_r \geq 0 \quad U^B \; free \; of \; sign$$

The approach of Kao and Hwang (2008) and Liang et al. (2008) cannot be extended to the variables return to scale assumption, because $Q_{j_o}^1 . Q_{j_o}^2$ cannot be converted into a linear form under the condition of $\gamma_d^A = \gamma_d^B$, due to the free variable U^A in the numerator of $Q_{j_o}^1$. On the other hand, using Wade and Joe (2014) approach, we have the variables return to scale overall efficiency as using the weights defined under the constant return to scale assumption

$$max \; \frac{\sum_{d=1}^{D} \gamma_d z_{dj_o} + U^A + \sum_{i=1}^{s} u_r y_{ij_o} + U^B}{\sum_{i=1}^{m} \nu_i x_j + \sum_{d=1}^{D} \gamma_d z_{dj_o}} \quad (3.18)$$

subject to

$$\frac{\sum_{d=1}^{D} \gamma_d z_{dj} + U^A}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1 \quad j = 1, 2, ..., n$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + U^B}{\sum_{d=1}^{D} \gamma_d z_{dj}} \leq 1, \quad j = 1, 2, ..., n$$

$\gamma_d, v_i, u_r \geq 0$ U^A, U^B free of sign

This is an input-oriented model. If we use output-oriented variables return to scale models, the weights will be defined as

$$w_1 = \frac{\sum_{d=1}^{D} \gamma_d z_{dj_o}}{\sum_{r=1}^{s} u_r y_{rj_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}}$$

$$w_2 = \frac{\sum_{r=1}^{s} u_r y_{rj_o}}{\sum_{r=1}^{s} u_r y_{rj_o} + \sum_{d=1}^{D} \gamma_d z_{dj_o}}$$

and equation (3.18) is equivalent to the following linear programming program

s D

maxXμdyrjo + U1 + Xπdzijo + U2 (3.19)

r=1 d=1 subject to

D m

Xπdzij − Xwixij + U 1 ≤ 0

d=1 i=1 s D

Xμryrj − Xπdzdj + U 2 ≤ 0

r=1 d=1 m D

X X

wixijo + πdzijo = 1

i=1 d=1

$\pi_d, \mu_r, w_i \geq 0$, j = 1,2,...,n U^1, U^2 free in sign

Once we obtain the overall efficiency, models similar to (3.13) and (3.15) can be developed to determine the efficiency of each stage. Specifically, assuming pre-emptive priority for stage 1, the following model determines that stage's efficiency (Q_o^{1*}), while maintaining the overall efficiency score at Q_o calculated from model (3.19).

$$Q1o* = \max \sum_{d=1}^{D} \pi dzdjo + U1 \qquad (3.20)$$

subject to

$$\sum_{d=1}^{D} \pi dzij + U\,1 - \sum_{i=1}^{m} wixij \le 0$$

$$\sum_{r=1}^{s} \mu ryrj - \sum_{d=1}^{D} \pi dzdj \le 0$$

$$(1 - Qo)\sum_{d=1}^{D} \pi dzijo + \sum_{i=1}^{s} \mu ryrjo + U1 + U2 = Qo$$

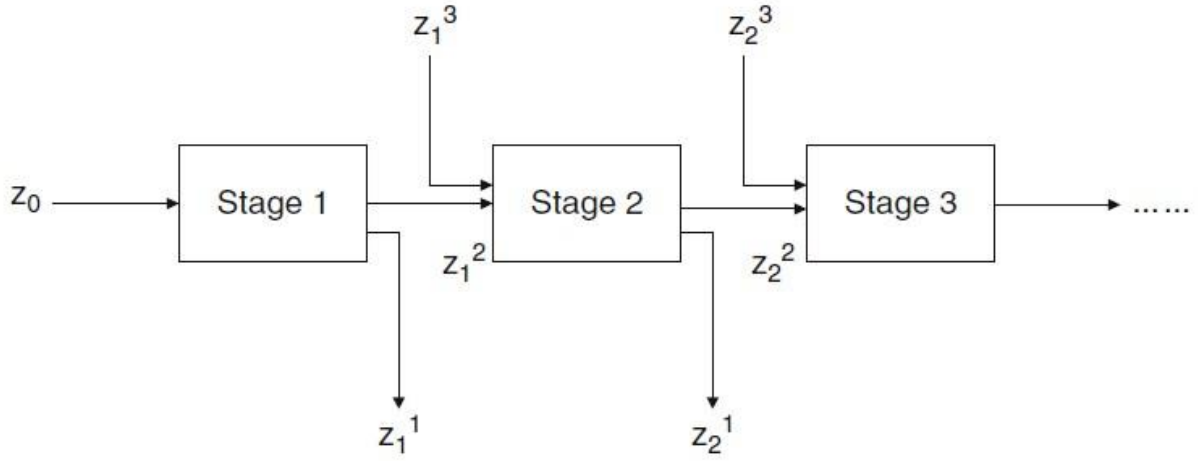


Figure 2. Serial multistage process

$$\sum_{i=1}^{m} w_i x_{ij_o} = 1$$

$\pi_d, \mu_r, w_i \ge 0, \quad j = 1, 2, ..., n, \quad U^1, U^2$ free in sign

Similarly, if stage 2 is to be given pre-emptive priority, the following model determines the efficiency (Q_j^{2*}) for that stage, while maintaining the overall efficiency score at Q_o calculated from equation (3.4).

$$Q_o^{2*} = max \sum_{r=1}^{s} \mu_d y_{rj_o} + U^2 \qquad (3.21)$$

subject to

$$\sum_{d=1}^{D} \pi_d z_{ij} + U^1 - \sum_{i=1}^{m} w_i x_{ij} \le 0$$

$$\sum_{r=1}^{s} \mu_r y_{rj} + U^2 - \sum_{d=1}^{D} \pi_d z_{dj} \le 0$$

$$\sum_{d=1}^{D} \pi_d z_{ij_o} + \sum_{i=1}^{s} \mu_r y_{rj_o} - Q_o \sum_{i=1}^{m} w_i x_{ij_o} + U^1 + U^2 = Q_o$$

$$\sum_{d=1}^{D} \pi_d z_{dj_o} = 1$$

$\pi_d, \mu_r, w_i \ge 0$, $j = 1,2,...,n$, U^1, U^2 free in sign

Once the efficiency score for one of the stages is calculated using (3.5) or (3.6), the score for the other stage can be derived in the similar manner as in the constant return to scale case.

ADDITION OF GENERAL MULTISTAGE PROCESSES

Serial Processes. considering the H-stage process, we denote the input vector to stage 1 by z_o. The output vectors from stage h (h = 1, ... , H) take two forms, namely z_h^1 and z_h^2. Here, z_p^1 represents output that leaves the process at this stage and is not passed on as input to the next stage. The vector z_p^2 represents the amount of output that becomes input to the next (h+1) stage. These types of intermediate measures are called links in Tone and Tsutsui (2009). In addition, there is the provision for new inputs z_p^3 to enter the process at the beginning of stage h + 1. Specifically, when h = 2, 3, ..., we define

1 z_{hr}^{j1} the r^{th} component (r = 1;...;R_p) of the R_p-dimensional output vector for DMU_j flowing from stage h, that leaves the process at that stage, and is not passed on as an input to stage h + 1.

2 z_{hk}^{j2} the k^{th} component (k = 1;...;S_h) of the S_h-dimensional output vector for DMU_j flowing from stage h, and is passed on as a portion of the inputs to stage h + 1.

3 z_{pi}^{j3} the i^{th} component (i = 1;...;I_h) of the I_h-dimensional input vector for DMU_j at the stage h + 1 , that enters the process at the beginning of that stage.

In the last stage H, all the outputs are viewed as z_{hr}^{j1}, as they leave the process. We denote the multipliers (weights) for the above factors as

1 u_{hr} is the multiplier for the output component z_{hr}^{j1} flowing from stage h.

2 γ_{hk} is the multiplier for the output component z_{hk}^{j2} at stage h, and is as well the multiplier for that same component as it becomes an input to stage h + 1.

3 v_{pi} is the multiplier for the input component z_{hi}^{j3} entering the process at the beginning of stage h + 1.

Therefore, when h = 2,3,..., the efficiency ratio for DMU_j (for a given set of multipliers) would be expressed as follows:

$$\theta_h = \frac{\left(\sum_{r=1}^{R_h} u_{hr} z_{hr}^{j1} + \sum_{k=1}^{s_h} \gamma_{hk} z_{hk}^{j2}\right)}{\left(\sum_{k=1}^{S_{h-1}} \gamma_{h-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_h} \nu_{h-1i} z_{h-1i}^{j3}\right.} \qquad (4.1)$$

!

Here, there are no outputs flowing into stage 1. The efficiency measure for stage 1 of the process (namely, h = 1) , for DMU_j becomes

$$\theta_1 = \frac{\left(\sum_{r=1}^{R_1} u_{1r} z_{1r}^{j1} + \sum_{k=1}^{s_1} \gamma_{1k} z_{1k}^{j2}\right)}{\sum_{i=1}^{Io} \nu_{oi} z_{oi}^{j}} \quad (4.2)$$

where z_{oi}^{j} are the only inputs to the first stage represented by the input vector z_o. We claim that the overall efficiency measure of the multistage process can reasonably be represented as a convex linear combination of the h (stage-level) measures, namely

H H

$\theta = {}^{X}w_h\theta_h$ where ${}^{X}w_h = 1$

h=1 h=1

The weights w_p are intended to represent the relative importance or contribution of the performances of individual stages h to the overall performance of the entire process. One reasonable choice for weights w_p is the proportion of total resources for the process that are devoted to stage h, and reflecting the relative size of that stage. To be more specific,

$$\sum_{i=1}^{Io} \nu_{oi} z_{oi}^{j} + \sum_{h=2}^{h} \left(\sum_{k=1}^{S_{h-1}} \gamma_{h-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_h} \nu_{h-1i} z_{h-1i}^{j3} \right)$$

represents the total size of or total amount of resources consumed by the entire process, and we define the w_h to be the proportion of the total input used at the h^{th} stage. We then have

$$w_1 = \frac{\sum_{i=1}^{Io} \nu_{oi} z_{oi}^{j}}{\sum_{i=1}^{Io} \nu_{oi} z_{oi}^{j} + \sum_{h=2}^{h} \left(\sum_{k=1}^{S_{h-1}} \gamma_{h-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_h} \nu_{h-1i} z_{h-1i}^{j3} \right)} \qquad (4.3)$$

$$w_h = \frac{\sum_{k=1}^{S_{h-1}} \gamma_{h-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_h} \nu_{h-1i} z_{h-1i}^{j3}}{\sum_{i=1}^{Io} \nu_{oi} z_{oi}^{j} + \sum_{h=2}^{h} \left(\sum_{k=1}^{S_{h-1}} \gamma_{h-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_h} \nu_{h-1i} z_{h-1i}^{j3} \right)}, \quad p > 1$$

. (4.4)

Thus, we can write the overall efficiency θ in the form

$$\theta = \frac{\sum_{h=1}^{H}\left(\sum_{r=1}^{R_h} u_{hr} z_{hr}^{j1} + \sum_{k=1}^{s_h} \gamma_{hk} z_{hk}^{j2}\right)}{\sum_{i=1}^{Io} \nu_{oi} z_{oi}^{j} + \sum_{h=2}^{h}\left(\sum_{k=1}^{S_{h-1}} \gamma_{h-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_h} \nu_{h-1i} z_{h-1i}^{j3}\right)}. \quad (4.5)$$

We then set out to optimize the overall efficiency θ of the multistage process, subject to the restrictions that the individual measures θ_h must not exceed unity, or in the linear programming format, after making the usual Charnes and Cooper transformation,

$$max \sum_{h=1}^{H}\left(\sum_{r=1}^{R_h} u_{hr} z_{hr}^{j1} + \sum_{k=1}^{s_h} \gamma_{hk} z_{hk}^{j2}\right) \quad !$$

(4.6)

subject to

□ Io h □Sh−1 Ih □□ X o X X o2 X o3

v □ oizoi + □ γh−1kzp−1k +vh−1izh−1i□□ = 1

i=1 h=2 k=1 i=1

R1 s1 ! Io

X j1 X j2 X j

u1rz1r + γ1kz1k ≤ voizoi

r=1 k=1 i=1

Rh sh ! □Sh−1 Ih □ X j1 X j2 X j2 X j3

uhrzhr + γhkzhk ≤ □ γh−1kzp−1k + vh−1izh−1i□ ∀j

r=1 k=1 k=1 i=1 uhr,γhr,vhi,voi ≥ 0.

Remarks:

- we should impose the restriction that the overall efficiency scores for each j should not exceed unity, but since these are redundant, this is unnecessary.
- more so, that the w_h, as defined above, are variables related to the inputs and the intermediate measures. By virtue of the optimization process, it can turn out that some $w_h = 0$ at optimality. To overcome this problem, one can impose bounding restrictions $w_h > c$, where c is a selected constant.

4.2. **Parallel Processes.** The model in the previous section to handle such strict serial processes is easily adapted to more general network structures. Specifically, the efficiency ratio for an overall process can be expressed as the weighted average of the efficiencies of the individual components. The efficiency of any given component is the ratio of the total output to the total input corresponding to that component. Again, the weight w_h to be applied to any component p is expressed as w_h = (component p input)/(total input across all components). There is no convenient way to represent a network structure that would lend itself to a generic mathematical representation analogous to equation (4.6) above. The sequencing of activities and the source of inputs and outputs for any given component will differ from one type of process to another. However, as a simple illustration, consider the following two examples of network structures.

Consider the process with an initial input vector z_o enters component 1. Three output vectors exit this component, that is z_1^1 leaves the process, z_1^2 is passed on as an input to component 2, and z_1^3 as an input to component 3. Additional inputs z_1^4 and z_1^5 enter components 2 and 3 respectively, from outside the process. Components 2 and 3 have z_2^1 and z_3^1, respectively as output vectors which are passed on as inputs to component 4, where a final output vector z_4^1 is the result. Then we have the following Efficiencies Component 1 efficiency ratio: $\theta_1 = (u_1 z_1^1 + \gamma_1^2 z_1^2 + \gamma_1^3 z_1^3)/\nu_o z_o$

Component 2 efficiency ratio: $\theta_2 = \gamma_2^1 z_2^1/(\gamma_1^2 z_1^2 + \nu_1 z_1^4)$

Component 3 efficiency ratio: $\theta_3 = \gamma_3^1 z_3^1/(\gamma_1^3 z_1^3 + \nu_2 z_1^5)$ Component 4 efficiency ratio: $\theta_4 = u_4 z_4^1/(\gamma_2^1 z_2^1 + \gamma_3^1 z_3^1)$

The total (weighted) input across all components is given by the sum of the denominators of θ_1 through θ_4, namely

$$I = \nu_o z_o + \gamma_1^2 z_1^2 + \nu_1 z_1^4 + \gamma_1^3 z_1^3 + \nu_2 z_1^5 + \gamma_2^1 z_2^1 + \gamma_3^1 z_3^1$$

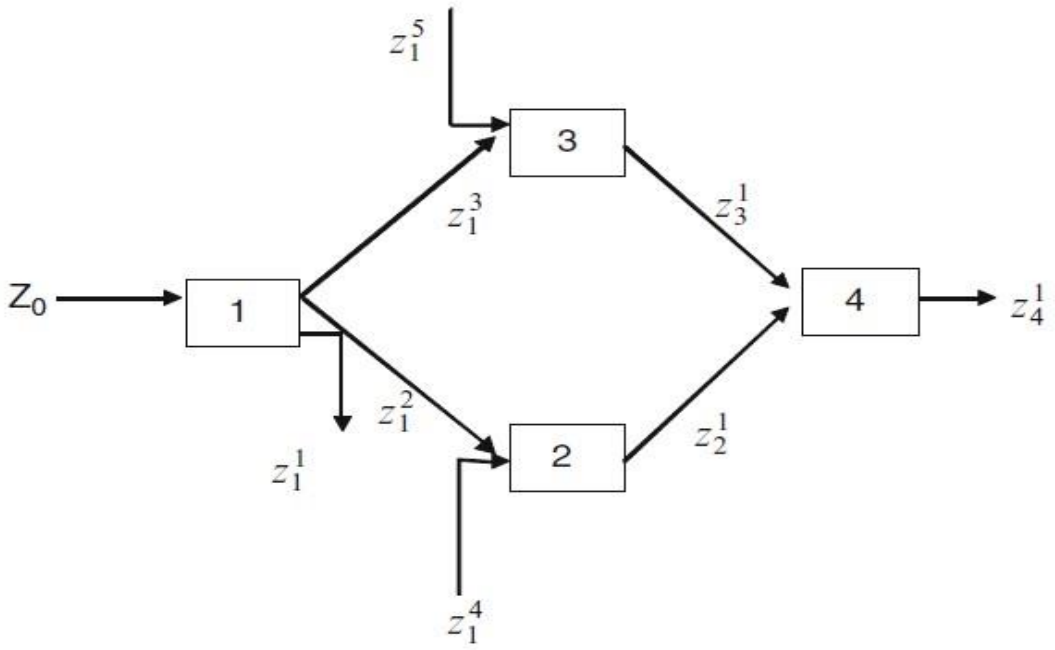


Figure 3. Parallel multistage process

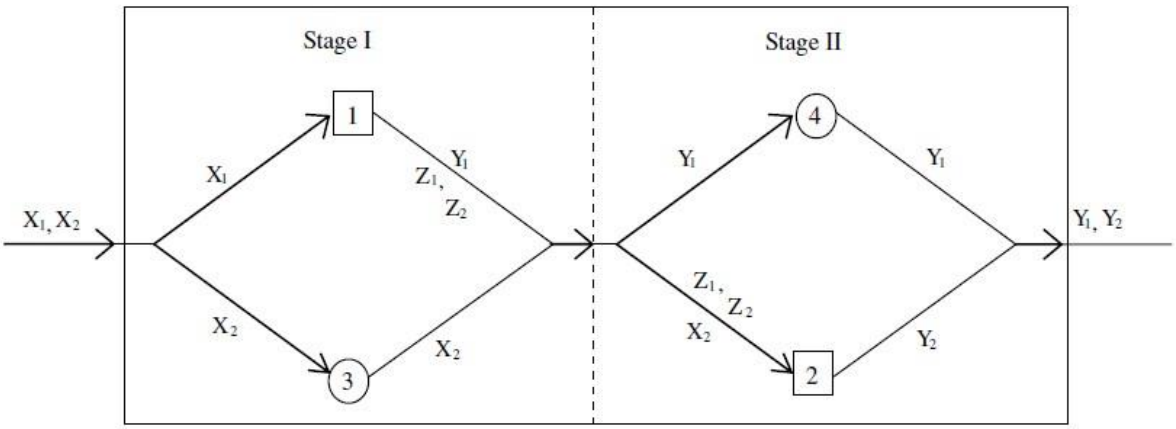


Figure 4. General multistage process

Thus, we now express the w_h as
$w_1 = v_o z_o / I$ w2 = (γ12z12 + v1z14)/I w3 = (γ13z13 + v2z15)/I w4 = (γ21z21 + γ31z31)/I

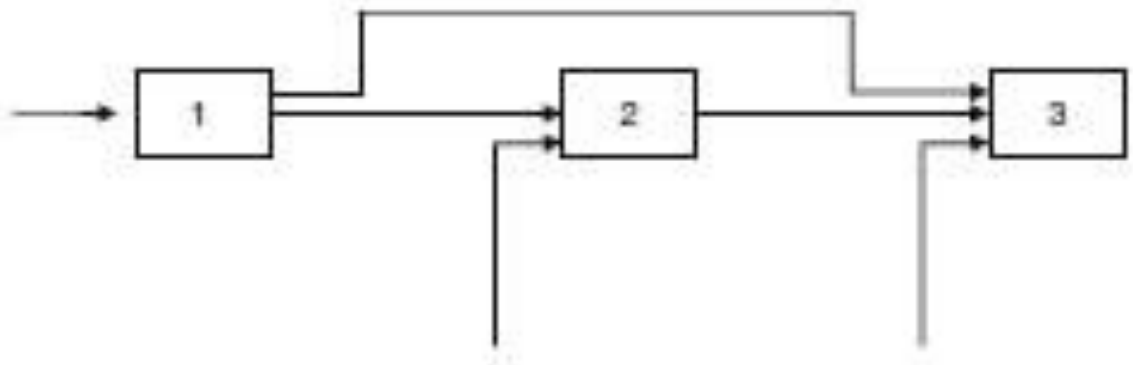


Figure 5. Non Immediate Successor Flow

Table 1: Data set in Tone and Tsutsui (2009)

DMU	stage 1 input 1 (X_1)	stage 2 input 2 (X_2)	output 2 (y_1)	stage 3 input 3 (x_3)	output3 (y_3)	intermediate measure link12 (z_1)	link23 (z_2)
A	0.838	0.277	0.879	0.962	0.337	0.894	0.362
B	1.233	0.132	0.538	0.443	0.18	0.678	0.188
C	0.321	0.045	0.911	0.842	0.198	0.836	0.207
D	1.483	0.111	0.57	0.467	0.491	0.869	0.16
E	1.592	0.208	0.086	1.073	0.372	0.693	0.407
F	0.79	0.139	0.722	0.545	0.253	0.966	0.269
G	0.451	0.075	0.509	0.366	0.241	0.647	0.257
H	0.408	0.074	0.619	0.229	0.097	0.756	0.103
I	1.864	0.061	1.023	0.691	0.38	1.191	0.402
J	1.222	0.149	0.769	0.337	0.178	0.792	0.187

therefore, the overall network efficiency ratio is given by

$$\theta = \sum_{h=1}^{4} w_h \theta_h = (u_1 z_1^1 + \gamma_1^2 z_1^2 + \gamma_1^3 z_1^3 + \gamma_2^1 z_2^1 + \gamma_3^1 z_3^1 + u_4 z_4^1)/I \qquad (4.7)$$

and one then proceeds, as in (4.6) above, to derive the efficiency of each DMU and its components.

4.3. **Non Immediate Successor Flows.** In the previous example all flows of outputs from a stage or component either leave the process entirely or enter as an input to an immediate successor stage. In non immediate successor flow, the same is true except that there is more than one immediate successor of stage 1.

Consider Figure 5, here, the inputs to stage 3 are of three types, namely outputs from stage 2, inputs coming from outside the process, and outputs from a previous, but not immediately previous stage. Again the above rationale for deriving weights w_h can be applied and a model equivalent to (4.6) solved to determine the decomposition of an overall efficiency score into scores for each of the components in the process.

5. Application

We finally apply our approach to the numerical example used in W.D.Cook and J. Zhu (2014). Table 1 provides the data. We have two intermediate measures or outputs flow from one stage to the other. Table 2 reports the results. In this case, if we do not impose a lower bound for the w_p($p = 1, 2, 3$), we have some $wp = 1$ at optimality (for DMUs B, D, I and J). Therefore, we impose $w_p > 0.1$ ($p = 1, 2, 3$) in model (4.6). Because our approach is different from Tone and Tsutsuis (2009) and our choice of weights introduces restrictions on the multipliers, our results are different from theirs.

Table 2: result on three- stage process

DMU	overall	stage 1	stage 2	stage 3	w_1	w_2	w_3
A	0.579	0.410	0.646	0.971	0.46	0.41	0.13
B	0.386	0.211	0.339	0.414	0.10	0.10	0.80
C	1.000	1.000	1.000	0.999	0.42	0.48	0.10
D	0.917	0.225	0.942	1.000	0.10	0.10	0.80
E	0.478	0.167	0.501	0.953	0.36	0.42	0.22
F	0.598	0.470	0.656	0.984	0.51	0.37	0.11
G	0.762	0.551	0.717	0.983	0.24	0.44	0.32
H	0.675	0.711	0.599	0.843	0.46	0.44	0.10
I	0.922	0.245	1.000	0.990	0.10	0.64	0.26
J	0.476	0.249	0.423	0.511	0.10	0.10	0.80

CONCLUSION

Original studies in DEA view systems as a whole, ignoring the performance of their component processes in calculating the relative efficiency of a set of production systems. The deficiencies are, firstly, that the efficiency score may not properly represent the aggregate performance of the processes of a system. Secondly, it does not show which process causes the low efficiency of an inefficient system. The existing models in network DEA partially improve these deficiencies. Since, in terms of the multipliers used, each process is independent, a mathematical relationship between the process efficiencies of the system and component processes at the same time is not revealed.

In order to identify the source of inefficiency, one can calculate the efficiency of each process independently. By introducing dummy processes, this paper transforms a network system into a series system, where each stage in the series is a parallel structure composed of a set of processes. Based on the series and parallel relationships in which the processes are connected, the efficiency, or inefficiency, of the system is decomposed into those of the component processes. Specifically, in the series structure the system efficiency is the product of the process efficiencies and for the parallel structure the inefficiency slack of the system is the sum of the inefficiency slacks of the component processes. Thus, the decisive process causing the low efficiency of the system can be identified. A difficulty frequently encountered in real world applications of DEA is that the number of DMUs that are available is usually limited. A small sample size often produces results which are misleading. Moreover, it produces a relatively large proportion of efficient DMUs which makes the subsequent rankings difficult.

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