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ANALYTICAL SOLUTIONS OF KLEIN-GORDON EQUATION FOR A CENTRAL INVERSE SQUARE POTENTIAL USING NIKIFOROV-UVAROV METHOD

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ABSTRACT

The radial Klein-Gordon Equation is solved for a central inverse square potential. The relativistic energy eigenvalues are given and the corresponding eigen functions are obtained in terms of the Laguerre polynomials by using the Nikiforov-Uvarov method which is based on solving the second-order linear differential equations by reduction to a generalized equation of hypergeometric-type.

INTRODUCTION

One of the interesting problems in high energy physics is to obtain exact and numerical solutions of the Klein Gordon, Duffin-Kemmer-Petiau and Dirac equations. These equations are frequently used to describe the particle dynamics in relativistic quantum mechanics, and are consequently referred to as relativistic wave equations. They predict the behavior of particles at high energies and velocities comparable to the speed of light. When a quantum system is in a strong potential field, the relativistic effects must be considered, which gives the correction for nonrelativistic quantum mechanics (Wang & Wong 1988), and the physics of such a system can only be described by Klein Gordon, Duffin-Kemmer-Petiau and/or Dirac equations. In the context of Quantum Field Theory (QFT), the equations determine the dynamics of quantum fields.
The solutions to these equations, universally denoted as Ψor Ψ, referred to as "wave functions" in the context of relativistic quantum mechanics, and "fields" in -the context of quantum field theory. The equations themselves are called “wave equations” or “field equations,” because they have the mathematical form of a wave equation or are generated from a Lagrangian density and the field-theoretic Euler-Lagrange equations. Among all three equations, the Klein-Gordon equation is the oldest, and consequently the first equation one encounters in a course on relativistic quantum mechanics. The Klein-Gordon equation (or Klein-Fock-Gordon equation) is a relativistic version of the Schrodinger equation, which describes scalar (or pseudoscalar) spinless particles. It has been dubbed an “equation with many fathers” because of the numerous authors to whom it was credited; notably, Schrodinger, de Broglie, Oscar Klein (Klein 1926), Walter Gordon (Gordon 1926) and Vladimir Fock. But historians of science have published that the Klein-Gordon equation was discovered in the notebooks of Schrodinger, and he had first derived it, before he made the discovery of the equation that now bears his name. He rejected it because he couldn't make it fit experimental data (the equation doesn't take into account the spin of the electron). The way he found his equation was by making simplification to the Klein-Gordon equation. The KG equation was later published independently by the authors for whom it is now named. Later it was revived and it has become commonly accepted that Klein-Gordon equation is the appropriate model to describe the wave function of the particle that is charge-neutral, has no spin and relativistic effects can't be ignored.
It is applied in the description of π-mesons and corresponding fields. The free Klein-Gordon equation is a linear homogeneous second-order partial differential equation with constant coefficients:

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$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{c^2\partial t^2}-\mu^2\right)\phi=0 \qquad 1.$$

where $\phi(x,t)$ is a pseudo-scalar function, in the general case, $\mu = \frac{mc}{h}$ and m is the rest mass of the particle.

If ϕ is a real function, then the Klein-Gordon equation describes neutral (pseudo-) scalar particles while when ϕ is complex it describes charged particles. The free Klein-Gordon in Eq. (1) can be derived by starting with the square of the identity from special relativity, i.e.

$$p^2c^2+m^2c^2=E^2 \qquad 2$$

where *p* and *E* are quantum mechanical operators:

$$p\rightarrow ih\nabla,\quad E\rightarrow ih\frac{\partial}{\partial t} \qquad 3$$

which, when quantized, gives

$$((-ih\nabla)^2c^2+m^2c^4))\psi=\left(ih\frac{\partial^2}{\partial t}\right)^2\psi \qquad 4$$

which simplifies to:

$$-h^2c^2\nabla^2\psi+m^2c^4\psi=-h^2\frac{\partial^2}{\partial t^2}\psi \qquad 5$$

Rearranging terms yields

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi-\nabla^2\psi+\frac{m^2c^2}{h^2}\psi=0 \qquad 6$$

Since all references to imaginary numbers have been eliminated from this equation, it can be applied to fields that are real valued as well as those that have complex values. Eq. 6 can be rewritten as Eq. 1, (Dariescu et *al.,* 2005)

The Klein-Gordon equation has attracted a great deal of attention since its discovery, especially in the studies of Solitons (Drazin, 1983) in condensed matter physics and in investigating the interaction of Solitons in a collisionless plasma. It finds applications in quantum field theory, dispersive wave phenomena, non-linear optics and in applied physical sciences. In plasma physics, together with Zakharov equation, it describes the interaction of Langmuir wave and the ion acoustic wave in a plasma, (Ozawa *et al.,* 1999 and Khusnutdinova et *al.,* 2003).

Klein-Gordon Equation with Equal Scalar and Vector Potentials

The radial Klein-Gordon equation may be obtained by the non relativistic Schrondinger equation, (Alhaidari et *al.,* 2006). It is seen to contain two major objects: the four-vector linear momentum operator and the scalar rest mass. This invariably allows one to introduce two types of potential: the four-vector potential $V(\vec{r})$ and the space-time scalar potential $S(\vec{r})$ from which one obtains the Klein-Gordon equation having the form:

$$\left[-\left(i\frac{\partial}{\partial t}-V(\vec{r})\right)^2-\vec{\nabla}^2+(S(\vec{r})+M)^2\right]\psi(\vec{r})=0 \qquad 7$$

However, it is noted by Yasuk et *al.,* (2006) that for time independent-potentials, writing the total wave function as $\psi(\vec{r},t)=e^{i\in t}\psi(\vec{r})$, where ε is the relativistic energy enables us to obtain the time-independent Klein-Gordon equation.

Consequently, we obtain:

$$\left[\nabla^2+(V(\vec{r})-\in)^2-(S(\vec{r})+M^2)\right]\psi(\vec{r})=0 \qquad 8$$

which is the three-dimensional KG equation with mixed vector and scalar potential, where M is the mass, ε is the energy and $S(\vec{r})$ and $V(\vec{r})$ are the scalar and vector potentials respectively. If we take $S(\vec{r})=\pm V(\vec{r})$, the Klein-Gordon equation becomes:

$$\left[\vec{\nabla}^2-2(\in\pm M)V(\vec{r})+\in^2-M^2\right]\psi(\vec{r})=0 \qquad 9$$

Eq. 9 according to Alhaidari *et al* (2006), describes a spin-0 scalar particle. It is the Schrodinger equation for the potential 2V in the non-relativistic limit. Thus, Alhaidari et al. (2006) conclude that only the choice S+V produces a nontrivial nonrelativistic limit with a potential function 2V, and not V (Yasuk 2006). Accordingly, it would be natural to scale the potential terms in Eq. (1.8) so that in the relativistic limit, the interaction potential becomes *V*, not *2V*. Therefore, they modify Eq. (1.8) to read as follows (Alhaidari 2006):

$$\left[\vec{\nabla}^2 + \left(\frac{1}{2}(\vec{r}) - \in\right)^2 - \left(\frac{1}{2}S(\vec{r}) + M\right)\right]\psi(\vec{r}) = 0 \qquad 10$$

Thus, Eq. 9 becomes Eq. 11

$$\left[\vec{\nabla}^2 - (\in \pm M)V(\vec{r}) + \in^2 - M^2\right]\psi(\vec{r}) = 0 \qquad 11$$

Separating variables of the Klein-Gordon Equation with Non-Central Potential

For $S(\vec{r}) = +V(\vec{r})$, if we take $V(\vec{r})$ as a general non-central potential, three dimensional Klein-Gordon equation can be separated into variables. In the spherical coordinates, the Klein-Gordon equation for, a particle in the existence of a general non-central Potential $V(r,\theta)$ becomes

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\partial\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\in^2 - M^2\right]\psi(r,\theta,\varphi) = 0 \qquad 12$$

where $V(r,\theta)$ is a general non-central potential (Yasuk 2006).

Radial Klein-Gordon Equation

To obtain the form of the equation which is required for this work, it is imperative to separate the wave equation (1.12) for a general non-central potential into variables. If one assigns the corresponding spherical total wave function as $\psi(r,\theta,\varphi) = \frac{1}{r}R(r)Y(\theta,\varphi)$, then by selecting $Y(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$, the wave equation is separated into variables and the following equations are obtained (Yasuk 2006):

$$\left[\frac{d^2}{dr^2} - \frac{\lambda}{r^2}(\in + M)V(r) + \in^2 - M^2\right]R(r) = 0 \qquad 13$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + Cot\theta\frac{d\Theta(\theta)}{d\theta^2} + \left[\lambda - \frac{m^2}{\sin^2\theta} - (\in + M)V(\theta)\right]\Theta(\theta) = 0 \qquad 14$$

$$\frac{d^2\Theta(\varphi)}{d\varphi^2} + m^2\Theta(\varphi) = 0 \qquad 15$$

where m^2 and A are the separation constants. The solution of equation 15 is the azimuthal angle solution,

$$\Phi_m = Ae^{im\varphi}, (m = 0 \pm 1, \pm 2 ...) \qquad 16$$

Equation 14 is polar equation which can also be solved by use of the Nikiforov-Uvarov method (Nikiforov and Uvarov 1988). Equation 13 is the Radial Klein-Gordon equation and consequently our equation of interest.

THEORETICAL FRAMEWORK

The field of relativistic quantum mechanics emerged out of the need to incorporate non-relativistic quantum mechanics with the well-established theory of relativity formulated by Einstein. This need for unification was motivated by three reasons. Firstly, there are many experimental phenomena which cannot be understood or explained within purely non-relativistic domain. Secondly, aesthetically and intellectually it would be profoundly unsatisfactory if relativity and quantum mechanics, two highly successful theories in their own right, could not be united. Finally, there are theoretical reasons why one would expect new phenomena to appear at relativistic velocities.

Particles are relativistic when their velocities approach the speed of light, *c* or, more intrinsically, when a particle's energy is large compared to its rest mass energy, mc^2. Photons are on the other hand never non-relativistic as they have zero rest mass and always travel at the speed of light.

The new phenomena that non-relativistic quantum mechanics failed to address, but which a marriage of quantum mechanics and relativity was hoped to address include:

A) Particle production: The phenomenon of particle production has as an example the production of electron-positron pairs by energetic y-rays in matter. To observe this, one needs collisions involving energies of order twice the rest mass energy of the electron.

B) Spin: While the phenomenon of electron spin has to be grafted artificially onto the non-relativistic Schrodinger equation, it emerges naturally from a relativistic treatment of quantum of quantum mechanics.

C) Vacuum instability: This is also inexplicable from the stand point of non-relativistic quantum mechanics.

The seminal works on this incorporation of quantum mechanics and special relativity were done by O. Klein and W. Gordon in papers published independently by both authors (Klein 1926; Gordon 1926).

Inspired by the work of Schrodinger (1926), whose derivation was based on the Hamiltonian formalism in classical mechanics, Oscar Klein, whose form of the equation is widely used nowadays, first proposed the following invariant wave equation for the relativistic motion in five dimensions:

$$\partial_A (g^{AB}\partial_B)\psi = 0 \qquad 17$$

(where $\partial_A = \partial / \partial x^A, g^{AB}$ is the symmetric metric tensor in general relativity representing the gravitational field, which describes the geometric properties of space time) which is similar to the differential Hamilton-Jacobi equation

$$g^{AB}\frac{\partial S}{\partial x^A \partial x^B} = 0 \qquad 18$$

The wave function $\psi = \psi(x,y)$ in equation 17 can in general be a complex quantity for which the action function *S(x,y)* is the phase of the wave. Thus we can write $\psi = \exp(iS/h)$ where the constant *h* has the same dimension as for mechanical action and turns out to be Planck-Dirac constant. Inserting this wave function into 17 and taking the classical limit $h \to 0,$ one then recovers the Hamilton-Jacob equation 18. But since $ps = mc$ where *m* is the mass of the particle in four dimensions, and $p_5 = const.,$ we must have $S(x,y) = psy + S(x)$ and can thus write

$$\psi(x,y) = e^{ips/h}\Phi(x) \qquad 19$$

where now $\Phi(x)$ should be Klein's wave function in four-dimensional space time.

Writing again $p_s = mc,$ one then finds from 17 that it must obey the differential equation

$$\left[\left(\partial_\mu - \frac{1}{h}eA_\mu\right)^2 + \left(\frac{mc}{h}\right)\right]\Phi(x) = 0 \qquad 20$$

This is the well-known Klein-Gordon equation for relativistic particles without spin.

Since its discovery, much work has been done in the field of high energy physics, with many authors proposing solutions to the Klein-Gordon equation for different potentials by use of various analytical and numerical methods. Notable works include those by Alhaidari et *al.,* (2006), who studied the three dimensional Dirac and Klein-Gordon equations with scalar and vector potentials of equal magnitude, and their result has provided a necessary basis for further investigation by other authors including Yasuk et *al* (2006), who found solution to the Klein-Gordon equation with a non-central equal scalar and vector potential having the form

$$V(r,\theta) = \frac{a}{r} + \frac{\beta}{r^2 \sin^2\theta} + \gamma\frac{\cos\theta}{r^2\sin^2\theta} \qquad 21$$

The KG equation has also been solved exactly by Ikot *et al* (2010) for the Hylleras potential given by

$$V(r) = D_e\left[1 - \frac{(1+a)(1+c)(1+b)}{(s+a)(s+c)(s+b)}\right] \qquad 22$$

where D_e is the dissociation energy, $S = e^{2(1+k)wr}$ and the intermediate quantities *a,b,c* defined by

$$a = \left(\frac{K-k_2}{1+k_2}\right), b = \left(\frac{K-k_1+k_2}{1+k_1+k_2}\right) c = \frac{K-k_1}{1+k_1} \qquad 23$$

Furthermore, Sharma *et al* (2011), understanding the role of potential theory in affording a deeper perspective on the problems of quantum field theory, took the initiative to construct a wider class of exactly solvable potentials for the Klein-Gordon and Dirac equations. Interestingly, the potential for which we attempt to solve the Klein-Gordon was derived in the work by Sharma et al. Literature shows that no work has before now been done using this potential, hence the originality of our work.

STATEMENT OF THE PROBLEM

By using various standard second-order differential equations (namely, the Laguerre equations, the Associated Legendre equations, Stokes equation and Whittaker equation), Sharma et *al* (2011) evaluated a wide class of potentials for which the Klein Gordon and Dirac equations could be solved exactly. One of the potentials derived in their work by making appropriate transformation to the Stokes equation was an inverse square potential having the form

$$V(r)=\frac{2a\sqrt{b}}{3r^2} \qquad 24$$

where r represents separation, and a and *b* are constants. A survey of literature reveals that while the Klein-Gordon equation has been exactly solved for a wide class of potentials namely, Hulthen potential (Qiang 2007), Woods-Saxon potential Poschl-Teller potential (Chen 2001), Pseudo-harmonic oscillator (Chen 2004), reflection less - type potential, Q-Deformed Morse Potential (Laachir 2014) and the shifted 1/N expansion, the solution of it for the potential having the form of equation (1.24) remains an open question. Therefore, this work is an enquiry into the form of the eigen functions and eigenvalues for the Klein-Gordon equation with the potential expressed in equation (1.24). In particular, we seek to solve the equation

$$\left[\frac{d^2}{dr^2}-\frac{\lambda}{r^2}-(\in+M)\frac{2a\sqrt{b}}{3r^2}+\in^2-M^2\right]R(r)=0 \qquad 25$$

using an analytical method known as the Nikiforov-Uvarov method (Nikiforov & Uvarov 1988).

IMPORTANT DEFINITIONS

In order to facilitate an understanding of this work, I shall define and elucidate certain key concepts and technical words that one is bound to come across in the course of reading.

1) **Central Potential:** A central potential is that which depends only on the distance *r* from the origin v(r). If spherical coordinates are used to parameterize three-dimensional space, a central potential does not depend on the angular variables *6* and 0. A typical example of central potential is the Coulomb potential between electrically charged particles:

$$V(r)=-\frac{Ze^2}{4\pi\in_0 r} \qquad 26$$

The central potential finds application in particle and high energy physics where it describes the quark interaction; in nuclear physics where it yields outstanding results in spectroscopy. In atomic physics, it describes the binding energy; in molecular physics, it describes intermolecular interactions and atomic pair correlation functions. (Onate 2016).

2) **Inverse square potential:** An inverse square law potential is a potential that defines an inverse function $1/r^2$ at large distances. It plays an important role in scattering theory.

3) **Hypergeometric function:** In Mathematics, the Gaussian or ordinary hypergeometric function is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE).

4) **Electromagnetic four-potential:** This is a relativistic vector function from which the electromagnetic field can be derived. It combines both an electric scalar potential and a magnetic vector potential into a single four-vector.

5) **Energy eigenvalue and eigenfunction:** In nonrelativistic quantum mechanics, energy eigenvalue is a specific value of energy for which a solution exists for the time-independent Schrodinger equation. Corresponding to each eigenvalue is an eigen function. The solution to the Schrodinger (or Schrodinger-like) equation for a given energy E_i involves also finding the specific function Ψ_i which describes that energy state.

6) **Laguerre polynomials:** The Laguerre polynomials are the solutions of the Laguerre's equation $xy''+(1-x)y'+ny=0.$ They are a polynomial sequence and may be defined by the Rodrigues formula

$$L_n(x)=\frac{1}{n!}\left(\frac{d}{dx}-1\right)^n x^n \qquad .27$$

OBJECTIVES OF THE STUDY

The objective of this study is to solve the Klein_Gordon equation for an inverse square potential via the Nikiforov-Uvarov method and:

i. obtain the exact energy eigenvalues of the equation;
ii. obtain the eigen functions in terms of the associated Laguerre functions.

LITERATURE REVIEW

There are basically two broad methods of solving the Schrodinger and Schrodinger-like (including the Klein-Gordon) equations, namely: numerical and analytical methods.

Numerical methods are used to find approximate solutions to Klein-Gordon equation with certain complicated potentials that would otherwise not be solved or for which an analytical solution would prove prohibitive and time-consuming to obtain. Also, nonlinear differential equations are easier solved by the use of numerical techniques. Numerical techniques are nowadays mostly performed on computers with the use of specialized softwares (example matlab, maple, mathematica, etc.). Some numerical techniques for solving the Klein-Gordon equation include: finite difference methods (fully implicit finite difference method (FIFDM) and exponential finite difference method (ExpFDM)), the inverse scattering method, the auxiliary equation method, Backlund transformation, the Wadatitrace method, Hirota bilinear forms, the spectral method, pseudo-spectral method, the tanh-sech method, the Adomian decomposition method, the sine-cosine method, Jacobi elliptical functions and the Riccati equation expansion method. (Aminataei & Soori 2013; El-Sayed 2003; Kaya & El-Sayed 2004; Sirendaoreji 2006 & 2007; Soori & Aminataei 2012; Wazwaz 2005 & 2008).

Despite the wide range of potentials for which the Klein-Gordon equation could be solved by use of numerical methods, numerical methods give merely approximate solutions and often obscure intrinsic features of the solution (Wang & Wang 2004:7), and a good numerical scheme is that which obtains results that are reasonably equivalent to closed form analytical solutions. Wong et al (2004) argue that "analytical solutions can serve as critical cross-references that help elucidate the fundamental and unexpected features of numerical solutions." Therefore, the practice is to compare results obtained numerically to those that have been found by analysis.

In the light of the above argument, it is reasonable to conclude that the best approach to finding solutions of equations is to first seek an analytical scheme that would address the problem, and to resort to a numerical scheme only if the search for an analytical solution is unfruitful.

EXISTING ANALYTICAL SCHEMES

Interestingly, many analytical methods have equally been developed over the years to solved the Klein-Gordon equation. Among those proposed in the literature are: homotopy analysis method (Alomari et *al* 2008), algebraic method (Akcay & Sever 2013), Frobenius method (Ita et *al* 2014), factorization method (Soylu et al 2008; Dong 2007), supersymmetric method (Onate et *al* 2016), Adomian decomposition method (ADM) presented by Deeba and Khun (2005) to solve Klein-Gordon equation of the form $U_{xx} + b_1 U + b_2 g(u) = f(x,t),$ and the modified Adomian decomposition method (MADM) employed by Wazwaz (2006) to solve non-linear Klein-Gordon equation. The following is a review of these analytical schemes:

A) Homotopy analysis method is a semi-analytical technique developed by Liao(Liao 2003) to solve nonlinear problems, and has no doubt had its share of success in the solution of many problems in science and engineering (Ayub *et al* 2003; Bataineh *et al* 2007).

It contains an auxiliary parameter *h* which provides a simple way to adjust and control the convergence region and the rate of convergence of the series solution and seeks to solve the differential equation of the form:

$$N[u(x,t)] = 0 \qquad 28$$

where *N* is a nonlinear operator, *x* and *t* denote the independent variables and *u* is an unknown function. HAM requires that one first constructs the so-called Zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)] \qquad 29$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is as earlier stated, an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function and $u_0(x,t)$ is an initial guess of $u(x,t)$.. Expanding $\phi(x,t;q)$ in Taylor series with respect to *q,* we have

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) q^m \qquad 30$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0} \qquad 31$$

Liao (2003) says that the convergence of the above series depends upon the auxiliary parameter *h,* and proved that if it is convergent at *q* = l, one obtains

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) \quad 32$$

which must be one of the solutions of the original nonlinear equation.

The homotopy analysis method was used by Alomari *et al* (2008) to solve various forms of the Klein-Gordon equation, including the linear form $u_{tt} - u_{xx} - u,$ subject to the initial conditions $u(x,0) = 1 + \sin(x), u_t(x,0) = 0,$ and they obtained the approximate solution $u(x,t) \cong \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!}...$ when $\hbar = -1,$ which in the limit of infinitely many terms yielded the closed form or exact solution

$$u(x,t) \cong \sin(x) + \cosh(t) \quad 33$$

While the homotopy method proves to be a good method of analysis of linear and nonlinear differential equations (especially of Klein-Gordon equation), it is only semi-analytic in the sense that $u_m(x,t)$ for $m \geq 1$ can only be solved by the use of symbolic computation softwares such as Maple and Mathematica.

B) Algebraic method: This method of analysis has been studied by Akcay and Server (2013). It seeks solution to the second order differential equation of the form

$$\frac{d^2}{ds^2} + \frac{(c_1 + c_2 s)d\psi}{s(1+c_3 s)ds} + \frac{1}{s^2(1+c_3 s)}\left[-\Lambda_1 s^2 + \Lambda_2 s - \Lambda_3 s\right]\psi = 0 \quad 34$$

where c_i and Λ_i are constants. Akcay and Sever (2012) have shown that differential equations of this parametric for-m have bound state solutions; and that when the parameter c_3 is not zero, the solutions are given in terms of Jacobi polynomials (the hyper-geometric functions), while when c_3 is zero they are given in terms of Laguerre polynomials (the confluent hype-geometric functions). The solutions are therefore studied in light of these conditions.

When c_3 is not zero the solutions can be written as

$$\psi(s) = (1 + c_3 s) - p_1 s^{q1} y(s) \quad 35$$

where p_1 and q_1 are some arbitrary constants and *y* is a new function to be determined. By substitution of (2.8) into (2.7), it is shown that one obtains

$$s(1+c_3 s)\frac{d^2 y(s)}{ds^2} + A(s)\frac{dy}{ds} + B(s)y(s) = 0 \quad 36$$

where

$$A(s) = 2_{q1}(1 + c_3 s) - 2p_1 c_3 s + c_1 + c_2 s \quad 37$$

and

$$B(s) = q_1(q_1 - 1)\frac{(1+c_3 s)}{s} - 2p_1 q_1 c_3 + \frac{p_1(p_1+1)c\,\overset{2}{_3}}{(1+c_3 s)}s + \frac{(c_1 + c_2 s)}{s(1+c_3 s)} \quad 38$$

$$\left[(p_1 - q_1)c_3 S\right] + \frac{(-\Lambda_1 s^2 + \Lambda_2 s - \Lambda_3}{s(1+c_3 s)}$$

Defining a new variable z = 1 + $2c_3s$ and writing equation (2.9) in terms of this new variable, Akcay and Sever obtained

$$(1-z^2)^2\frac{d^2 y}{dz^2} + (1-z^2)\left[\beta - \alpha - (a + \beta + 2)z\right]\frac{dy}{dz} + R(z)y = 0 \quad 39$$

where

$$\alpha = 2q_1 + c_1 - 1, \beta = -2p_1 - c_1 + \frac{c_2}{c_3} - 1, R(z) = r_1 z^2 + r_2 z + r_3 \quad 40$$

The coefficients of the polynomial *R* contain only the parameters c_i, Λ_i, q_1 and p_1. Akcay and Sever (2013) also showed that in order to have bound state solutions, the following set of conditions must be satisfied:

$$q_1 = \left(\frac{1-c_1}{2} \pm \sqrt{\left(\frac{1-c_1}{2}\right)^2 + \Lambda_3} \right) \quad 41$$

$$p_1 = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + H} \quad 42$$

where

$$D = \frac{c_2}{c_3} - c_1 - 1, \quad H = \frac{\Lambda_1}{c_3^2} + \frac{\Lambda_2}{c_3} + \Lambda_3 \quad 43$$

Using these equations for q_1 and p_1 Akcay and Sever rewrote eq. (2.12) as

$$(1-z^2)\frac{d^2}{dz^2} + [\beta - \alpha - (\alpha+\beta+2)]\frac{dy}{dz} + n(n+\alpha+\beta+1)y = 0 \quad 44$$

where n is a positive integer. The last equation is the well-known Jacobi's differential equation and its solutions are the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$.

Thus, they found that the case for $c_3 \neq 0$ gives the wave functions

$$\psi(s) = (1+c_3 S)^{-p_1} S^{q1} P_n^{(\alpha,\beta)}(1+2c_3 s) \quad 45$$

In addition, they obtained the following equation from which the energy eigenvalues can be evaluated:

$$(q_1 - p_1)^2 + \left(\frac{c_2}{c_3} + 2n - 1\right)(q_1 - p_1) + n\left(n + \frac{c_2}{c_3} - 1\right) = \frac{\Lambda_1}{c_3^2} \quad 46$$

Applying a similar analysis to the case when $c_3 = 0$, i.e. for the equation

$$\frac{d^2\psi}{ds^2} + \frac{(c_2 + +c_2 s}{s} + \frac{1}{s^2}\left[-\Lambda_1 s^2 + \Lambda_2 s - \Lambda_3\right]\psi = 0, \quad 47$$

they obtained the wavefunction in terms of Laguerre polynomials $L_n^k(Z)$, and expressed it as

$$\Psi(s) = \exp(-p_2 S) S^q L_n^k([2p_2 - c_2]S) \quad 48$$

In this case, their expression for the energy eigenvalues was

$$c_1 p_2 - q_2(c_2 - 2p_2) - \Lambda_2 = n(c_2 - 2p_2) \quad 49$$

C) Frobenius method: This method of analysis finds the solution of a differential equation in the form of series, either a whole series, a Laurent series, or a series involving contribute exhibitors (Ita *et al* 2014). The only difference among these situations is the property of regularity of the equation coefficients. The method requires us to put the Klein-Gordon equation in the form

$$y''(x) + p(x)y'(x) + Q(x)y(x) = 0 \quad 50$$

Suppose a regular singular point x_0 singular functions *P(x)* and *Q(x)* and using the Fuck's theorem, the solutions of the differential equation can be written in the form:

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r} \quad 51$$

The indicial equation is obtained for

r(r-1)+P(0)r+Q(0)=0 52

From the indicial equation, the values for r are obtained, and for each value of r, the values of a_k are determined, and then the solutions of the differential equation can be found.

The Klein-Gordon equation, owing to its significance in the field of theoretical high energy physics and other areas of Physics and applied mathematics, has enjoyed a surge of interest among researchers in recent time. Literature is replete with proposed solutions of the equation for different potentials of interest by the use of various analytical and numerical techniques.

The radial KG equation has been solved by Antia *et al* (2014) for a combination of two unequal potentials: Hulthen scalar and Yukawa vector given by

$$S(r) = -\frac{S_0 e^{-zar}}{1-e^{-zar}} - \frac{S_1 e^{-ar}}{r} \text{ and } V(r) = -\frac{V_o e^{-2ar}}{1-e^{-2ar}} - \frac{v_1 e^{-ar}}{r} \qquad 53$$

respectively. They obtained an expression for the energy spectrum of a system described by the equation and showed that for different limiting cases of their potential, the energy spectrum changed corresponding to these cases. Furthermore, they obtained the radial wavefunction for the system as

$$R(s) = N_{n1} S^{\frac{\mu}{2}} (1-S) \frac{1+v}{2} P_n^{(\mu,v)}(1-2s) \qquad 54$$

Where

$$\mu - 2\sqrt{-\sqrt{\frac{(E_{nl}^2 - m^2)}{4\alpha^2}}}, \qquad 55$$

and

$$v = 2\sqrt{\frac{1}{4} + \left[\frac{S_0^2 - V_0^2}{4\alpha^2}\right]} + \frac{1}{\alpha}(S_0 S_1 - V_0 V_1) + S_1^2 + 1(1+1) \qquad 56$$

The beauty in their work lies in the ease with which they manipulated this combined potential to obtain other kinds of potentials, resulting in eigenvalues and eigenfunctions corresponding to those potentials. For example, when they set $V_0 = 0$, $S_0 = 0$, $S_1 = V_1$ and $\alpha \to 0$ in (2.26), they obtained the well-known Coulomb potential (Greiner 2000) $V(r) = -\frac{v_1}{r}$. Corresponding to this potential are the eigenvalues expressed as

$$E_{nl}^2 - m^2 = -\left[\frac{(E_{nl} + m)V_1}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + 1(1+1)}\right)}\right] \qquad 57$$

Also recently, Egrifes and Sever found solution for the (l+l)-dimensional time-independent KG equation for the generalized Hulthen potential

$$S_q(x) = -S_0 \frac{e^{\alpha x}}{1 - qe^{-\alpha x}} \qquad 58$$

obtaining the exact energy eigenvalues as

$$E_n(q,\alpha,S_0) = \pm \frac{1}{4qk_n(q,\alpha,S_0} \sqrt{(k_n^2(q,\alpha,S_0) - 4S_0^2((2S_0 + 4qm)^2 - k_n^2(q,\alpha,S_0)} \qquad 59$$

where

$$k_n(q,\alpha,S_0) = \sqrt{q^2\alpha^2 + 4S_0^2} + q\alpha(2n+1) \qquad 60$$

They referred to *q* as the deformation parameter which determines the shape of the potential, and noted that the potential transformed for specific values of *q.* Hence, for *q* = 0, one obtained the exponential potential, for *q=l,* the generalized Hulthen potential transformed to the standard Hulthen potential, and for *q*=-1, we get the Wood-Saxon potential. The KG equation has equally been solved by other authors for some of the above named potentials.

For the one-dimensional Rosen-Morse type potential given as

$$V(x) = -V_1 \sec h_q^2(\alpha x) - V_2 \tanh_q(\alpha x), \qquad 61$$

where $1 \ge q \ge -1$

is the deformation parameter, Akbarieh and Motavali obtained the exact energy equation for the s-wave bound state of the Klein-Gordon equation as

They noted that the deformed hyperbolic functions which they gave as

$$\sinh_q x = \frac{e^x - qe^{-x}}{2}, Cosh_q x = \frac{e^x + qe^{-x}}{2}, \tanh_q x = \frac{\sinh_q x}{\cosh_q x} h$$

$$\coth_q x = \frac{\cosh_q x}{\sinh_q x}, \sec h_q x = \frac{1}{\cosh_q x}, \cos ech_q x = \frac{1}{\sinh_q x} \qquad 62$$

allowed them to investigate the effect of the deformation parameter *q* on the energy levels and the corresponding wave functions. They also explained in their work, like
Egrifes and Sever did, that their deformed Rosen-Morse potential is transformed into the standard Rosen-Morse potential form for *q = l,* into the standard Eckart potential for *q* =l, and into the exponential potential for *q* = 0.
Solution of the equation has also been found by Laachir and Laaribi (2014) for the q-deformed Morse potential

$$V(x) = V_0(e^{-2\alpha x} - 2qe^{-\alpha x}) \qquad 63$$

and given in terms of Laguerre polynomials as

$$\psi(s) = B_n S^\gamma e^{-\frac{s}{2}} L_h^{2\gamma}(s) \qquad 64$$

where $\gamma = \beta - n - \frac{1}{2}$ and $\beta = \frac{qv}{2}$. When normalized, Eq. (2.38) becomes

$$\psi(S) = \sqrt{\frac{n!}{(n+2\gamma}} S^\gamma e^{-\frac{s}{2}} L_n^{2\gamma}(S) \qquad 65$$

The energy eigenvalues obtained were expressed as

$$E_n^2 - m^2 = -\frac{8(E+m)}{v^2} V_0 \in_n \qquad 66$$

METHODS

While the choice between numerical and analytical tools remains largely based on the difficulty level of the problem, as we would normally be inclined to use analytical method since it reveals more the physics of the problem, and go for numerical only when the problem is beyond the use of analytical methods, the choice between two or more analytical schemes is not so easy, as all available analytical schemes seem to have their advantages and disadvantages. However, one analytical method whose use has dominated popular literature is the Nikiforov-Uvarov method.

THE NIKIFOROV-UVAROV FORMALISM

The Nikiforov-Uvarov (NU) method is based on solving the hypergeometric-type second-order differential equations by means of the special orthogonal functions (Nikiforov & Uvarov 1988). For a given potential, the Schrodinger or Schrodinger-like equations in spherical coordinates are reduced to a generalized equation of hypergeometric-type with an appropriate coordinate transformation r→s and then they can be solved systematically to find the exact or particular solutions. The main equation which is closely associated with the method is given by Nikiforov & Uvarov in the following form:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\partial(s)}{\sigma^2(s)}\psi(s) = 0, \qquad 67$$

where σ(s) and $\tilde{\sigma}(s)$ are polynomials of at second-degree, $\tilde{\tau}(s)$ is a first-degree polynomial and Ψ(s) is a function of the hypergeometric-type.

By taking $\psi(s) = \phi(s)y(s)$ and choosing an appropriate function $\phi(s)$, Eq. (3.1) is reduced to a comprehensible form;

$$y''(s) = \left(2\frac{\phi'(s)}{\phi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)}\right) y'(s) + \left(\frac{\phi''(s)}{\phi(s)} + \frac{\phi'(s)\tilde{\tau}(s)}{\phi(s)\,\sigma(s)} + \frac{\partial(s)}{\sigma^2(s)}\right) y(s) = 0 \qquad 68$$

The coefficient of y'(s) is taken in the form $\tau(s)/\sigma(s)$, , where $\tau(s)$ is a polynomial of degree at most one, i.e.,

$$2\frac{\phi'(s)}{\phi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)} = \frac{\tau(s)}{\sigma(s)}, \qquad 69$$

and hence the most regular form is obtained as follows,

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \qquad 70$$

where

$$\pi(s) = \frac{1}{2}\left[\tau(s) - \tau(s)\right]. \qquad 71$$

The most useful demonstration of eq. 71 is

$$\tau(s) = \tau(s) + 2\pi(s). \qquad 72$$

The new parameter 71(5) is a polynomial of degree at most one. In addition, the term $\phi''(s)/\phi(s)$ which appears in the coefficient of y(s) in Eq. (3.2) is arranged as follows:

$$\frac{\phi''(s)}{\phi(s)} = \left(\frac{\phi'(s)}{\phi(s)}\right)' + \left(\frac{\phi'(s)}{\phi(s)}\right)^2 = \left(\frac{\pi(s)}{\sigma(s)}\right)' + \left(\frac{\pi(s)}{\sigma(s)}\right)^2. \qquad 73$$

In this case, the coefficient of y(s) is transformed into a more suitable form by taking the equality given in Eq. (3.4)

$$\frac{\phi''(s)}{\phi(s)} = \frac{\phi'(s)}{\phi(s)}' + \frac{\tilde{\sigma}(s)}{\sigma(s)} = \frac{\bar{\sigma}(s)}{\sigma^2(s)} \qquad 74$$

where

$$\bar{\sigma}(s) = \tilde{\sigma}(s) + \pi^2(s) + \pi(s)\left[\tilde{\tau}(s) - \sigma'(s)\right] + \pi'(s)\sigma(s) \text{ -} \qquad 75$$

Substituting the right-hand sides of Eq. 69 and Eq. 74 into Eq. 68, an equation of hypergeometric-type is obtained as follows:

$$y''(s) + \frac{\tau(s)}{\sigma(s)} y'(s) + \frac{\sigma(s)}{\sigma^2(s)} y(s) = 0 \qquad 76$$

As a consequence of the algebraic transformations mentioned above, the functional form of Eq. 67 is protected in a systematic way. If the polynomial $\bar{\sigma}(s)$ in Eq. 76 is divisible by $\sigma(s)$, i.e.,

$$\sigma(s) = \lambda\sigma(s), \qquad 77$$

where λ is a constant, Eq. (3.10) is reduced to an equation of hyper--geometric-type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \qquad 78$$

and so its solution is given as a function of hypergeometric-type. To determine the polynomial $\pi(s)$, Eq. 75 is compared with Eq. 77 and then a quadratic equation for $\pi(s)$ is obtained as follows,

$$\pi(s) + \pi(s)\left[\tilde{\tau}(s) - \sigma'(s)\right] + \tilde{\sigma}(s) = k\sigma(s) = 0, \qquad 79$$

where

$$k = \lambda - \pi'(s) \qquad 80$$

The solution of this quadratic equation for $\pi(s)$ yields the following equation:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \qquad 81$$

In order to obtain the possible solutions according to the plus and minus signs of Eq. 81, the parameter *k* within the square root sign must be known explicitly. To provide this requirement, the expression under the square root sign has to be the square of a polynomial, since $\pi(s)$ is a polynomial of degree at most one. In this case, an equation of the quadratic form is available for the constant *k.* Setting the discriminant of this quadratic equal to zero, the constant *k* is determined clearly. After determining *k,* the polynomial $\pi(s)$ is obtained from Eq. 81, and then $\tau(s)$ and λ are also obtained by using Eq. 71 and Eq. 80, respectively.

A common trend that has been followed to generalize the solutions of Eq. 78 is to show that all the derivatives of hypergeometric-type functions are also of the hypergeometric-type. For this purpose, Eq.78 is differentiated by using the representation $v_1(s) = y'(s)$

$$\sigma(s)v_1''(s) + \tau_1(s)v_1'(s) + \mu_1 v_1(s) = 0 \qquad 82$$

where $\tau_1(s) = \tau(s) + \sigma'(s)$ and $\mu_1 = \lambda + \tau'(s)$, $\tau_1(s)$ is a polynomial of degree at most one and μ_1 is a parameter that is independent of the variable *s.* It is clear that Eq. 82 is an equation of hyper-geometric-type. By taking $u(s) = y''(s)$ as a new representation, the second derivative of Eq. 81 becomes:

$$\sigma(s)u_2''(s) + \tau_2(s)u_2'(s) + \mu_2\mu_2(s) = 0, \quad 83$$

where

$$\tau_2(s) = \tau_1(s) + \sigma'(s) = \tau(s)2\sigma'(s), \quad 84$$

$$\mu_2 = \mu_1 + \tau_1'(s) = \lambda + 2\tau'(s)\sigma''(s) \quad 85$$

In a similar way, an equation of hyper-geometric-type can be constructed as a family of particular solutions of Eq. (3.12) by taking $v_n(s) = y^{(n)}(s)$;

$$\sigma(s)v_n''(s) + \tau_n(s)v_n'(s) + \mu_n u_n(s) = 0 \quad 86$$

and here the general recurrence relations for $\tau_n(s)$ and μ_n are found as follows, respectively,

$$\tau_n(s) = \tau(s) + n\sigma'(s) \quad 87$$

$$\mu_n \lambda + n\tau'(s) = \frac{n(n-1)}{2}\sigma''(s), \quad 88$$

and then Eq. 76 has a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. To obtain an eigenvalue solution through the NU method, the relationship between λ and λ_n must be set up by means of Eq. 80 and 88. $y_n(s)$ is the hyper-geometric-type function whose polynomial solutions are given by the Rodrigues relation

$$y_n(s) = \frac{B_n d^n}{\rho(s)ds^n}\left[\sigma^n(s)p(s)\right], \quad 89$$

where B_n is a normalization constant and the weight function $\rho(s)$ must satisfy the condition below:

$$(\sigma(s)\rho(s))' = \tau(s)\rho(s) \quad 90$$

It could be facilitative to introduce a "guide" to figure out the solution of Klein-Gordon equation in a faster way. To obtain the unknown radial function R(r) in Eq. (1.25) and the energy eigenvalue *E* of the Klein-Gordon equation by means of the NU method, we shall look at the following ten-step guide;

1) Reduce the differential equation that satisfies the KG equation into the differential equation given in Eq. (3.1),

2) Compare each equations and determine the values of polynomials $\tilde{\tau}_n(s), \sigma(s)$ and $\tilde{\sigma}(s)$. this stage, don't forget to make some abbreviations in the original differential equation,

3) Arrange the polynomial $\pi(s)$ given in Eq. (3.15) by inserting the polynomials $\tilde{\tau}_n(s), \sigma(s)$ and $\tilde{\sigma}(s)$ we have found in the second stage and compose an equation of quadratic form under the square root sign of the $\pi(s)$,

4) Set up the discriminant of this quadratic equal to zero, using the expression $\Delta = b^2 - 4ac = 0$ and find two roots regarding with the *k,* i.e., $k\pm$,

5) Substitute these values of *k* into the $\pi(s)$ and obtain the four possible forms of *n(s).* Now we have two forms of the $\pi(s)$ for k_+ and two forms for *k-.*

6) Try to find a negative derivative of the $\tau(s)$ given in Eqn. 72 using the four forms of the $\pi(s)$ and keep this form to use it in the further stages because that would be physically valid.

7) Recall Eqn. 80 for λ and Eq 88 for λ_n, and compare them with each other, i.e., $\lambda = \lambda_n$, and so it would the energy spectrum.

8) Insert the values of $\sigma(s)$ and $\pi(s)$ into Eq. (3.4), so the result would be the functional -form of $\phi(s)$,

9) Satisfy Eq. 91 with the weight function *p(s)* and obtain the hypergeometric-type function $y_n(s)$ which can be given by the Rodrigues relation in Eq. 90.

10) Combine $\phi(s)$ and $y_n(s)$ to form $\psi(s)$, , and so it would be the radial wave function R_n.

RESULTS AND ANALYSIS

Recall the radial part of the Klein-Gordon equation

$$\left[\frac{d^2}{dr^2}-\frac{\lambda}{r^2}-(\in+M)V(r)+\in^2 M^2\right]R(r)=0 \qquad 91$$

With the potential $V(r)=\frac{2\alpha\sqrt{b}}{3r^2}$ explicitly indicated, Eq. 91 can be written as

$$\left[\frac{d^2}{dr^2}-\frac{\lambda}{r^2}-(\in+M)\frac{2\alpha\sqrt{b}}{3r^2}+\in^2-M^2\right]R(r)=0. \qquad 92$$

This equation can be further arranged as

$$R''+\left[(\in^2-M^2)r^2-\frac{2}{3}(\in+M)\alpha\sqrt{b}-\lambda\right]\frac{R(r)}{r^2}=0 \qquad 93$$

To make our mathematics comparable with Eq. 67, we choose a function in the form of $R(r)\equiv\psi(s),$ where the transformation $r\to s$ is valid. Therefore, Eq. 93 simplified yields

$$\psi''(s)+(-\eta^2r^2-\zeta^2-\lambda)\frac{\psi(s)}{s^2}=0 \qquad 94$$

with

$$\in^2-M^2=-\eta^2,\frac{2}{3}(\in+M)\alpha\sqrt{b}=\zeta^2,\quad \lambda=l(l+1) \qquad 95$$

Eq. 94 is now comparable with Eq. 67 and then the following expressions are obtained:

$$\tilde{\sigma}=-\eta^2S^2-\zeta^2-\lambda\sigma=s \qquad 96$$

We are able to find out possible solutions of the polynomial

$$\psi''(s)+(-\eta^2r^2-\zeta^2-\lambda)\frac{\psi(s)}{s^2}=0 \qquad 97$$

$$k^2-(4\eta^2\zeta^2+4\eta^2\lambda+\eta^2)=0 \qquad 98$$

$$k_{\pm}=\pm2\sqrt{\eta^2}\sqrt{\zeta^2}+(l+1/2)^2 \qquad 99$$

When the two values of k given in Eq. 99 are substituted into Eq. 97, hence the four possible forms of $\pi(s)$ are obtained as:

$$\pi(s)=\begin{cases}\left(\frac{1}{2}\pm\sqrt{\eta^2s}+\sqrt{\zeta^2+(l+1/2)^2,}\right) & k_+=2\sqrt{\eta^2}\sqrt{\zeta^2+(l+1/2)^2}\\ \frac{1}{2}\pm\left(\sqrt{\eta^2}s-\sqrt{\zeta^2+(l+1/2)^2}\right), & k_-=-2\sqrt{\eta^2}\sqrt{\zeta^2+(l+1/2)^2}\end{cases} \qquad 100$$

In order to make derivative of the polynomial $\tau(s)$ to be negative, we must select the most suitable form of the polynomial $\pi(s)$. Therefore, the most suitable expression of $\pi(s)$ is chosen as:

$$\pi(s)=\frac{1}{2}-\left(\sqrt{\eta^2}s-\sqrt{\zeta^2+(l+1/2)}\right) \qquad 101$$

Corresponding to $k_-=-2\sqrt{\eta^2}\sqrt{\zeta^2+(l+1/2)^2}$. By using $\pi(s)$ given in Eq 101 and remembering that $\tilde{\tau}=0,$ we can obtain the expression:

$$\tau(s)=1-2\sqrt{\eta^2}S-\sqrt{\zeta^2+(l+1/2)^2} \qquad 102$$

And the derivative of this expression would be negative i.e., $\tau' = -2\sqrt{\eta^2} < 0$. The expression $\lambda = k\_ + \pi'(s)$ in Eq. (3.14) and $\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s)$ in Eq. (3.23) are obtained as follows

$$\lambda = -\sqrt{\eta^2}\left(2\sqrt{\zeta^2 + (l+1/2)^2} + 1\right) \quad 103$$

$$\lambda = 2N\sqrt{\eta^2} \quad 104$$

When we compare these expressions, $\lambda = \lambda_n$, and inserting the values of η and ξ we can obtain the exact energy eigenvalues of radial Klein-Gordon equation with a central potential

$$-\sqrt{\eta^2}\left(2\sqrt{\zeta^2 + (l+1/2)^2} + 1\right) = 2N\sqrt{\eta^2} \quad 105$$

$$\sqrt{\zeta^2 + (l+1/2)^2} + \frac{1}{2} = -N \quad 106$$

$$\zeta^2 = (N+1/2)^2 - (l+1/2)^2 \quad 107$$

$$\frac{2}{3}(\in + M)\alpha\sqrt{b} = (N+1/2)^2 - (l+1/2)^2 \quad 108$$

$$\in_{N1} = \frac{3}{2}\frac{\left[N^2 + N) - (l^2 + l)\right]}{\alpha\sqrt{b}} - M \quad 109$$

Where N denotes the radial quantum number.

To find the corresponding eigenfunctions for the radial equation, the polynomial solution of the hypergeometric-type function $y_n(s)$ depends on the determination of the weight function $\rho(s)$. Thus, using Eq. (3.4), we obtain

$$\frac{\phi'(s)}{\phi(s)} = \frac{d\phi(s)}{ds}\frac{1}{\phi(s)} = \frac{\frac{1}{2} - \sqrt{\eta^2}s + \sqrt{\zeta^2 + (l+1/2)^2}}{s} \quad 110$$

$$\frac{d\phi(s)}{ds}\frac{1}{\phi(s)} = \frac{\frac{1}{2} - \sqrt{\eta^2}s + \sqrt{\zeta^2 + (l+1/2)^2}}{s} - \sqrt{\eta^2} \quad 111$$

$$\int \frac{d\phi(s)}{\phi(s)} = \int \left(\frac{\frac{1}{2} - \sqrt{\eta^2}s + \sqrt{\zeta^2 + (l+1/2)^2}}{s} - \sqrt{\eta^2}\right)ds \quad 112$$

$$\log\phi(s) = \left(\frac{1}{2} + \sqrt{\zeta^2 + (l+1/2)^2}\right)\log s - \sqrt{\eta^2 S} \quad 113$$

$$\phi(s) = S^{\frac{1}{2}} + \sqrt{\zeta^2 + (l+1/2)^2}\exp(-\sqrt{\eta^2 s}) \quad 114$$

On the other hand, to find the solution for $y_n(s)$ we should first obtain the weight function $\rho(s)$ given in Eq. 90 can be written in a simple form and obtained as

$$\frac{d\rho(s)}{ds}\frac{1}{\rho(s)} = \frac{\tau(s)-\sigma'(s)}{\sigma(s)} = \frac{1-2\sqrt{\eta^2}s+2\sqrt{\zeta^2+(l+1/2)^2}-1}{s} \quad 115$$

$$\frac{d\rho(s)}{ds}\frac{1}{\rho(s)} = \frac{2\sqrt{\zeta^2+(1+1/2)^2}}{s} - 2\sqrt{\eta^2} \quad 116$$

$$\int \frac{d\rho(s)}{\rho(s)} = \int\left(\frac{2\sqrt{\xi^2+(1+1/2)^2}}{s} - 2\sqrt{\eta^2}\right)ds \quad 117$$

$$\log\rho(s) = \log 2\sqrt{\zeta^2+(l+1/2)^2} - 2\sqrt{\eta^2 s} \quad 118$$

$$\rho(s) = S^2\sqrt{\xi^2+(1+1/2)^2}e^{-2}\sqrt{\eta^2}s \quad 119$$

Substituting $\rho(s)$ into Eq. 89 allows us to obtain the polynomial $y_n(s)$ as follows:

$$y_n(s) = B_n e^2\sqrt{\eta^2}sS^{-2\sqrt{\xi^2+(l+1/2)^2}}\frac{d^n}{ds^n}\left(e^{-2}\sqrt{\eta^2 s}S^{n+2}\sqrt{\xi^2+(l+1/2)^2}\right) \quad 120$$

It is shown from the Rodrigues' formula of the associated Laguerre polynomials

$$L_n^{\sqrt{\xi^2+(l+1/2)^2}}\left(2\sqrt{\eta^2}\right) = \frac{1}{n!}e^{2\sqrt{\eta^2 s}}S^{-2-2\sqrt{\xi^2+(1+1/2)^2}}\frac{d^n}{ds^n}\left(e^{-2\sqrt{\eta^2 s}}S^{n+2\sqrt{\xi^2+(1+1/2)^2}}\right) \quad 121$$

Where 1/n! = B_n. But $y_n(S) \equiv L_n^{\sqrt{\xi^2+(l+1/2)^2}}\left(2\sqrt{\eta^2}\right)$. By using $\psi(S) = \phi(S)y_n(S)$, we obtain

$$\psi(S) = N_{nl}S^{\frac{1}{2}\sqrt{\xi^2+(1+1/2)^2}}e^{-\sqrt{\eta^2}S}L_n^{\sqrt{\xi^2+(l+1/2)^2}}\left(2\sqrt{\eta^2}\right) \quad 122$$

Where N_{nl} is a normalization constant and the Ψ(s) represents the radial wavefunction R(r) through the transformation $s \rightarrow r$.

DISCUSSION

An exact solution of the Klein-Gordon equation is not practical except for the simplest of potential energy eigenfunctions. In most cases of practical interest, we can just settle for an approximate solution. To overcome various types of problems in quantum mechanics, we have to apply several methods of approximation to solve Schrodinger, Klein-Gordon, Dirac and other Schrodinger-like equations appropriately. One of such method is the Nikiforov-Uvarov method introduced by A. F. Nikiforov and V. B. Uvarov. This method has the advantage that it can be used to solve the Klein-Gordon equation with a nonzero angular momentum, unlike other analytical methods.

The solution of the Klein-Gordon equation for an inverse square potential is similar to the solution of the Schrodinger equation for an inverse fourth power potential (Sharma *et al.,* 2010). Our solution is comparable to that obtained by Yasuk *et al* (2006), where the energy eigenvalue depends on the radial quantum number *N,* and our expression for eigenfunctions is in terms of the Laguerre polynomial.

CONCLUSION

This work presents the Nikiforov-Uvarov method to the calculation of the non-zero angular momentum solutions of the relativistic Klein-Gordon equation. Exact eigenvalues and eigen functions for the Klein-Gordon equation in the presence of a central inverse square potential is derived. The radial wave functions are found in terms of Laguerre polynomials. The methods presented in this study is general and worth extending to the solution of other interaction problems.

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