# A MODIFIED DESIGN STRESS EQUATION BASED ON MAXIMUM SHEAR STRESS THEORY

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## ABSTRACT

The aim of this study was to formulate a new design stress equation for the analysis of two-dimensional plane elements using the maximum shear stress yield criterion known as the Tresca yield criterion. The existing Tresca yield equation was modified for a two-dimensional plane element to obtain a new general applied stress equation in terms of yield stress and stress factor. To get the specific applied stress equation for the twelve plate types considered, the polynomial shape parameters were solved to obtain the specific stress factor equation. To get the numerical value of the stress factor, a 1m by 1m mild steel of yield stress 250MPa, with deflection values taken at the point of maximum deflection, was analyzed. The numerical values of the stress factor for each plate type obtained were observed to be all less than unity, giving rise to a very high applied stress for plate types with no free edge and a moderately high stress for plate types with one free edge. This revealed the weakness of the existing yield criterion that may lead to failure. To improve this, a correction factor to take care of this limitation was introduced to the stress factor to have a factor of safety that will result in a design stress which is less than the yield stress of the material, by so doing ensuring safety. Therefore, the new design stress equation was found appropriate for predicting stress for the design of plane rectangular plates with one free edge, while this yield criterion is not advisable for use with plates without a free edge.

## 1.0: INTRODUCTION

Yield criteria are specific to different types of materials and are used to predict when yielding will occur under various loading conditions. Yield criteria are mathematical expressions that describe the relationship between the applied stress in a material and the beginning of plastic deformation (Chatti, 2019). To avoid permanent deformation in structures leading to failure, engineers consider yield criterion very important when designing structures or components. Yielding occurs when a material's stress exceeds a certain threshold, known as the yield strength. According to Muhammad Rehan (2023), yield criteria help engineers predict the point at which materials will start to deform plastically. The knowledge of the basic strength of materials has shown that beyond the yield point, the material will deform plastically (Onwuka, 2001; Ryder, 1961; Khurmi & Khurmi, 2013; Ugural, 1999; Singh, 2009).

Callister & Rethwisch (2020) stated that yield criterion analysis helps to optimize material utilization and decrease superfluous material expenses.
According to Ross (1987), the problem with maximum principal stress theory, maximum principal strain energy theory, and total strain energy theory is that they did not consider the effect of hydrostatic stresses in the failure of a material. He further stated that, based on experimental observations, whether or not a solid piece of material is a soft and ductile or hard and brittle material, the materials do not suffer an elastic breakdown in the condition when it is subjected to a large uniform external pressure, irrespective of the fact that the yield stress is grossly exceeded. Yu & Xue (2022) said that Von Mises claims that yielding happens when the corresponding stress crosses a certain level. Benham & Warnock, (1976), state that a yield criterion such as that of Tresca or von Mises which is based on principal stress difference would seem to be the most logical.

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They further stated that the maximum shear stress theory, even though not quite so consistent as von Mises, gives fairly reasonable predictions and is sometimes used in design by virtue of its simpler mathematical form. Also, Ross (1987) said that there are no shear stresses in a hydrostatic stress condition, and this accounts for why low-strength materials survive intact under a large value of water pressure. So, it can be concluded that for elastic failure to take place, the material must distort (change shape), and for this to happen, shear stress must exist. Roostaei & Jahed (2022) stated that maximum shear stress criterion is frequently used to describe brittle materials and states that yield occurs when the maximum shear stress exceeds a predetermined level and reaches maximum shear stress in a uniaxial tension specimen at yield. Also, Moy (1981), Lee (1977), and Save et al, 1997;), all agreed that the Tresca and Von Mises yield criteria are mostly used for metallic materials because, they considered the fact that shear controls yield, and they gave an accurate prediction of the onset of yield in ductile materials. With this fact in mind,

Ibearugbulem et al (2020) and Adah et al. (2023) proposed a yield criterion based on maximum strain energy theory as Equation (1) involving shear share stress, $\tau_{xy}$.

$$\sigma_x{}^2 + 2\tau_{xy}{}^2 + \sigma_y{}^2 + 2\upsilon\left(\tau_{xy}{}^2 - \sigma_x\sigma_y\right) = f_y{}^2 \qquad (1)$$

Where $\upsilon$ is the Poisson ratio, $\sigma_x$ and $\sigma_y$ are principal stresses along x- and y- directions respectively, and $f_y$ is the yield stress of the material.

The challenge with the maximum shear stress theory of yield is that even though its results are better than others except for von Mises, the experimental data still defer in some magnitude from the predicted result and are not quite consistent. The purpose of this work was to correct this limitation and develop a design stress equation with a factor of safety that can predict adequately the safe stress to avoid failure of plane structures.

## 2.0: Derivation of Applied Stress Equation

The maximum shear stress theory generally known as the Tresca yield criterion states that elastic failure will take place when the maximum shear stress at a point equals the maximum shear stress obtained in a specimen, made from the same material, to the simple uni-axial test ie

$$\sigma_x - \sigma_z = f_y \qquad (2)$$

In the 2-D case, $\sigma_z = 0$     hence

$$\sigma_x - \sigma_y = f_y \qquad (3)$$

Recall the stress in a plane along x- and y- axes is given as

$$\sigma_x = \frac{-Ez}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right) \qquad (4a)$$

$$\sigma_y = \frac{-Ez}{1-\upsilon^2}\left(\upsilon\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (4b)$$

But the displacement shape function is expressed as (Ibearugbulem et al. 2014)

$$w = Ah \qquad (5)$$

Substituting eqn (5) and the non-dimensional parameters; x = aR, y = bQ, z = St, into equation (4) yields

$$\sigma_x = \frac{-EAz}{(1-\upsilon^2)a^2}\left(\frac{\partial^2 h}{\partial R^2} + \frac{\upsilon}{z^2}\frac{\partial^2 h}{\partial Q^2}\right) \qquad (6)$$

$$\sigma_y = \frac{-EAz}{(1-\upsilon^2)a^2}\left(\upsilon\frac{\partial^2 h}{\partial R^2} + \frac{1}{z^2}\frac{\partial^2 h}{\partial Q^2}\right) \qquad (7)$$

Where the aspect ratio $S = \frac{b}{a}$,  a and b are the plate dimensions along the x- and y- axes respectively.

substitute Eqn 6 and Eqn (7) into Eqn (3) yields

$$\frac{-EAz}{(1-\upsilon^2)a^2}\left(\frac{\partial^2 h}{\partial R^2} + \frac{\upsilon}{z^2}\frac{\partial^2 h}{\partial Q^2}\right) - \frac{EAz}{(1-\upsilon^2)a^2}\left(\upsilon\frac{\partial^2 h}{\partial R^2} + \frac{1}{z^2}\frac{\partial^2 h}{\partial Q^2}\right) = f_y \qquad (8)$$

$$\frac{-EAZ}{(1-\upsilon^2)a^2}\left[\left(\frac{\partial^2 h}{\partial R^2} + \frac{\upsilon}{z^2}\frac{\partial^2 h}{\partial Q^2}\right) + \left(\upsilon\frac{\partial^2 h}{\partial R^2} + \frac{1}{z^2}\frac{\partial^2 h}{\partial Q^2}\right)\right] \leq f_y \qquad (9)$$

$$\frac{-EAz}{(1-\upsilon^2)a^2}\left[n_{1} + n_2\right] = f_y \qquad (10)$$

where

$$n_1 = \frac{\partial^2 h}{\partial R^2} + \frac{\upsilon}{z^2}\frac{\partial^2 h}{\partial Q^2}, \ n_{2} = \upsilon\frac{\partial^2 h}{\partial R^2} + \frac{1}{z^2}\frac{\partial^2 h}{\partial Q^2} \qquad (11)$$

From eqn   (3)

$$\sigma_x\left(1 - \frac{\sigma_y}{\sigma_x}\right) = f_y \qquad (12)$$

$$\sigma_x\left(1 - m_1\right) = f_y \qquad (13)$$

where

$$m_{1} = \frac{\sigma_y}{\sigma_x} \qquad (14)$$

Substitute eqn (4) into eqn (14) yield

$$m_1 = \cfrac{\cfrac{-\epsilon\,\text{A}z}{(1-\upsilon^2)a^2}\left[\upsilon\dfrac{\partial^2 h}{\partial R^2} + \dfrac{1}{\mathcal{Z}^2}\dfrac{\partial^2 h}{\partial Q^2}\right]}{\cfrac{-\epsilon\,\text{A}z}{(1-\upsilon^2)a^2}\left[\dfrac{\partial^2 h}{\partial R^2} + \dfrac{\upsilon}{\mathcal{Z}^2}\dfrac{\partial^2 h}{\partial Q^2}\right]}$$

$$m_1 = \cfrac{\upsilon\dfrac{\partial^2 h}{\partial R^2} + \dfrac{1}{\mathcal{Z}^2}\dfrac{\partial^2 h}{\partial Q^2}}{\dfrac{\partial^2 h}{\partial R^2} + \dfrac{\upsilon}{\mathcal{Z}^2}\dfrac{\partial^2 h}{\partial Q^2}} \qquad (15)$$

This reduces to

$$m_1 = \frac{n_2}{n_1} \qquad (16)$$

From eqn (13)

$$\sigma_x = \frac{f_y}{1 - m_1} \qquad (17)$$

Substituting Equation (16) into Equation (17) yields

$$\sigma_x = \frac{f_y}{\left(1-\frac{n_2}{n_1}\right)} = \frac{f_y}{F_T} \qquad (18)$$

$$\sigma_x = \frac{f_y}{F_T} \qquad (19)$$

Where the Tresca stress factor $F_T$ is

$$F_T = 1 - \frac{n_2}{n_1} \qquad (20)$$

## 2.1 Evaluation of Stress Factor Equations

Polynomial displacement shape profiles for the different plate types in Table 1 (Ibearugbulem et al. 2014) will be used to evaluate n1 and $n_2$ leading to stress factor.

**Table 1:** The polynomial displacement shape profiles

| Plate Type | Shape Profile, h |
|---|---|
| CCSS | $(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$ |
| CCCC | $(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$ |
| SSSS | $(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)$ |
| CSSS | $(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$ |
| CSCS | $(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$ |
| CCCS | $(1.5R^2 - 2.5R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$ |
| SSFS | $(R-2R^3+R^4)(\frac{7}{3}Q-\frac{10}{3}Q^3+\frac{10}{3}Q^4-Q^5)$ |
| SCFS | $(1.5R^2-2.5R^3+R^4)(\frac{7}{3}Q-\frac{10}{3}Q^3+\frac{10}{3}Q^4-Q^5)$ |
| CSFS | $(R-2R^3+R^4)(2.8Q^2-5.2Q^3+3.8Q^4-Q^5)$ |
| CCFS | $(1.5R^2-2R^3+R^4)(2.8Q^2-5.2Q^3+3.8Q^4-Q^5)$ |
| SCFC | $(R^2-2R^3+R^4)(\frac{7}{3}Q-\frac{10}{3}Q^3+\frac{10}{3}Q^4-Q^5)$ |
| CCFC | $(R^2-2R^3+R^4)(2.8Q^2-5.2Q^3+3.8Q^4-Q^5)$ |

Source: Ibearugbulem, et al. (2014)

Where
CCCC - a plate clamped on all four edges
CCSS- a plate clamped/fixed on at two edges, and simply supported at the other two edges, and so on.

The n-values for the various plate types will be evaluated as follows.

*From Table* 1, for CCSS

$$h = (1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4) = h_x * h_y \qquad (21)$$

$$\frac{\partial^2 h}{\partial R^2} = \frac{\partial^2 h_x}{\partial R^2} * h_y = (3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4) \qquad (22a)$$

$$\frac{\partial^2 h}{\partial Q^2} = h_x * \frac{\partial^2 h_y}{\partial Q^2} = (1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2) \qquad (23b)$$

Substitute Equations (22) into Equations (11) yields

$$n_1 = \left[(3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4) + \frac{\upsilon}{ꝗ^2}(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2)\right] \qquad (24)$$

$$n_2 = \left[\upsilon(3 - 15R + 12R^2)(1.5Q^2 - 2.5Q^3 + Q^4) + \frac{1}{ꝗ^2}(1.5R^2 - 2.5R^3 + R^4)(3 - 15Q + 12Q^2)\right] \qquad (25)$$

At the point of maximum deflection, R = Q = 0.5. Substitute these values of R and Q in Equations (24) to (25), we have

$$n_1 = \left(-0.1875 - \frac{0.1875\upsilon}{ꝗ^2}\right) \qquad (26)$$

$$n_2 = \left(-0.1875\upsilon - \frac{0.1875}{ꝗ^2}\right) \qquad (27)$$

Substituting Equation (26) and (27) into Equation (20) yields the stress factor as Equation (28)

$$F_T = \left[1 - \frac{\left(-0.1875\upsilon - \frac{0.1875}{ꝗ^2}\right)}{\left(-0.1875 - \frac{0.1875\upsilon}{ꝗ^2}\right)}\right] \qquad (28)$$

Similarly, the rest of the 11 plate types contained in Table 1, were evaluated but with R=0.5 and Q =1 for plate types with a free edge.

The stress factor equations for the twelve plates are presented in Table 3.

## 2.2 Numerical Application

Consider a structural steel square plate with the following properties. $\upsilon = 0.3$, a =1m, $f_y$ =250MPa.
The numerical results obtained from yield criterion equations in Table 3 are presented in Table 4.

## 3.0: Results and Discussions

The results of the formulated equations are presented in Tables 2 to Table 3 for general applied stress equation and stress factor equation respectively.

**Table 2:** General Formulated Applied Stress Equation Based on the Tresca Yield Criterion

| SN | DESCRIPTION | EQUATIONS |
|---|---|---|
| 1 | Applied Stress | $\sigma_x = \dfrac{f_y}{F_T}$ |
| 2 | Tresca Stress Factor | $F_T = (1 - \dfrac{n_2}{n_1})$ |

**Table 3:** Tresca Stress Factor, $F_T$, Equations

| Plate Type | $\sigma_x = \dfrac{f_y}{F_T}$ |
|---|---|
| | $F_T = (1 - \dfrac{n_2}{n_1})$ |
| CCSS | $\left[1 - \dfrac{(-0.1875 \jmath^2 \upsilon - 0.1875)}{(-0.1875 \jmath^2 - 0.1875\upsilon)}\right]$ |
| CCCC | $\left[1 - \dfrac{(-0.0625 \jmath^2 \upsilon - 0.0625)}{(-0.0625 \jmath^2 - 0.0625\upsilon)}\right]$ |
| SSSS | $\left[1 - \dfrac{(-0.9375 \jmath^2 \upsilon - 0.9375)}{(-0.9375 \jmath^2 - 0.9375\upsilon)}\right]$ |
| CSSS | $\left[1 - \dfrac{(-0.375 \jmath^2 \upsilon - 0.46875)}{(-0.375 \jmath^2 - 0.46875\upsilon)}\right]$ |
| CSCS | $\left[1 - \dfrac{(-0.1875 \jmath^2 \upsilon - 0.3125)}{(-0.1875 \jmath^2 - 0.3125\upsilon)}\right]$ |
| CCCS | $\left[1 - \dfrac{(-0.09375 \jmath^2 \upsilon - 0.125)}{(-0.09375 \jmath^2 - 0.125\upsilon)}\right]$ |
| SSFS | $1 - \upsilon$ |
| SCFS | $1 - \upsilon$ |
| CSFS | $1 - \upsilon$ |
| CCFS | $1 - \upsilon$ |
| SCFC | $1 - \upsilon$ |
| CCFC | $1 - \upsilon$ |

The numerical results for n-values are presented in columns 2 and 3 for $n_1$ and $n_2$, respectively in Table 4. The results for the stress factor of safety for the various plate types are presented in column 4, and the calculated applied stress using the yield stress of 250MPa are presented in column 5.

**Table 4:** Applied Stress from Yield Criterion Analysis

| Plate type | $\sigma_d = \dfrac{\sigma_x}{F_S} \le f_y;$ | | $\jmath = \dfrac{b}{a} = 1;\ f_y = 250\text{MPa}$ | | |
|---|---|---|---|---|---|
| | $n_1$ | $n_2$ | $F_T$ | $\sigma_x$(MPa) | $\sigma_d$(MPa) |
| CCSS | -0.24375 | -0.24375 | 0.13333333 | 1875 | NA |
| CCCC | -0.08125 | -0.08125 | 0 | ∞ | NA |
| SSSS | -1.21875 | -1.21875 | 0 | ∞ | NA |
| CSSS | -0.51563 | -0.58125 | -0.12727273 | -1964 | NA |
| CSCS | -0.28125 | -0.36875 | -0.31111111 | -804 | NA |
| CCCS | -0.13125 | -0.15313 | -0.16666667 | -1500 | NA |
| SSFS | -4 | -1.2 | 0.7 | 357 | 238 |
| SCFS | -2 | -0.6 | 0.7 | 357 | 238 |
| CSFS | -1.2 | -0.36 | 0.7 | 357 | 238 |
| CCFS | -0.6 | -0.18 | 0.7 | 357 | 238 |
| SCFC | -1.33333 | -0.4 | 0.7 | 357 | 238 |
| CCFC | -0.4 | -0.12 | 0.7 | 357 | 238 |

NA= Not applicable

From Table 4, the values of the stress factor for the various plate types are inconsistent just like it was observed by (Benhan and Warnock, 1976). The values of the stress factor for plates are all less than one, given the applied stress to be too high than the yield stress for plates with no free edge, and moderately high for plates with one free edge. The applied stresses obtained for CCCC and SSSS plates are even at infinity. That of CSCS and CSSS is negative. This implies that the Tresca yield criterion is may not suitable for the analysis of plates with no free edge. Hence, this theory is not advisable for the design of plates with no free edge. However, for

plates with one free edge, the stress factor is 0.7 for all the six plate types considered, resulting in an applied stress of 357MPa. This value is higher than the yield stress of the structural steel considered which is not safe. Therefore, for design purposes, a a safety factor, $F_s$ greater than one has been proposed, to have a design stress less than the yield stress. Dividing the applied stress by 1.5 gives a stress that is less than 250MPa. Therefore, the proposed factor of safety will be 1.5 for the plates with a free edge which will reduce the design stress to 238MPa for a yield stress of 250MPa for mild steel. This will make the structure safe. The design stress for plate types with one free edge based on the Tresca yield criterion is given in Equation (29) and the values are presented in column 6 of Table 4.

$$\sigma_d = \frac{\frac{f_y}{1 - \frac{n_2}{n_1}}}{F_s} = \frac{\sigma_x}{F_s} \leq f_y \qquad (29)$$

Equation (29) is the new modified design yield stress equation for rectangular plates based on the maximum shear stress theory of yield. Also, the Tresca stress factor is independent of the Poisson ratio. Meaning it is not a function of the material property. This study has revealed the weakness of this criterion and has offered suggestions that will ensure the safety of these structures.

**4.0: CONCLUSION AND RECOMMENDATIONS**
The current work has contributed to the volume of knowledge in the field of yield criterion analysis of materials and it is concluded that:

1. This work has developed a new applied stress equation for the analysis of plate structures.
2. Has shown the limitation of the original Tresca yield equation and has corrected the stress factor to obtain a safety factor, leading to the new design stress equation for plates with one free edge.
3. Since the the predicted design stresses from the new equation for all plate types were less than the yield stress of the materials (for mild steel of 250MPa), its implies that the new equation now is suitable for for predicting the design stress of plate structures.
4. The work has revealed that this criterion is not suitable for rectangular plate analysis with no free edge and design for plates with no free edge, therefore preventing failure and economic waste.
5. Hence, the current equations from this work are adequate for yield criterion analysis and design of plate structures with one free edge..

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