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ANINVESTIGATION OF WINDING NUMBER OF A CLOSED PLANAR CURVE

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ABSTRACT

This paper examines winding number of a closed planar curve through various aspects. It is related to a range on an arc in the complex plane and a point not in the range. Functions of such natures are considered to be continuous real-valued functions. Conclusion was drawn by naming a number of areas of applications.

KEYWORDS AND PHRASES: winding number, plane, closed curve, loops, homotopy, rotation, continuous deformation

INTRODUCTION

In mathematics, the winding number of a closed curve in the plane around a given point is an integer that represents the total number of counter clock wise rotations the curve does around the point. The winding number is determined by the curve's orientation, and is negative if the curve goes clockwise around the point. In vector calculus, complex analysis, geometric topology, differential geometry, and physics, particularly string theory, winding numbers play an important role and they are key objects of study in algebraic topology.

Several numerical qualities that assume only integer values are associated with every planar polygon P (or, more widely, with certain types of closed curves in the plane). The rotation number of P (also known as the "tangent winding number") and the winding number of P with respect to a point in the plane are the most well-known of these (Branko and Shephard 1990).

The rotation number $\omega(\gamma)$ of a regular closed planar curve is simply the number of complete turns the tangent vector to the curve makes when passing once around the curve; and the winding number, $\omega(\gamma, p)$ of a closed curve with respect to a point p is the number of times the curve winds around the point.

BASIC DEFINITIONS

Winding Number of a Curve

The winding number of a smooth closed planar curve around a given point is the total number of times that curve travels counter-clockwise around the point.

Winding number is invariant under regular homotopy, and may thus be regarded as an integer-valued homomorphism from the group of regular homotopy classes of smooth planar curves (Whitney 1937).

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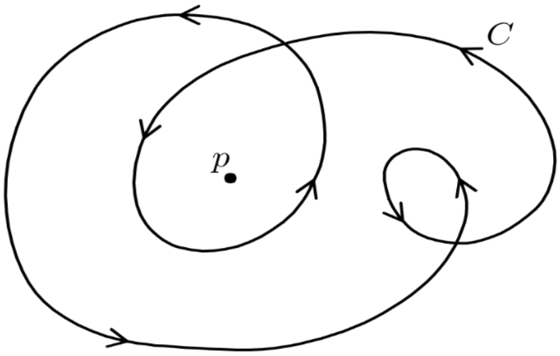

The winding number of the curve denoted by $\omega(\gamma, O)$ is the total amount of turning of vector $\overrightarrow{OP}$, $p \in \gamma$. This can be represented as the sum of the turns of individual segments of the curve.

Winding Number

If $\sigma: [a, b] \to \mathbb{C}$ is a curve in $\mathbb{C}$ and ω is not in the image of σ, then we may find differentiable functions, $r(t) > 0$ and $\theta(t) \in \mathbb{R}$ such that

$$\sigma(t) = \omega + r(t)e^{i\theta(t)} \qquad (1.1)$$

The function $\theta(t)$ is unique up to an additive constant, $2\pi k$, for some $k \in \mathbb{Z}$ and therefore

$$\Delta \arg(\sigma) := \Delta\theta = \theta(b) - \theta(a)$$

is well defined and independent of the choice of θ. Moreover, if σ is a loop so that $\sigma(b) = \sigma(a)$, then $e^{i\,\theta(b)} = e^{i\,\theta(a)}$ which implies that $\theta(b) - \theta(a) = 2\pi \cdot n$ for some $n \in \mathbb{Z}$ (Wesenberg 2020).

Definition

Given a continuous loop $\sigma: [a, b] \to \mathbb{C}$ and ω is not in the image of σ, the winding number of σ around ω is defined by

$$N_\sigma(\omega) := \frac{1}{2\pi} \Delta \arg(\sigma) := \frac{1}{2\pi}[\theta(b) - \theta(a)] \in \mathbb{Z}$$

where $\theta(t)$ is any continuous choice of angle so that eqn. (1.1) holds with $r(t) = |\sigma(t) - \omega|$ (The University of California, San Diego 2020).

Remarks

It follows from the definition that

$$N_\sigma(\omega) = N_{\sigma-\omega}(0)$$

Obviously, if σ is written as in eqn. (1.1), then

$$\sigma(t) - \omega = 0 + r(t)e^{i\,\theta(t)} \Longrightarrow N_{\sigma-\omega}(0) = \frac{1}{2\pi}\Delta\,\theta = N_\sigma(\omega)$$

as seen in (The University of California, San Diego 2020).

Computing the winding numbers of some curves

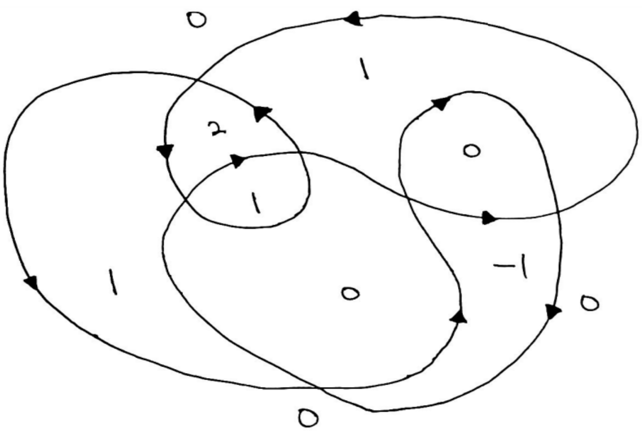

Figure 2: Computing the winding number of a closed curve $\mathbb{C}$.

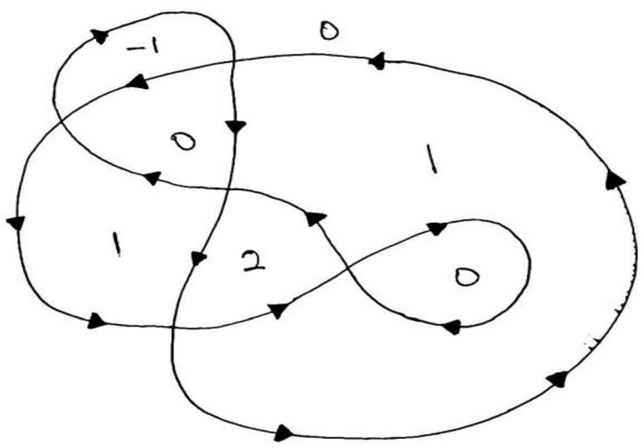

Figure 3: Finding the winding number of the closed curve at several points.

Properties of the winding number

Two closed plane curves c_0, c_1 not meeting at the origin (parameterized over $[0,2\pi]$) are said to be freely homotopic if there exists a continuous map

$$H: [0,1] \times [0, 2\pi] \to \mathbb{R}^2 \backslash \{0\}$$

so that:

i. For a fixed $s \in [0, 1] : H(s, 0) = H(s, 2\pi)$;

ii. $F(0, t) = c_0(t)$, $F(1, t) = c_1(t)$. Thus, the closed curves $c_s(t) = F(s, t)$ deform c_0 to c_1, without hitting the origin.

1. If c_0, c_1 are freely homotopic closed plane curves in $\mathbb{R}^2 \backslash \{0\}$, then $\gamma(c_0) = \gamma(c_1)$.
2. Conversely, if c_0, c_1 are closed plane curves not hitting the origin and $\gamma(c_0) = \gamma(c_1)$, then they are freely homotopic.
3. $c(t) = (\cos xt, \sin xt); t \in [0, 2\pi]$ satisfies $\gamma(c) = x, \forall x \in \mathbb{Z}$;
4. If c_0, c_1 are closed curves in $\mathbb{R}^2 \backslash \{0\}$ and the line segment in $\mathbb{R}^2$ from $c_0(t)$ to $c_1(t)$ does not include the origin (for any t), then $\gamma(c_0) = \gamma(c_1)$. In particular, this is true if $|c_0(t) - c_1(t)| < |c_0(t)|$ for each $t \in [0, 2\pi]$.

The proof of property 1 follows from the invariance of line integrals of closed 1-forms under free homotopy.

We prove property (2.) below;

Assume first c_0 and c_1 take values in the unit circle, and let $\varphi_1, \varphi_2 : [0, 2\pi] \to \mathbb{R}$ be choices of angle functions for c_0, c_1 respectively. Consider:

$\varphi(s, t) = (1 - s)\varphi_0(t) + s\varphi_2$, $H(s, t) = (\cos \varphi(s, t),\ \sin \varphi(s, t))$, $s \in [0, 1]$, $t \in [0, 2\pi]$.

Note that:

$\varphi(s, 2\pi) - \varphi(s, 0) = (1 - s)2\pi\gamma(c_0) + s2\pi\gamma(c_1) = 2\pi\gamma$, since $\gamma(c_0) = \gamma(c_1) := \gamma$.

Thus, the curves $c_s(t) = H(s, t)$ on the unit circle are closed (with the same winding number γ), and deform c_0 to c_1.

Refer to (The University of Tennessee 2020) for the above properties of the winding numbers.

Other properties of the winding number of a curve includes:

a. $\omega(\gamma, p)$ is an integer.
b. $\omega(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
c. $\omega(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
d. $\omega(\gamma, p)$ is additive in γ.
e. $\omega(\gamma, p)$ is linear in γ.

Refer to (Vin de Silva 2017) for more on the other properties of the winding number.

Theorem 1

The winding number, $\omega(\gamma)$ of γ is given by

$$\omega(\gamma) = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i)\,.$$

It is obvious that the winding number of γ is just the integral of its curvature, divided by 2π. On the contrary, by the Gauss-Bonnet formula, the Euler characteristic of each of the S_i is (up to a sign, and when multiplied by 2π) equal to the integral of the curvature of those segments of γ that form the boundary of S_i plus the sum of the external angles at the crossing points of γ that occur in the boundary of S_i. It suffices then to notice that each crossing of γ occurs in the boundary of precisely two domains S_i, and that the corresponding pairs of external angles each sum to zero (Mcintyre and Cairns 1993).

Theorem 2
The winding number, $\omega(\gamma)$ of γ is given by

$$\omega(\gamma) = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i)$$

if Q is the torus T^2, and

$$\omega(\gamma) = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i) \quad (mod|\chi(Q)|),$$

otherwise.
Refer to (Mcintyre and Cairns 1993) for the proof of Theorem 2.

Theorem 3
Suppose $\gamma, \eta : [a,b] \to \mathbb{R}^2$ are closed curves, $m \in \mathbb{R}^2$ and

$$|\gamma(t) - \eta(t)| < |\gamma(t) - m| + |\eta(t) - m| \; for\; all\; t \in [a,b].$$

Then m is not either γ or η and $\omega(\gamma; m) = \omega(\eta; m)$.
The intuitive meaning and proof of Theorem 3 can be found in (Baker 1991).

Corollary 1
Suppose $\gamma : [a,b] \to \mathbb{R}^2$ is a closed curve, $m, n \in \mathbb{R}^2$ and

$$|m - n| < |\gamma(t) - m| \; for\; all\; t \in [a,b].$$

Then $m \notin [\gamma], n \notin [\gamma]$, and $\omega(\gamma; m) = \omega(\eta; n)$.

Proof: Since $|(\gamma(t) - n) - (\gamma(t) - m)| < |\gamma(t) - n| + |\gamma(t) - m| \; for\; all\; t \in [a,b]$, it follows from Theorem 3 that $\omega(\gamma - n; 0) = \omega(\gamma - m; 0)$ and the rest follows. Baker provided the complete proof in (Baker 1991).

Corollary 2
Suppose $\gamma : [a,b] \to \mathbb{R}^2$ is a closed curve, $m \in \mathbb{R}^2$ and $|\gamma(t)| < |m| \; for\; all\; t \in [a,b]$.
Then $\omega(\gamma; m) = 0$.

Proof: Let $\eta(t) = -m \; for\; all\; t \in [a,b]$. Then $|(\gamma(t) - m) - \eta(t)| < |\eta(t)| \le |\gamma(t) - m| + |\eta(t)| for\; all\; t \in [a,b]$. Hence by Theorem 3, $\omega(\gamma - p; 0) = \omega(\eta; 0)$. But $\omega(\eta; 0) = 0$ and $\omega(\gamma - m; 0) = \omega(\gamma; m)$.

We now aim to prove that $\omega(\gamma; m)$ depends continuously on γ (in a sense now to be defined). Suppose U is a non-empty open subset of $\mathbb{R}^2$ and $\gamma, \eta: [a,b] \to \mathbb{R}^2$ are closed curves in U (i.e., $[\gamma], [\eta] \subseteq U$). We say γ is homotopic to η in U provided there exists a function $H: [a,b] \times [0,1] \to \mathbb{R}^2$ such that

i. H is continuous on $[a,b] \times [0,1]$,
ii. $H(t,\lambda) \in U \; for\; every\; t \in [a,b]$ and every $\lambda \in [0,1]$,
iii. $H(a,\lambda) = H(b,\lambda) for\; every\; \lambda \in [0,1]$, and
iv. $H(t,0) = \gamma(t) \; and\; H(t,1) = \eta(t) \; for\; every\; t \in [a,b]$.

Such an H is called a homotopy in U. With H as in the definition of homotopy, for $0 \le \lambda \le 1$ let $H_\lambda(t) = H(t,\lambda) \; for\; a \le t \le b$. Then it follows that H_λ is a closed curve in U for each $\lambda \in [0,1], H_0 = \gamma$ and $H_1 = \eta$. Intuitively, as λ varies continuously from 0 to 1, H_λ varies continuously from λ to η with $[H_\lambda] \subseteq U$ for every $\lambda \in [0,1]$. Thus H can be thought of as a continuous deformation of λ to η inside U. Property i. implies that if $\lambda_k \to \lambda_0$ in $[0,1]$ then $H_{\lambda_k} \to H_{\lambda_0}$ uniformly on $[a,b]$. This can be proved by noting that i. implies that H is uniformly continuous. Note that if $\gamma, \eta : [a,b] \to \mathbb{R}^2$ are closed curves then γ is homotopic to η in $\mathbb{R}^2$; simply let $H(t,\lambda) = (1-\lambda)\gamma(t) + \lambda\eta(t) for\; a \le t \le b\; and\; 0 \le \lambda \le 1$. The proof can be found in (Baker 1991).

Concluding with areas where Applications of Winding Numbers can used

The concept of the winding number can be applied in vast areas. Few of such areas are listed below:

a. Topological proof of the Fundamental Theorem of Algebra.
b. Degree of a map of the circle.
c. Brouwer fixed point theorem for the disk in $\mathbb{R}^2$
d. Borsuk-Ulam Theorem
e. Index of a planar vector field on the boundary of a disk.

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