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CONSTRUCTING HOMEOMORPHISMS OF THE CANTOR TERNARY SET

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ABSTRACT

In this paper, we present a generalized form of the Cantor ternary set by studying the cantor $\frac{1}{2\lambda+1}$ middle set where $1 \leq \lambda < \infty$ and its fractal dimension. The paper also introduces the Heine-Borel set and shows that the cantor set and its generalised $\frac{1}{2\lambda+1}$ middle set where $1 \leq \lambda < \infty$ are Heine-Borel sets.

1. INTRODUCTION

The Cantor set is a compact, perfect, nowhere dense and totally disconnected subset of the real line with Lebesgue measure zero. It was discovered by the Russian born German mathematician Georg Ferdinand Ludwig Philip Cantor (1845 - 1918) in 1883 and hence, it was named after him [10], [5].

The Cantor set has been defined in many ways and also been constructed in many different forms [See, [3],[11], [6], [12], [7],[4] and [1]]. However, it is not only the construction of the Cantor set that has seen a lot of versions, the determination of the endpoints has also been worked on by many researchers (see, [1] and [2]). Hornuvo and Obeng-Denteh established a recurrence relation for the middle k^{th} of the Cantor ternary set where they defined k to be k = 2y + 1, for y = 1, 2, 3, ... [1]. From their construction the middle k^{th} was given by the recurrence relation

$$\{B\}_{n=1}^{\infty} = \frac{\left(\frac{k+1}{2}\right)-1}{k} B_{n-1} \cup \left[\frac{(k+1)}{2} + \frac{\left(\frac{k+1}{2}\right)-1}{k} B_{n-1}\right], n = 1, 2, 3, \ldots$$

The Cantor set happens to be one of the most interesting mathematical phenomenon which seems to permeate every aspect of mathematics from topology, real analysis, fractal geometry, probability theory, dynamical systems and many other areas even outside mathematics.

In 1975, Benoit B. Mandelbrot introduced the term fractal to describe mathematical and natural objects that are self-similar. And since then, fractals have become an essential tool to physical, natural and social scientist [13]. Fractal objects are mainly characterized by their self-similarity and non-integer dimensions. The cantor set is a fractal because it possesses the characteristics of a fractal by having a non-integer dimension and self -similar qualities. In this paper, we study the cantor $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ and $\frac{1}{11}$ sets and in general the cantor $\frac{1}{2\lambda+1}$ middle set where $1 \leq \lambda < \infty$. We will also state and prove some properties relating to this generalized cantor set.

2. PRELIMINARIES

2.1 Construction of the cantor ternary set

Definition: *Quantum level (q): in this paper we define the quantum level of a cantor set as the gap left in the iteration process during the construction of the cantor-like sets.*

We now proceed to construct the Cantor $\frac{1}{3}^{rd}$ middle set; it is constructed from the unit interval by a sequence of deletion operations sometimes called iterations. The Cantor middle third set is constructed by deleting the middle thirds from the closed unit interval on the real line.

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We start with the closed set $B_0 = [0,1]$ and let B_1 be the set obtained by deleting the middle $\frac{1}{3}$ of B_0, such that B_1 consists of the two closed intervals $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$. Again, deleting the middle $\frac{1}{3}$ from each of the intervals in B_1, we obtain B_2 which consist of four closed intervals $\left[0,\frac{1}{9}\right]$, $\left[\frac{2}{9},\frac{3}{9}\right]$, $\left[\frac{6}{9},\frac{7}{9}\right]$ and $\left[\frac{8}{9},1\right]$. Continuing the iteration processes this way, we obtain a set of nested sequence$B_{n+1} \subset B_n$. From the construction, we can see that the number of closed intervals increases from B_0 to B_n starting from 2^0 in B_0, 2^1 in B_1, 2^2 in B_2, 2^3 in B_3 …, to 2^n in $B_n$of the n^{th} iteration. Hence, we obtain 2^n closed intervals of length $\left(\frac{1}{3}\right)^n$.

The cantor set is thus defined as the set of points remaining as the number of iterations tends to infinity. And so, we define the cantor ternary set as

$$B^3 = \bigcap_{n=0}^{\infty} B_n$$

The geometric construction of the cantor one-third set is illustrated in fig 1 below

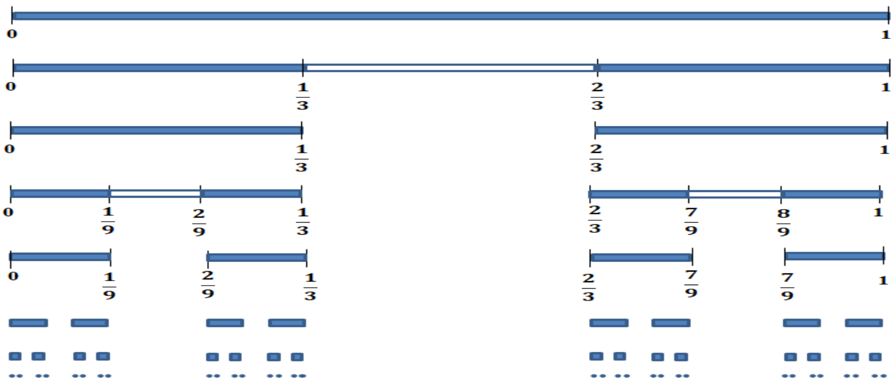

Figure 1

We now show that the total length of the Cantor ternary set is 0, and that the total length of the open intervals removed is 1. From the first iteration, an interval of $\frac{1}{3}$ is removed fromB_0. We removed $\frac{2}{9}$ during the second iteration and $\frac{4}{27}$ in the third iteration. As the iteration process continues, the length of the removed interval at each iteration level forms a geometric series $\frac{1}{3},\frac{2}{9},\frac{4}{2},\frac{8}{81}$, ... with the first term $a = \frac{1}{3}$ and a common ratio of$r = \frac{2}{3}$. This series has a sum $S_\infty = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$. This shows that the total length of the intervals removed is 1. However, we started with $B_0 = [0,1]$ with a total length of 1, Hence subtracting 1 from 1 gives the length of 0 which is the length of the Cantor ternary set.

We now put forward a general relation as n approaches infinity.

$$B_\infty^3 = \left[0,\frac{1}{3^n}\right] \cup \left[\frac{2}{3^n},\frac{3}{3^n}\right] \cup \left[\frac{(2.3^q)}{3^n},\frac{(2.3^q)+1}{3^n}\right] \cup \left[\frac{(2.3^q)+2}{3^n},\frac{(2.3^q)+3}{3^n}\right] \cup \left[\frac{(2.3^{q+1})}{3^n},\frac{(2.3^{q+1})+1}{3^n}\right] \cup \left[\frac{(2.3^{q+1})+2}{3^n},\frac{(2.3^{q+1})+3}{3^n}\right] \cup \left[\frac{(2.3^{q+1})+6}{3^n},\frac{(2.3^{q+1})+7}{3^n}\right] \cup \left[\frac{(2.3^{q+1})+8}{3^n},\frac{(2.3^{q+1})+9}{3^n}\right] \cup \left[\frac{(2.3^{q+2})}{3^n},\frac{(2.3^{q+2})+1}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+2}{3^n},\frac{(2.3^{q+2})+3}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+6}{3^n},\frac{(2.3^{q+2})+7}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+8}{3^n},\frac{(2.3^{q+2})+9}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+18}{3^n},\frac{(2.3^{q+2})+19}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+20}{3^n},\frac{(2.3^{q+2})+21}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+24}{3^n},\frac{(2.3^{q+2})+25}{3^n}\right] \cup \left[\frac{(2.3^{q+2})+26}{3^n},\frac{(2.3^{q+2})+27}{3^n}\right] \cup \ldots$$

Where n is the number of iterations and q is the quantum level. For example, when $n = 3$ and $q = 1$

$$B_n^3 = \left[0,\frac{1}{27}\right] \cup \left[\frac{2}{27},\frac{3}{27}\right] \cup \left[\frac{6}{27},\frac{7}{27}\right] \cup \left[\frac{8}{27},\frac{9}{27}\right] \cup \left[\frac{18}{27},\frac{19}{27}\right] \cup \left[\frac{20}{27},\frac{21}{27}\right] \cup \left[\frac{24}{27},\frac{25}{27}\right] \cup \left[\frac{26}{27},\frac{27}{27}\right]$$

Which can be simplified to be

$$B_n^3 = \left[0,\frac{1}{27}\right] \cup \left[\frac{2}{27},\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{7}{27}\right] \cup \left[\frac{8}{27},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{19}{27}\right] \cup \left[\frac{20}{27},\frac{7}{9}\right] \cup \left[\frac{8}{9},\frac{25}{27}\right] \cup \left[\frac{26}{27},1\right]$$

2.2. Construction of the cantor $\frac{1}{2\lambda+1}$ set for some selected λ.

When $\lambda = 1$, we have $\frac{1}{3}$ which is the Cantor ternary set and when $\lambda = 2$, we have $\frac{1}{5}$.

Now let $B_0^5 = [0,1]$. Divide [0,1] into five equal open intervals and remove the 2nd and 4th open intervals

$$\left(\frac{1}{5},\frac{2}{5}\right),\left(\frac{3}{5},\frac{4}{5}\right)$$

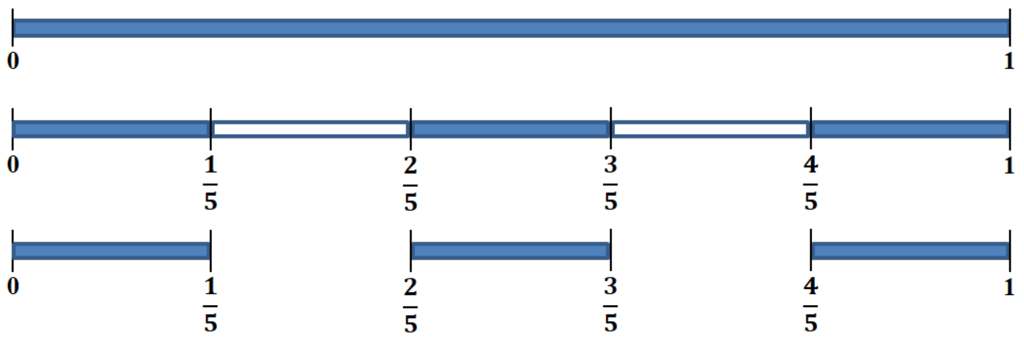

Figure 2: The first iteration for $\lambda = 2$

The process leads to the creation of B_1^5 which is given as

$$B_1^5 = \left[0,\frac{1}{5}\right] \cup \left[\frac{2}{5},\frac{3}{5}\right] \cup \left[\frac{4}{5},1\right]$$

It is easy to see that B_1^5 is the union of $3^1 = 1$ disjoint closed intervals. Next, divide each closed interval of B_1^5 into five and remove from each interval the 2nd and 4th open intervals. Now, let B_2^5 be the set remaining from B_1^5 after removing

$$\left(\frac{1}{25},\frac{2}{25}\right),\left(\frac{3}{25},\frac{4}{25}\right),\left(\frac{11}{25},\frac{12}{25}\right),$$
$$\left(\frac{13}{25},\frac{14}{25}\right),\left(\frac{21}{25},\frac{22}{25}\right),\left(\frac{23}{25},\frac{24}{25}\right)$$

From B_1^5 . Then we have

$$B_2^5 = \left[0,\frac{1}{25}\right] \cup \left[\frac{2}{25},\frac{3}{25}\right] \cup \left[\frac{4}{25},\frac{1}{5}\right] \cup \left[\frac{2}{5},\frac{11}{25}\right] \cup \left[\frac{12}{25},\frac{13}{25}\right] \cup \left[\frac{14}{25},\frac{3}{5}\right] \cup \left[\frac{4}{5},\frac{21}{25}\right] \cup \left[\frac{22}{25},\frac{23}{25}\right] \cup \left[\frac{24}{25},1\right]$$

Here, we note that B_2^5 is the union of $3^2 = 9$ disjoint closed intervals. If the process continues then we can construct B_{n+1}^5 from B_n^5 inductively. Now divide each of the 3^n disjoint closed intervals of B_n^5, and remove from each one of them the 2nd and the 4th open intervals. It is now clear that what is left from B_n^5 is B_{n+1}^5 and that B_{n+1}^5 is the union of 3^{n+1} disjoint closed intervals. It is, therefore easy to see that $B_{n+1}^5 \subset B_n^5$ is true for all n. Hence, we define the Cantor $\frac{1}{5}$th set on [0,1] as

$$B^5 = \bigcap_{n=0}^{\infty} B_n$$

We now show that, the total length of the Cantor $\frac{1}{5}$th set is 0, and that the total length of the open intervals removed is 1. From the first iteration, an interval of $\frac{2}{5}$ is removed from B_0. We removed $\frac{6}{25}$ during the second iteration and $\frac{18}{125}$ in the third iteration. As the iteration process continues, the length of the removed interval at each iteration level forms a geometric series $\frac{2}{5}, \frac{6}{25}, \frac{18}{125}, \ldots$ with the first term $a = \frac{2}{5}$ and a common ratio of $r = \frac{3}{5}$. This series has a sum $S_\infty = \frac{\frac{2}{5}}{1-\frac{3}{5}} = 1$. This shows that the total length of the intervals removed is 1. However, since we started with $B_0 = [0,1]$ with a total length of 1, subtracting 1 from gives 0. This means the Cantor $\frac{1}{5}$th set has a length of 0.

We now put forward a general relation as n approaches infinity.

$$B_\infty^5 = \left[0,\frac{1}{5^n}\right] \cup \left[\frac{2}{5^n},\frac{3}{5^n}\right] \cup \left[\frac{4}{5^n},\frac{5}{5^n}\right] \cup \left[\frac{(2.5^q)}{5^n},\frac{(2.5^q)+1}{5^n}\right] \cup \left[\frac{(2.5^q)+2}{5^n},\frac{(2.5^q)+3}{5^n}\right] \cup \left[\frac{(2.5^q)+4}{5^n},\frac{(2.5^q)+5}{5^n}\right] \cup \left[\frac{(2.5^q)+10}{5^n},\frac{(2.5^q)+11}{5^n}\right] \cup$$
$$\left[\frac{(2.5^q)+12}{5^n},\frac{(2.5^q)+13}{5^n}\right] \cup \left[\frac{(2.5^q)+14}{5^n},\frac{(2.5^q)+15}{5^n}\right] \cup \left[\frac{(2.5^{q+1})}{5^n},\frac{(2.5^{q+1})+1}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+2}{5^n},\frac{(2.5^{q+1})+3}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+4}{5^n},\frac{(2.5^{q+1})+5}{5^n}\right] \cup$$
$$\left[\frac{(2.5^{q+1})+10}{5^n},\frac{(2.5^{q+1})+11}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+12}{5^n},\frac{(2.5^{q+1})+13}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+14}{5^n},\frac{(2.5^{q+1})+15}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+20}{5^n},\frac{(2.5^{q+1})+21}{5^n}\right] \cup$$
$$\left[\frac{(2.5^{q+1})+22}{5^n},\frac{(2.5^{q+1})+23}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+24}{5^n},\frac{(2.5^{q+1})+25}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+50}{5^n},\frac{(2.5^{q+1})+51}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+52}{5^n},\frac{(2.5^{q+1})+53}{5^n}\right] \cup$$
$$\left[\frac{(2.5^{q+1})+54}{5^n},\frac{(2.5^{q+1})+55}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+60}{5^n},\frac{(2.5^{q+1})+61}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+62}{5^n},\frac{(2.5^{q+1})+63}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+64}{5^n},\frac{(2.5^{q+1})+65}{5^n}\right] \cup$$
$$\left[\frac{(2.5^{q+1})+70}{5^n},\frac{(2.5^{q+1})+71}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+72}{5^n},\frac{(2.5^{q+1})+73}{5^n}\right] \cup \left[\frac{(2.5^{q+1})+74}{5^n},\frac{(2.5^{q+1})+75}{5^n}\right] \cup \ldots$$

Where n is the number of iterations and q is the quantum level. For example, when $n = 2$ and $q = 1$ we have

$$B_2^5 = \left[0,\frac{1}{5^2}\right] \cup \left[\frac{2}{5^2},\frac{3}{5^2}\right] \cup \left[\frac{4}{5^2},\frac{5}{5^2}\right] \cup \left[\frac{(2.5^1)}{5^2},\frac{(2.5^1)+1}{5^2}\right] \cup \left[\frac{(2.5^1)+2}{5^2},\frac{(2.5^1)+3}{5^2}\right] \cup \left[\frac{(2.5^1)+4}{5^2},\frac{(2.5^1)+5}{2}\right] \cup \left[\frac{(2.5^1)+10}{5^2},\frac{(2.5^1)+11}{5^2}\right] \cup$$
$$\left[\frac{(2.5^1)+12}{5^2},\frac{(2.5^1)+13}{5^2}\right] \cup \left[\frac{(2.5^1)+14}{5^2},\frac{(2.5^1)+15}{5^2}\right]$$

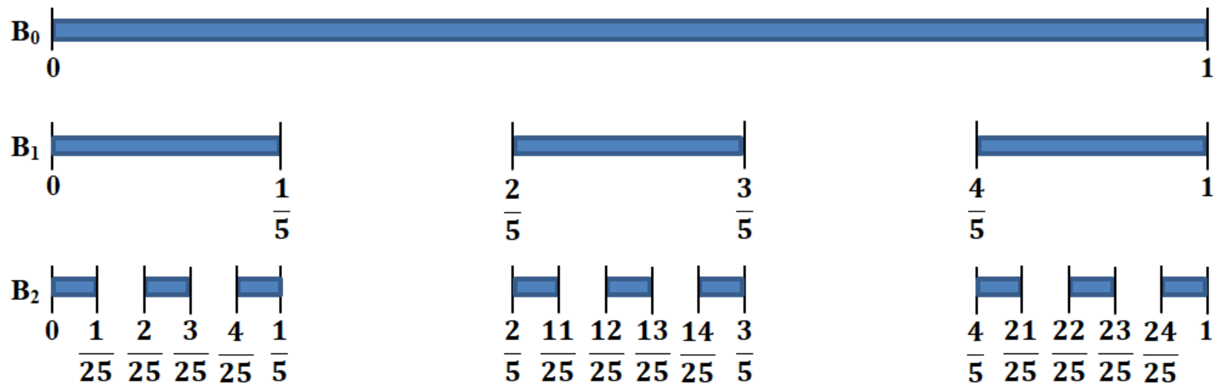

Figure 3: The first two iteration for $\lambda = 2$

When $\lambda = 3$, we let $B_0^7 = [0,1]$. Divide [0,1] into seven equal open intervals and remove the 2nd, 4th and the 6th open intervals

$$\left(\frac{1}{7},\frac{2}{7}\right),\left(\frac{3}{7},\frac{4}{7}\right),\left(\frac{5}{7},\frac{6}{7}\right)$$

This gives the closed intervals left in [0,1] as

$$B_1^7 = \left[0,\frac{1}{7}\right] \cup \left[\frac{2}{7},\frac{3}{7}\right] \cup \left[\frac{4}{7},\frac{5}{7}\right] \cup \left[\frac{6}{7},1\right]$$

Hence B_1^7 becomes the union of 4^1 disjoint closed intervals. Next, divide each closed interval of B_1^7 into seven and remove from each interval the 2nd, 4th and 6th open intervals

$$\left(\frac{1}{49},\frac{2}{49}\right),\left(\frac{3}{49},\frac{4}{49}\right),\left(\frac{5}{49},\frac{6}{49}\right),\left(\frac{15}{49},\frac{16}{49}\right),$$
$$\left(\frac{17}{49},\frac{18}{49}\right),\left(\frac{19}{49},\frac{20}{49}\right),\left(\frac{29}{49},\frac{30}{49}\right),\left(\frac{31}{49},\frac{32}{49}\right),$$
$$\left(\frac{33}{49},\frac{34}{49}\right),\left(\frac{43}{49},\frac{44}{49}\right),\left(\frac{45}{49},\frac{46}{49}\right),\left(\frac{47}{49},\frac{48}{49}\right)$$

Again, let B_2^7 be the set remaining after removing the open intervals above from B_1^7. Hence, we have

$$B_2^7 = \left[0,\frac{1}{49}\right] \cup \left[\frac{2}{49},\frac{3}{49}\right] \cup \left[\frac{4}{49},\frac{5}{49}\right] \cup \left[\frac{6}{49},\frac{1}{7}\right] \cup \left[\frac{2}{7},\frac{15}{49}\right] \cup \left[\frac{16}{49},\frac{17}{49}\right] \cup \left[\frac{18}{49},\frac{19}{49}\right]$$
$$\cup \left[\frac{20}{49},\frac{3}{7}\right] \cup \left[\frac{4}{7},\frac{29}{49}\right] \cup \left[\frac{30}{49},\frac{31}{49}\right] \cup \left[\frac{32}{49},\frac{33}{49}\right] \cup \left[\frac{34}{49},\frac{5}{7}\right] \cup \left[\frac{6}{7},\frac{43}{49}\right] \cup \left[\frac{44}{49},\frac{45}{49}\right]$$
$$\cup \left[\frac{46}{49},\frac{47}{49}\right] \cup \left[\frac{48}{49},1\right]$$

Here, we note that B_2^7 is the union of 4^2 disjoint closed intervals. If the process continues then we can construct B_{n+1}^7 from B_n^7 inductively. Now divide each of the 4^n disjoint closed intervals of B_n^7, and remove from each one of them the 2nd, 4th and the 6th open intervals. It is now clear that what is left from B_n^7 is B_{n+1}^7 and that B_{n+1}^7 is the union of 4^{n+1} disjoint closed intervals. We notice here that $B_{n+1}^7 \subset B_n^7$ is true for all n. Hence, we define the Cantor $\frac{1}{7}$th set on [0,1] as

$$B^7 = \bigcap_{n=0}^{\infty} B_n$$

Following the discussion so far, it is easy to see that the total length of the Cantor $\frac{1}{7}$th set is 0, and that the total length of the open intervals removed is 1. From the first iteration, an interval of $\frac{3}{7}$ was removed from B_0. $\frac{12}{49}$ was removed during the second iteration while $\frac{48}{343}$ was removed during the third iteration. As the iteration process continues, the length of the removed interval at each iteration level forms a geometric series $\frac{3}{7}$, $\frac{12}{49}$, $\frac{48}{343}$ . . . with the first term $a = \frac{3}{7}$ and a common ratio of $r = \frac{4}{7}$. This series has a sum $S_\infty = \frac{\frac{3}{7}}{1-\frac{4}{7}} = 1$. This shows that the total length of the intervals removed is 1. However, since we started with $B_0 = [0,1]$ with a total length of 1, this means the Cantor $\frac{1}{7}$ set has a length of 0.

We now put forward a general relation as n approaches infinity.

$$B_\infty^7 = \left[0,\frac{1}{7^n}\right] \cup \left[\frac{2}{7^n},\frac{3}{7^n}\right] \cup \left[\frac{4}{7^n},\frac{5}{7^n}\right] \cup \left[\frac{6}{7^n},\frac{7}{7^n}\right] \cup \left[\frac{(2.7^1)}{7^n},\frac{(2.7^1)+1}{7^n}\right] \cup \left[\frac{(2.7^1)+2}{7^n},\frac{(2.7^1)+3}{7^n}\right] \cup \left[\frac{(2.7^1)+4}{7^n},\frac{(2.7^1)+5}{7^n}\right] \cup \left[\frac{(2.7^1)+6}{7^n},\frac{(2.7^1)+7}{7^n}\right] \cup \left[\frac{(2.7^1)+14}{7^n},\frac{(2.7^1)+15}{7^n}\right] \cup \left[\frac{(2.7^1)+16}{7^n},\frac{(2.7^1)+17}{7^n}\right] \cup \left[\frac{(2.7^1)+18}{7^n},\frac{(2.7^1)+19}{7^n}\right] \cup \left[\frac{(2.7^1)+20}{7^n},\frac{(2.7^1)+21}{7^n}\right] \cup \left[\frac{(2.7^1)+28}{7^n},\frac{(2.7^1)+29}{7^n}\right] \cup \left[\frac{(2.7^1)+30}{7^n},\frac{(2.7^1)+31}{7^n}\right] \cup \left[\frac{(2.7^1)+34}{7^n},\frac{(2.7^1)+35}{7^n}\right] \cup \left[\frac{(2.7^2)}{7^n},\frac{(2.7^2)+1}{7^n}\right] \cup \ldots$$

When $\lambda = 4$, we have $\frac{1}{9}$. Now, we let $B_0^9 = [0,1]$ and divide [0,1] into nine equal open intervals and remove the 2nd, 4th, 6th and the 8th open intervals.

$$\left(\frac{1}{9},\frac{2}{9}\right),\left(\frac{3}{9},\frac{4}{9}\right),\left(\frac{5}{9},\frac{6}{9}\right),\left(\frac{7}{9},\frac{8}{9}\right)$$

This gives the closed intervals left in [0,1] as

$$B_1^9 = \left[0,\frac{1}{9}\right],\left[\frac{2}{9},\frac{3}{9}\right],\left[\frac{4}{9},\frac{5}{9}\right],\left[\frac{6}{9},\frac{7}{9}\right],\left[\frac{8}{9},1\right]$$

Hence B_1^9 becomes the union of 5^1 disjoint closed intervals. Next, divide each closed interval of B_1^9 into nine and remove from each interval the 2nd, 4th, 6th and 8th open intervals

$$\begin{gathered}\left(\frac{1}{81},\frac{2}{81}\right),\left(\frac{3}{81},\frac{4}{81}\right),\left(\frac{5}{81},\frac{6}{81}\right),\left(\frac{7}{81},\frac{8}{81}\right)\\ \left(\frac{19}{81},\frac{20}{81}\right)\left(\frac{21}{81},\frac{22}{81}\right)\left(\frac{23}{81},\frac{24}{81}\right),\left(\frac{25}{81},\frac{26}{81}\right)\\ ,\left(\frac{37}{81},\frac{38}{81}\right),\left(\frac{39}{81},\frac{40}{81}\right),\left(\frac{41}{81},\frac{42}{81}\right),\left(\frac{43}{81},\frac{44}{81}\right),\\ \left(\frac{55}{81},\frac{56}{81}\right),\left(\frac{57}{81},\frac{58}{81}\right),\left(\frac{59}{81},\frac{60}{81}\right),\left(\frac{61}{81},\frac{62}{81}\right),\\ \left(\frac{73}{81},\frac{74}{81}\right),\left(\frac{75}{81},\frac{76}{81}\right),\left(\frac{77}{81},\frac{78}{81}\right),\left(\frac{79}{81},\frac{80}{81}\right)\end{gathered}$$

Again, let B_2^9 be the set remaining from B_1^9 after removing the open interval. Hence, we have

$$\begin{gathered}B_2^9 = \left[0,\frac{1}{81}\right] \cup \left[\frac{2}{81},\frac{3}{81}\right] \cup \left[\frac{4}{81},\frac{5}{81}\right] \cup \left[\frac{6}{81},\frac{7}{81}\right] \cup \left[\frac{8}{81},\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{19}{81}\right] \cup \left[\frac{20}{81},\frac{21}{81}\right] \cup \left[\frac{22}{81},\frac{23}{81}\right]\\ \cup \left[\frac{24}{81},\frac{25}{81}\right] \cup \left[\frac{26}{81},\frac{3}{9}\right] \cup \left[\frac{4}{9},\frac{37}{81}\right] \cup \left[\frac{38}{81},\frac{39}{81}\right] \cup \left[\frac{40}{81},\frac{41}{81}\right] \cup \left[\frac{42}{81},\frac{43}{81}\right] \cup \left[\frac{44}{81},\frac{5}{9}\right]\\ \cup \left[\frac{6}{9},\frac{55}{81}\right] \cup \left[\frac{56}{81},\frac{57}{81}\right] \cup \left[\frac{58}{81},\frac{59}{81}\right] \cup \left[\frac{60}{81},\frac{61}{81}\right] \cup \left[\frac{62}{81},\frac{7}{9}\right] \cup \left[\frac{8}{9},\frac{73}{81}\right]\\ \cup \left[\frac{74}{81},\frac{75}{81}\right] \cup \left[\frac{76}{81},\frac{77}{81}\right] \cup \left[\frac{78}{81},\frac{79}{81}\right] \cup \left[\frac{80}{81},1\right]\end{gathered}$$

Here, we note that B_2^9 is the union of 5^2 disjoint closed intervals. If the process continues then we can construct B_{n+1}^9 from B_n^9 inductively. Now divide each of the 5^n disjoint closed intervals of B_n^9, and remove from each one of them the 2nd, 4th, 6th and the 8th open intervals. It is now clear that what is left from B_n^9 is B_{n+1}^9 and that B_{n+1}^9 is the union of 5^{n+1} disjoint closed intervals. We notice here that $B_{n+1}^9 \subset B_n^9$ which is true for all n. Hence, we define the cantor $\frac{1}{9}$th set on [0,1] as

$$B^9 = \bigcap_{n=0}^{\infty} B_n$$

Following the discussion so far, it is easy to see that the total length of the Cantor $\frac{1}{9}$th set is 0, and that the total length of the open intervals removed is 1. From the first iteration, an interval of $\frac{4}{9}$ was removed from B_0 . $\frac{20}{81}$ was removed during the second iteration while $\frac{100}{729}$ was removed during the third iteration. As the iteration process continues, the length of the removed interval at each iteration level forms a geometric series $\frac{4}{9}, \frac{20}{81}, \frac{100}{729}$ . . . with the first term $a = \frac{4}{9}$ and a common ratio of $r = \frac{5}{9}$. This series has a sum $S_\infty = \frac{\frac{4}{9}}{1-\frac{5}{9}} = 1$. This shows that the total length of the intervals removed is 1. However, since we started with $B_0^9 = [0,1]$ with a total length of 1, we can subtract 1 from 1. This means the Cantor $\frac{1}{9}$ set has a length of 0.

We now put forward a general relation as n approaches infinity.

$$B_\infty^9 = \left[0, \frac{1}{9^n}\right] \cup \left[\frac{2}{9^n}, \frac{3}{9^n}\right] \cup \left[\frac{4}{9^n}, \frac{5}{9^n}\right] \cup \left[\frac{6}{9^n}, \frac{7}{9^n}\right] \cup \left[\frac{8}{9^n}, \frac{9}{9^n}\right] \cup \left[\frac{(2.9^1)}{9^n}, \frac{(2.9^1)+1}{9^n}\right] \cup \left[\frac{(2.9^1)+2}{9^n}, \frac{(2.9^1)+3}{9^n}\right] \cup \left[\frac{(2.9^1)+4}{9^n}, \frac{(2.9^1)+5}{9^n}\right] \cup \left[\frac{(2.9^1)+6}{9^n}, \frac{(2.9^1)+7}{9^n}\right] \cup \left[\frac{(2.9^1)+8}{9^n}, \frac{(2.9^1)+9}{9^n}\right] \cup \left[\frac{(2.9^1)+18}{9^n}, \frac{(2.9^1)+19}{9^n}\right] \cup \left[\frac{(2.9^1)+20}{9^n}, \frac{(2.9^1)+21}{9^n}\right] \cup \left[\frac{(2.9^1)+22}{9^n}, \frac{(2.9^1)+23}{9^n}\right] \cup \left[\frac{(2.9^1)+24}{9^n}, \frac{(2.9^1)+25}{9^n}\right] \cup \left[\frac{(2.9^1)+26}{9^n}, \frac{(2.9^1)+27}{9^n}\right] \cup \left[\frac{(2.9^1)+36}{9^n}, \frac{(2.9^1)+37}{9^n}\right] \cup \left[\frac{(2.9^1)+38}{9^n}, \frac{(2.9^1)+39}{9^n}\right] \cup \left[\frac{(2.9^1)+40}{9^n}, \frac{(2.9^1)+41}{9^n}\right] \cup \left[\frac{(2.9^1)+42}{9^n}, \frac{(2.9^1)+43}{9^n}\right] \cup \left[\frac{(2.9^1)+44}{9^n}, \frac{(2.9^1)+45}{9^n}\right] \cup \left[\frac{(2.9^1)+54}{9^n}, \frac{(2.9^1)+55}{9^n}\right] \cup \left[\frac{(2.9^1)+56}{9^n}, \frac{(2.9^1)+57}{9^n}\right] \cup \left[\frac{(2.9^1)+58}{9^n}, \frac{(2.9^1)+59}{9^n}\right] \cup \left[\frac{(2.9^1)+60}{9^n}, \frac{(2.9^1)+61}{9^n}\right] \cup \left[\frac{(2.9^1)+62}{9^n}, \frac{(2.9^1)+63}{9^n}\right] \cup \ldots$$

When $\lambda = 5$, we have $\frac{1}{11}$. Now, we let $B_0^{11} = [0,1]$ and divide [0,1] into eleven equal open intervals and remove the 2nd, 4th, 6th, 8th and the 10th open intervals

$$\left(\frac{1}{11}, \frac{2}{11}\right), \left(\frac{3}{11}, \frac{4}{11}\right), \left(\frac{5}{11}, \frac{6}{11}\right),$$
$$\left(\frac{7}{11}, \frac{8}{11}\right), \left(\frac{9}{11}, \frac{10}{11}\right)$$

Now, after removing these open intervals, we obtain B_1^{11} which is represented by the following six closed intervals

$$B_1^{11} = \left[0, \frac{1}{11}\right], \left[\frac{2}{11}, \frac{3}{11}\right], \left[\frac{4}{11}, \frac{5}{11}\right],$$
$$\left[\frac{6}{11}, \frac{7}{11}\right], \left[\frac{8}{11}, \frac{9}{11}\right], \left[\frac{10}{9}, 1\right]$$

Hence B_1^{11} becomes the union of 6^1 disjoint closed intervals. Next, divide each closed interval of B_1^{11}into eleven equal open intervals and remove from each interval the 2nd, 4th, 6th, 8th and 10th open intervals. This leads to B_2^{11} with 36 closed intervals.

Following from the discussion so far, we note that B_2^{11} is the union of 6^2 disjoint closed intervals. If the process continues then we can construct B_{n+1}^{11} from B_n^{11} inductively. Now divide each of the 6^n disjoint closed intervals of B_n^{11}into eleven equal open intervals and remove from each one of them the 2nd, 4th, 6th, 8th and the 10th open intervals. What will be left in B_n^{11} is B_{n+1}^{11} which is the union of 6^{n+1} disjoint closed intervals. We notice here that $B_{n+1}^{11} \subset B_n^{11}$ which is true for all. Hence, we define the cantor $\frac{1}{11}$th set on [0,1]

$$B^{11} = \bigcap_{n=0}^{\infty} B_n$$

We now show that the total length of the Cantor $\frac{1}{11}$th set is 0, and that the total length of the open intervals removed is 1. From the first iteration, an interval of $\frac{5}{11}$ was removed from B_0. $\frac{30}{121}$ was removed during the second iteration while $\frac{180}{1331}$ was removed during the third iteration. As the iteration process continues, the length of the removed interval at each iteration level forms a geometric series $\frac{5}{11}, \frac{30}{121}, \frac{180}{1331}$, . . . with the first term $a = \frac{5}{11}$ and a common ratio of $r = \frac{6}{11}$.

This series has a sum $S_\infty = \frac{\frac{5}{11}}{1-\frac{6}{11}} = 1$. This shows that the total length of the intervals removed is 1. However, the length of $B_0^{11} = [0,1]$ is 1. Hence, subtracting 1 from 1 gives 0 which is the length of the Cantor $\frac{1}{11}$th set.

We now put forward a general relation as n approaches infinity.

$B_\infty^{11} = \left[0, \frac{1}{11^n}\right] \cup \left[\frac{2}{11^n}, \frac{3}{11^n}\right] \cup \left[\frac{4}{11^n}, \frac{5}{11^n}\right] \cup \left[\frac{6}{11^n}, \frac{7}{11^n}\right] \cup \left[\frac{8}{11^n}, \frac{9}{11^n}\right] \cup \left[\frac{10}{11^n}, \frac{11}{11^n}\right] \cup \left[\frac{(2.11^1)}{11^n}, \frac{(2.11^1)+1}{11^n}\right] \cup \left[\frac{(2.11^1)+2}{11^n}, \frac{(2.11^1)+3}{11^n}\right] \cup \left[\frac{(2.11^1)+4}{11^n}, \frac{(2.11^1)+5}{11^n}\right] \cup \left[\frac{(2.11^1)+6}{11^n}, \frac{(2.11^1)+7}{11^n}\right] \cup \left[\frac{(2.11^1)+8}{11^n}, \frac{(2.11^1)+9}{11^n}\right] \cup \left[\frac{(2.11^1)+10}{11^n}, \frac{(2.11^1)+11}{11^n}\right] \cup \left[\frac{(2.11^1)+22}{11^n}, \frac{(2.11^1)+23}{11^n}\right] \cup \left[\frac{(2.11^1)+24}{11^n}, \frac{(2.11^1)+25}{11^n}\right] \cup \left[\frac{(2.11^1)+26}{11^n}, \frac{(2.11^1)+27}{11^n}\right] \cup \left[\frac{(2.11^1)+28}{11^n}, \frac{(2.11^1)+29}{11^n}\right] \cup \left[\frac{(2.11^1)+30}{11^n}, \frac{(2.11^1)+31}{11^n}\right] \cup \left[\frac{(2.11^1)+32}{11^n}, \frac{(2.11^1)+33}{11^n}\right] \cup \left[\frac{(2.11^1)+44}{11^n}, \frac{(2.11^1)+45}{11^n}\right] \cup \left[\frac{(2.11^1)+46}{11^n}, \frac{(2.11^1)+47}{11^n}\right] \cup \left[\frac{(2.11^1)+48}{11^n}, \frac{(2.11^1)+49}{11^n}\right] \cup \left[\frac{(2.11^1)+50}{11^n}, \frac{(2.11^1)+51}{11^n}\right] \cup \left[\frac{(2.11^1)+52}{11^n}, \frac{(2.11^1)+53}{11^n}\right] \cup \left[\frac{(2.11^1)+54}{11^n}, \frac{(2.11^1)+55}{11^n}\right] \cup \ldots \cup \left[\frac{(2.11^2)}{11^n}, \frac{(2.11^2)+1}{11^n}\right] \cup \ldots$

3. The Fractal Dimension of the Cantor Set

The cantor set is a self-similar object. This means that a small portion of it looks like the whole object. The cantor set can be characterized by means of its dimension. This can be done by defining dimension in terms of scaling properties of shape [13]. We know that in the construction of the cantor set, the magnification factor is 3, since each new iteration, the line segment produced must be stretched three times to be as long as the line segment of the previous iteration.

At each iteration, there are two small lines segments that can be scaled up to reproduce the original line segment. Hence, the dimension of the cantor $\frac{1}{3}rd$ set is given as:

$$3^D = 2$$

Where D is the dimension, 3 is the magnification factor and 2 is the number of copies. Now taking logarithm of both sides and solving for D, we obtain:

$$D = \frac{\log 2}{\log 3} \approx 0.6309 \approx 0.631$$

Hence, the cantor $\frac{1}{3}rd$ set has a fractal dimension of 0.6309.

For $\lambda = 2$, We have $\frac{1}{5}th$, in this case the number of copies is 3 and the magnification factor is 5. Hence, we get:

$$D = \frac{\log 3}{\log 5} \approx 0.6826 \approx 0.638$$

For $\lambda = 3$, We have $\frac{1}{7}th$, here, we have the number of copies to be 4 and the magnification factor is 7. Hence, we get:

$$D = \frac{\log 4}{\log 7} \approx 0.7124 \approx 0.712$$

For $\lambda = 4$, We have $\frac{1}{9}th$, here, we have the number of copies to be 5 and the magnification factor is 9. Hence, we get:

$$D = \frac{\log 5}{\log 9} \approx 0.7324 \approx 0.732$$

When $\lambda = 5$, We have $\frac{1}{11}th$ cantor set, here, we have the number of copies as 6 and the magnification factor to be 11. Hence, we have:

$$D = \frac{\log 6}{\log 11} \approx 0.7472 \approx 0.747$$

4. Generalizing the Cantor $\frac{1}{2\lambda+1}$ Middle Set

From the construction of $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ and $\frac{1}{11}$, we generalize the cantor middle $\frac{1}{2\lambda+1}$ set for $0 < \frac{1}{2\lambda+1} < 1$, where $1 \leq \lambda < \infty$. We can now construct the cantor middle $\frac{1}{2\lambda+1}$ set by following similar iteration process used in the above constructions. Start with the closed interval $B_0 = [0,1]$ and remove the middle open interval $\left(\frac{1}{2\lambda+1}, \frac{2}{2\lambda+1}\right), \left(\frac{3}{2\lambda+1}, \frac{4}{2\lambda+1}\right) \ldots \left(\frac{2(2\lambda+1)^{q+1}}{2\lambda+1}, \frac{2(2\lambda+1)^{(q+1)}+1}{2\lambda+1}\right)$ where q is as defined above. After n iteration, we obtain

$$\left[0, \frac{1}{(2\lambda+1)^n}\right] \cup \left[\frac{2}{(2\lambda+1)^n}, \frac{3}{(2\lambda+1)^n}\right] \cup \left[\frac{4}{(2\lambda+1)^n}, \frac{5}{(2\lambda+1)^n}\right] \cup \ldots \cup \left[\frac{2(2\lambda+1)^{q+1}}{(2\lambda+1)^n}, \frac{2(2\lambda+1)^{q+1}}{(2\lambda+1)^n}\right]$$

Where n and q are as defined above. Hence, we define the cantor middle as

$$B^{\frac{1}{2\lambda+1}} = \bigcap_{n=0}^{\infty} B_n$$

4.1. The Fractal Dimension of the Cantor Middle $\frac{1}{2\lambda+1}$ Set

Generalizing for $\frac{1}{2\lambda+1}$ for $1 \leq \lambda < \infty$, it is easy to see that the number of copies after n iterations is $(\lambda+1)^n$ and the magnification factor is $(2\lambda+1)^n$. Hence, the dimension of the cantor $\frac{1}{2\lambda+1}th$ set is:

$$D = \frac{\log(\lambda+1)^n}{\log(2\lambda+1)^n} = \frac{\log \lambda + 1}{\log 2\lambda + 1} < 1$$

Definition 2: *A set $B \subset \mathbb{R}$ is called a cantor set if the following conditions are satisfied*

1. *B is non-empty*
2. *B is compact*
3. *B is nowhere dense*
4. *B has no isolated points*
5. *B is a perfect set*

Theorem 1: *Every non-empty infinite subset B of the real line is homeomorphic to the Cantor set if*

1. *it is compact*
2. *it is totally disconnected*
3. *it has no isolated points*
4. *it is a perfect set*

Proof: Let m and M be the infimum and the supremum of B respectively. Then the set $B^c = [m, M]$ is the complement of B which is the union of countably many open intervals. Again, if we let C be the cantor one-third middle set, then its complement is $C^c = [0,1] - C$ which is also a countable union of open intervals. We now define a function F such that F maps B homeomorphically onto C as $F: [m, M] \longrightarrow [0,1]$. Hence, we have $F: [m, M] - B \longrightarrow [0,1] - C$ Clearly, both $B^c = [m, M] - B$ and $C^c = [0,1] - C$ are disjoint union of infinitely but countably many open intervals. Now we denote the collection of open intervals whose union is $B^c = [m, M] - B$ by Γ and that of $C^c = [0,1] - C$ by Ψ. We can now construct a function $\phi: \Gamma \longrightarrow \Psi$ and let $\tau_1 \in \Gamma$ be an interval of maximal length and define $\phi(\tau_1) = \left(\frac{1}{3}, \frac{2}{3}\right)$ where $\left(\frac{1}{3}, \frac{2}{3}\right) \in C^c$ Next, we choose two intervals τ_{21} and τ_{22} such that τ_{21} is to the left of τ_1 and τ_{22} is to the right of τ_1 such that they have maximal length in the set Γ. Hence, we have $\phi(\tau_{21}) = \left(\frac{1}{9}, \frac{2}{9}\right)$ and $\phi(\tau_{22}) = \left(\frac{7}{9}, \frac{8}{9}\right)$. Clearly, we can extend the definition of our constructed function ϕ on the entire members of Γ since Γ contains only finitely many sets of length greater than some fixed $\varepsilon > 0$ and since any two intervals in Γ or Ψ have different end points because both B and C are perfect sets. Hence, our constructed function ϕ is bijective and order preserving in the sense that if τ is to the left of τ_3 then $\phi(\tau)$ is to the left of $\phi(\tau_3)$. For every $\tau \in \Gamma$, we define a unique increasing map $F|_\tau: \tau \longrightarrow \phi(\tau)$ such that τ mapped bijectively onto $\phi(\tau)$. We know that both B and C are totally disconnected and hence are nowhere dense and so there exist a continuation $\eta: [m, M] \longrightarrow [0,1]$. Now from the construction of ϕ, the continuation $\eta: [m, M] \longrightarrow [0,1]$ is a well-defined function and is given by $\eta(x) = sup\{\eta(y): y \notin B,\ y \leq x\}$. Let $\varphi = \eta|_B$, then $\varphi: B \longrightarrow C$ is a monotonic increasing continuous bijective map. Hence, we show that the inverse $\mu = \varphi^{-1}$exist and is continuous so without loss of generality, we can say μ is also a monotonically increasing function. Let $x \in C \subset \mathbb{R}$, then the sequence $x_n \longrightarrow x$ has a subsequence which is monotonic. Hence, x_n is monotonic increasing. Clearly, $y = \lim_{n\to\infty} \mu(x_n) = \sup \mu(x_n) \leq \mu(x)$. We now assume that $y < \mu(x)$. Since B is closed, we have $y \in B$ and $\mu^{-1}(y) < x$, hence $y < x_n$ for large n and by monotonicity $y \subset \mu(x_n)$ which contradicts the statement $y = \lim_{n\longrightarrow\infty} \mu(x_n)$. Thus $\mu(x_n) \longrightarrow \mu(x)$ and continuity is achieved.

All endpoints of the closed intervals at any stage of the iteration are members of the Cantor set. Hence, we state and prove the following theorem.

Theorem 2: Let B_∞ is the Cantor set as $\frac{1}{2\lambda+1}$ turns to infinity and let E_n be the set of endpoints of the closed intervals in the Cantor $\frac{1}{2\lambda+1}$ set such that $E_\infty = \bigcup_n^\infty E_n$ is the union sets of E_n. Then an endpoint $y \in B_\infty$ if and only if $y \in E_\infty \subset B_\infty$.

Proof: It is clear that $E_n \subset E_\infty$ since E_∞ is the union of sets of E_n. Then there exist an open set ψ such that $y \in \psi \subset E_n$. Since ψ is open and contained in E_n it follows that $\psi \in E_\infty$. Hence, $y \in \psi$ implies that $y \in E_\infty$. Again, if we let $y \in E_\infty$ and allow $\psi = E_\infty$, then ψ is an open set such that $y \in \psi \subset E_n$. Since E_∞ is the collection of endpoints of the cantor set and the cantor set contains it endpoints, it implies that $E_\infty \subset B_\infty$.

Definition 3: *A subset A of a metric space Γ is nowhere dense if its closure has an empty interior*.

Lemma 1: *For each $n \in \mathbb{N}$, if B_n is defined in the cantor $\frac{1}{2\lambda+1}$ middle set, then there are $(\lambda+1)^n$ closed intervals in B_n each of which has a length of $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^n$, where $1 \leq \lambda < \infty$.*

Proof: Let $B_0 = [0,1]$ with a length of 1. We start the iteration by removing $\frac{1}{2\lambda+1}$ from B_0 and obtain $(\lambda+1)$ closed intervals of lengths $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{n}$, Suppose we continue the iteration k –times and obtain B_k , then there are $(\lambda+1)^k$ intervals remaining in B_k and each of these intervals will have $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k}$. Now we need to show that there are $(\lambda+1)^{k+1}$ intervals remaining in B_{k+1} each of length $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k+1}$. It is clear that the removing $\frac{1}{2\lambda+1}$ from a closed interval divides the closed interval into $(\lambda+1)$ closed intervals, hence in moving from B_k to B_{k+1} we have $(\lambda+1)(\lambda+1)^k = (\lambda+1)^{k+1}$ intervals in B_{k+1}. Now by our assumption, each interval in B_k has a length of $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k}$ and since we removed the middle $\frac{1}{2\lambda+1}$ portion of each interval in B_k to create B_{k+1}, the intervals left in B_{k+1} is

$$\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k} - \frac{1}{2\lambda+1}\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k} = \frac{\left(1-\frac{1}{2\lambda+1}\right)^{k+1}}{(2\lambda)^k}$$

because this is the amount of interval left in 2λ intervals, the length of each remaining interval is

$$\frac{1}{2\lambda}\left[\frac{\left(1-\frac{1}{2\lambda+1}\right)^{k+1}}{(2\lambda)^k}\right] = \left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k+1}$$

Lemma 2: *For each $n \in \mathbb{N}$, if B_n is defined in the cantor $\frac{1}{2\lambda+1}$ middle set, then the length of each interval in B_n is $\left(\frac{\lambda+1}{2\lambda+1}\right)^n$ and it approaches zero as n approaches infinity for all $1 \le \lambda < \infty$.*

Proof: Let $B_0 = [0,1]$ with length 1. We remove the open interval $\frac{1}{2\lambda+1}$ from B_0 and obtain $\lambda+1$ closed intervals with a total of length of $(\lambda+1)\left(\frac{1}{2\lambda+1}\right) = \frac{\lambda+1}{2\lambda+1}$. Suppose there are $\lambda+1$ close intervals that remain in B_k then the total intervals will be $(\lambda+1)^k$ and each of these intervals will have $\left(\frac{1}{2\lambda+1}\right)^k$ as its length. We need to show that the total length of the intervals is $\left(\frac{\lambda+1}{2\lambda+1}\right)^{k+1}$. It is easy to see that removing $\frac{1}{2\lambda+1}$ from a closed interval divides the closed interval into $\lambda+1$ closed intervals, hence in moving from B_k to B_{k+1} we have $(\lambda+1)(\lambda+1)^k = (\lambda+1)^{k+1}$ intervals in B_{k+1}. by lemma 1, each of the remaining intervals in B_k has a length of $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k+1}$ Hence the total length of the intervals in B_{k+1} is

$$(\lambda+1)^{k+1}\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{k+1} = \left(\frac{\lambda+1}{2\lambda+1}\right)^{k+1}$$

Where $1 \le \lambda < \infty$. Since $0 < \frac{1}{2\lambda+1} < 1$, $\left(\frac{\lambda+1}{2\lambda+1}\right)^n$ converges to 0 as n increases without bound. Therefore, the total length of the intervals in B_n approaches 0 as n goes to infinity.

Theorem 3: Given that $1 \le \lambda < \infty$, let B be defined by removing the middle $\frac{1}{2\lambda+1}$ portion of the real unit closed interval [0,1], then B is a cantor set.

To prove this, we need to show that it satisfies the four conditions in our cantor set definition.

Proof: 1. From the construction, each time we remove $\frac{1}{2\lambda+1}$ of B_n to create B_{n+1} there remain exactly two points and since B is the intersection of each of B_n, B contains the endpoints of each subinterval and therefore B is non-empty. Again, it is easy to see that 0 is in each B_n. Hence B is non-empty.

2. We need to show that B is close and bounded. B is the intersection of closed sets and so it follows that B is closed. Again, we know that B is contained in [0,1] and [0,1] is bounded it follows that B is bounded since the subset of a bounded set is also bounded, hence that B is compact.

3. To show that B is nowhere dense, we must show that it contains no open intervals. Assume that B contains an open interval with length ξ. However, from lemma 1, each interval in B_n after n iterations is $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{n}$, for $1 \le \lambda\infty$. We then find an n_0 such that $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{n_0} < \xi$, for $1 \le \lambda\infty$, that is the length of each of the closed intervals in B_n is less

than ξ, implying that the entire interval ξ cannot be contained in B_{n_0} and so B contains no interval and hence B is nowhere dense.

4. We must show that every point in B is a limit point. Suppose $x \in B$, we show that for all $\varepsilon > 0$, there exist $x_1 \in B$ such that $|x - x_1| < \varepsilon$ and x is not equal to x_1. From lemma 1, let $x \in B$ and $\varepsilon > 0$, choose n such that $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{n} < \varepsilon$. Since $x \in B$ and $B = \cap B_n$, then $x \in B_n$. Let Γ be a component of B_n such that $x \in \Gamma$ and the length of Γ be $\left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{n}$, this implies that $\Gamma \cap B_{n+1}$ has two components Γ_0 and Γ_1 and x must be in either one of the two components but not both. Let assume $x \in \Gamma_0$ then Γ_0 must contain a point in B say x_1. Then x_1 is an endpoint of Γ_0. Hence, $|x - x_0| \leq \left(\frac{1-\frac{1}{2\lambda+1}}{2\lambda}\right)^{n} < \varepsilon$.

5. From 2, B is closed and from 4 B has no isolated point hence B is a perfect set. And the proof is complete.

5. The Cantor Set is Heine-Borel Set

In this section we introduce and define what we call the Heine-Borel set and show that the cantor set is a Heine-Borel set which implies that the cantor $\frac{1}{2\lambda+1}$ set is a Heine-Borel set as well. It should be noted that what we call Heine-Borel set is a derivative of the well-known Heine-Borel theorem.

Theorem 4 [14]. A subset of $\mathbb{R}^k$ is compact if and if only it is closed and bounded.

Proof: Suppose that $A \subset \mathbb{R}^k$ is compact, then A is a compact subset of a Hausdorff space, hence A is closed. Since $\mathbb{R}^k$ is covered by the open boxes $(-n, n)^k$ with $n \in \mathbb{N}$, finitely many of these boxes must cover A. So, A is bounded as well. Conversely, suppose that $A \subset \mathbb{R}^k$ is closed and bounded, then there is a positive integer N such that A is contain in the closed box $[-N, N]$. We note that this box is compact because a finite product of compact spaces is compact by induction. In particular, A is a closed subset of a compact space, hence also compact.

Definition 4: Let a non-empty set Ω be a closed subset of $\mathbb{R}^k$, then a non-empty subset Γ of Ω is said to be a Heine-Borel set if it is closed and bounded in Ω.

Lemma 3: The cantor $\frac{1}{2\lambda+1}$ middle set is a Heine-Borel set.

Proof: From theorem 2, B is non-empty and bounded in [0,1]. Again [0,1] is closed and bounded in R and therefore B is a Heine- Borel set.

Definition 5: A set is a Borel set if it can be formed from an open or closed set by repeatedly taking countable unions, countable intersections and relative complements.

Lemma 4: The cantor $\frac{1}{2\lambda+1}$ middle set is a Borel set.

Proof: Since arbitrary intersection of closed set is closed and the Cantor $\frac{1}{2\lambda+1}$ set is the intersection of closed set. It is by definition a closed set and therefore a Borel set.

6. CONCLUSION

In this paper we have been able to construct a family of cantor middle sets using $\frac{1}{2\lambda+1}$ as the generator for $1 \leq \lambda < \infty$ we calculated their fractal dimensions and concluded that, these fractal dimensions form an increasing sequence which converges to 1 as λ approaches infinity. We have also defined a set from the Heine-Borel theorem called the Heine-Borel set and have shown that the cantor $\frac{1}{2\lambda+1}$ middle set is a Heine-Borel set. Again, we have shown that for any $\lambda = 1, 2, 3, \ldots n$, the cantor middle $\frac{1}{2\lambda+1}$ set is homeomorphic to the cantor middle $\frac{1}{3}$ set.

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