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THE CLASS OF (n, m) POWER -A-QUASI-HYPONORMAL OPERATORS IN SEMI-HILBERTIAN SPACE

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ABSTRACT

The concept of K-quasi-hyponormal operators on semi-Hilbertian space is defined by Ould Ahmed Mahmoud Sid Ahmed and Abdelkader Benali in [7].This paper is devoted to the study of new class of operators on semi-Hilbertian space $(\mathcal{H}, \|.\|_A)$called (n, m)power-A-quasi-hyponormal denoted$[(n, m)QH]_A$.We give some basic properties of these operators and some examples are also given .An operator $T \in \mathcal{B}_A(\mathcal{H})$is said to be (n, m) power-A- quasi-hyponormal for some positive operator $A$and for some positive integers $n$and $m$if $T^\sharp((T^\sharp)^m T^n - T^n(T^\sharp)^m)T \geq_A 0$or equivalently $AT^\sharp((T^\sharp)^m T^n - T^n(T^\sharp)^m)T \geq 0$

KEYWORDS: Semi-Hilbertian space, A-positive, A-normal, A-hyponormal, A-quasi-hyponormal.

INTRODUCTION AND PRELIMINARIES RESULTS

A bounded linear operator T on a complex Hilbert space is (A, K) quasi-hyponormal operator if $T^K(T^\sharp T - TT^\sharp)T^k T \geq_A 0$. In the year 2016 the authors Ould Ahmed Mahmoud Sid Ahmed and Abdelkader Benali introduced the class of (A, K)quas-hyponormal operators and studied some proprietes of this class .From definition it is easily seen that this class contains the class of quasi-hyponormal operators. For more details see [7].The purpose of this paper is to study the class of (n, m) power-A-quasi-hyponormal operators in semi-hilbertian spaces. This manuscript has been organized in two sections. In section 1 we give notation and results about the concept of A-adjoint operators that will be useful in the sequel. In Section 2we introduce a new concept of quasi-hyponormality of operators in semi-Hilbertian space $(\mathcal{H}, \langle .|.\rangle_A)$called (n, m) power-A-quasi-hyponormal operator and we investigate various structural properties of this class of operators with some examples studied .Moreover the product, direct sum, tensor product and the sum of finite numbers of this type are discussed. Also we study the relationship between this class and the other kinds of classes of operators in semi-Hilbertian spaces.

We start by introducing some notations. Throughout this paper $\mathcal{H}$denotes a complex Hilbert space with inner product $\langle .|.\rangle_A$, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on$\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+$is the cone of positive operators of $\mathcal{B}(\mathcal{H})$difined as

$$\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) : \langle Tx|x\rangle \geq 0, \forall x \in \mathcal{H}\}$$

For every $T \in \mathcal{B}(\mathcal{H})$,its null is denoted by $\mathcal{N}(T)$, its range is denoted by $\mathcal{R}(T)$,its closure of range is denoted by $\overline{\mathcal{R}(T)}$ and its adjoint operator by T^*.The closed linear subspace $\mathcal{M}$ is called invariant subspace of T, if satisfying $T\mathcal{M} \subset \mathcal{M}$.In addition if $\mathcal{M}$ also is invariant subspace of T^*, then $\mathcal{M}$ is called a reducing subspace of T. We denote the orthogonal projection onto a closed linear subspace $\mathcal{M}$by $P_\mathcal{M}$.Note that for $A \in \mathcal{B}(\mathcal{H})^+$, the functional

$$\langle .|.\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad \langle u|v\rangle_A = \langle Au|v\rangle$$

is a semi-inner product on $\mathcal{H}$. By $\|.\|_A$we denote the semi-norm induced by $\langle .|.\rangle_A$ i.e$\|u\|_A = \langle u|u\rangle_A^{\frac{1}{2}} = \langle Au|u\rangle^{\frac{1}{2}}$.Observe that $\|u\|_A = 0$ if and only if $u \in \mathcal{N}(A)$, then $\|.\|_A$ is a norm if and only if A is an injective operator and the semi-normed space $(\mathcal{H}, \|.\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. The above semi-norm induces a semi-norm on the subspace $\mathcal{B}^A(\mathcal{H})$ of

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$\mathcal{B}(\mathcal{H})$. $\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) / \exists c > 0, \|Tu\|_A \leq c\|u\|_A, \forall x \in \mathcal{H}\}$

Indeed, if $T \in \mathcal{B}^A(\mathcal{H})$, then $\| T \|_A = sup \left\{ \frac{\|Tu\|_A}{\|u\|_A}, u \in \overline{\mathcal{R}(A)} \wedge u \neq 0 \right\}$

Operator in $\mathcal{B}^A(\mathcal{H})$, is called$A$-bounded operator.

From now A denoted a positive operator on $\mathcal{H}$,that is $A \in \mathcal{B}(\mathcal{H})^+$.

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$, is called an A-adjoint operator of T if for every u , $v \in \mathcal{H}$, we have $\langle Tu|v\rangle$ = $\langle u|Sv\rangle$ that is $AS = T^*A$,if T is an A-adjoint of itself, then T is called an A-selfadjoint operator $AS = T^*A$,. Generally, the existence of an$A$ adjoint operator is not guaranteed .The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem [21].We have that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \ / \ \mathcal{R}(T^*A) \subset \mathcal{R}(A)\}$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A-adjoint operator of T, namely the reduced solution of equation $AX = T^*A$, This operator is denoted by $T^\#$.Therefore , $T^\# = A^\dagger T^*A$ and $AT^\# = T^*A$, $\mathcal{R}(T^\#) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{N}(T^\#) = \mathcal{N}(T^*A)$

Note that in which $A^\dagger$ is the Moore-Penrose inverse of A. For more details see ([4],[5],[6]). In thenext proposition we collect some properties of $T^\#$. and its relationship with the semi-norm $\|T\|_A$. For the proof see ([4],[5]).

Proposition 1.1 Let$T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold

1. $T^\# \in \mathcal{B}_A(\mathcal{H})$,$(T^\#)^\# = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and$((T^\#)^\#)^\# = T^\#$
2. If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^\# = S^\# T^\#$
3. $TT^\#$ and $T^\#T$ are A-seladjoint
4. $\|T\|_A = \|T^\#\|_A = \|T^\#T\|_A^{\frac{1}{2}} = \|TT^\#\|_A^{\frac{1}{2}}$
5. $\|S\|_A = \|T^\#\|_A$ for every $S \in \mathcal{B}_A(\mathcal{H})$ which is an A -adjoint of T
6. If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$

The classes of normal,quasinormal,isometries,hyponormal,quasihyponormal and m-isometries on Hilbert spaces have been generalized to semi-Hilbert spaces by many authors in ([4],[5],[13],[16],[20]) and other papers

In the following definition we collect the notions of some classes of operators

Definition 1.1 Any operators$T \in \mathcal{B}_A(\mathcal{H})$. is called

1. A-normal if $TT^\# = T^\#T$

2. A-isometry if $T^\#T = P_{\overline{\mathcal{R}(A)}}$

3. A-unitary if if $T^\#T = TT^\# = P_{\overline{\mathcal{R}(A)}}$

4. (A, n)-hyponormal if $T^\#T^n \geq_A T^nT^\#$

5. (n, m) power A-hyponormal if $(T^\#)^mT^n \geq_A T^n(T^\#)^m$

6. A-quasinormal if $TT^\#T = T^\#T^2$

The following definition and results are useful for our study.

Definition 1.2 We say that$T \in \mathcal{B}(\mathcal{H})$ is an A-positive if $AT \in \mathcal{B}(\mathcal{H})^+$ or equivalently$\langle Tu|u\rangle_A \geq 0, \forall u \in \mathcal{H}$.We note $T \geq_A 0$

Example 1.1 If$T \in \mathcal{B}_A(\mathcal{H})$, then$T^\#T$ and $TT^\#$are A-positive i.e

$$T^\#T \geq_A 0 \text{ and } TT^\# \geq_A 0$$

Remark 1.1 We can define a order relation by$T \geq_A S \Leftrightarrow T - S \geq_A 0$

Lemma 1.2([7])Let$T, S \in \mathcal{B}(\mathcal{H})$such that$T \geq_A S$ and let$R \in \mathcal{B}_A(\mathcal{H})$, then the following properties hold

1.$R^\#TR \geq_A R^\#SR$

2.$RTR^\# \geq_A RSR^\#$

3. If R is$A$-selfadjoint then $RTR \geq_A RSR$

Lemma 1.3 Let$T, S \in \mathcal{B}(\mathcal{H})$are A-positive operators, if T commutes with S then TS is A-positive

Proof. Let$T, S \in \mathcal{B}(\mathcal{H})$are A-positive . Since T is A-positive then exists only one operator R is A-positive such that $R^2 = T$ i.e $R = T^{\frac{1}{2}}$ and commutes with S

Then for all $u \in \mathcal{H}$ we have$\langle TSu|u\rangle_A$= $\langle R^2Su|u\rangle_A = \langle RSu|Ru\rangle_A$=$\langle S(Ru)|Ru\rangle_A \geq 0$

Hence $TS \geq_A 0$

2. Class of (n, m)Power -A-Quasi-Hyponormal Operators in Semi-Hilbertian Space

Hyponormal and$K$-quasi-hyponormal operators on Hilbert spaces and semi-Hilbert spaces have received considerable attention in the current literaturein ([7],[10],[17],[18],[19]) .From which our inspiraton comes. In this section, we introduce the concept of (n, m)power-A-quasi-hyponormal Operators in semi-Hilbertian Space.

Definition 2.1 An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (n,m) power- A-Quasi-hyponormal operator for positive integers n,m if $T^{\#}((T^{\#})^m T^n - T^n (T^{\#})^m)T \geq_A 0$
i.e $AT^{\#}((T^{\#})^m T^n - T^n (T^{\#})^m)T \geq 0$
We denote the set of all (n,m) power-A-Quasi-hyponormal operators by$[(n,m)QH]_A$

Remark 2.1 We observed this results

1. if $n = m = 1$ then $(1,1)$power-A-Quasi-hyponormal be commes -A-Quasi-hyponormal i.e $[(1,1)QH]_A = [QH]_A$

2. from lemma $(1,2)$it is clear that every (n,m)power A-hyponormal operator is (n,m)power A-quasihyponormal operator i.e $[(n,m)H]_A \subset [(n,m)QH]_A$

The following examples show that there exits a (n,m) power-A-quasi-Hyponormal operator for some positive integers $n$and m but is not (n,m) power-A-hyponormal

Example 2.1 let$T = \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
be operators acting on two dimensional Hilbert space $\mathbb{C}^2$. A simple calculation shows that
$A \geq 0$, $T^* = \begin{pmatrix} -3 & 0 \\ 2 & 0 \end{pmatrix}$, and$T^{\#} = \begin{pmatrix} -3 & 0 \\ 4 & 0 \end{pmatrix}$
It is easy to chek that $T^{\#}((T^{\#})^2 T - T(T^{\#})^2)T \geq_A 0$ and $(T^{\#})^2 T - T(T^{\#})^2 \ngeq_A 0$then T is of class $[(1,2)QH]_A$ but is not of class $[(1,2)H]_A$

Example 2.2 Let $S = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}andA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$
A simple computation shows that
$S^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $S^{\#} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$
It is easily to see that $S \in [(3,2)QH]_A$ but $S \notin [(3,2)H]_A$
In the folloing theorem we give the characterization of (n,m) power -A-quasi-hyponormal

Theorem 2.1 Let $\mathrm{T} \in \mathcal{B}_{\mathrm{A}}(\mathcal{H})$.then T is (n,m) power-A-hyponormal operateur for some positive integers n and m if and only if T satisfing the following condiction
$\langle T^{n+1}u | T^{m+1}u \rangle_A \geq \langle (T^{\#})^m Tu | (T^{\#})^n Tu \rangle_A$, $\forall u \in \mathcal{H}$
Proof. Assume that $T$is a (n,m) power-A-quasi-hyponormal operator then

$$AT^{\#}((T^{\#})^m T^n - T^n (T^{\#})^m)T \geq_A 0 \Leftrightarrow \langle AT^{\#}((T^{\#})^m T^n - T^n (T^{\#})^m)Tu|u \rangle \geq 0 , \forall u \in \mathcal{H}$$
$$\Leftrightarrow \langle A(T^{\#})^{m+1} T^{n+1} u|u \rangle \geq \langle AT^{\#} T^n T^{\# m} Tu|u \rangle$$
$$\Leftrightarrow \langle (T^*)^{m+1} AT^{n+1} u|u \rangle \geq \langle T^* AT^n T^{\# m} Tu|u \rangle$$
$$\Leftrightarrow \langle AT^{n+1} u|T^{m+1} u \rangle \geq \langle AT^n T^{\# m} Tu|Tu \rangle$$
$$\Leftrightarrow \langle AT^{n+1} u|T^{m+1} u \rangle \geq \langle T^{\# m} Tu|(T^*)^n ATu \rangle$$
$$\Leftrightarrow \langle AT^{n+1} u|T^{m+1} u \rangle \geq \langle T^{\# m} Tu|AT^{\# n} Tu \rangle$$
$$\Leftrightarrow \langle T^{n+1} u|T^{m+1} u \rangle_A \geq \langle T^{\# m} Tu|T^{\# n} Tu \rangle_A$$

The proof is complete

Remark 2.2 a) If $n = m$ then

$$T \in [(n,n)QH]_A \Leftrightarrow \langle T^{n+1}u|T^{n+1}u \rangle_A \geq \langle T^{\# n}Tu|T^{\# n}Tu \rangle_A$$
$$\Leftrightarrow \|T^{n+1}u\|_A \geq \|T^{\# n}Tu\|_A$$

b) If $n = m = 1$ then, $T \in [QH]_A \Leftrightarrow \|T^2 u\|_A \geq \|T^{\#} Tu\|_A$

Lemma 2.2 *Let* $\mathrm{T} \in \mathcal{B}_{\mathrm{A}}(\mathcal{H})$ be an (n,n) power- A -Quasi-hyponormal operator such that $TT^{\#} = T^{\#}T$ i.e (T is A-normal) then,$\|T^{\# n}\|_A^2 \leq \|T^{\# n-1}Tu\|_A \|T^{n+1}u\|_A$
Proof. Assume $T \in [(n,n)QH]_A$ and $TT^{\#} = T^{\#}T$ then

$$\|T^{\# n}\|_A^2 = \langle T^{\# n}u|T^{\# n}u \rangle_A = \langle AT^{\#}T^{\# n-1}u|T^{\# n}u \rangle$$
$$= \langle AT^{\# n-1}u|TT^{\# n}u \rangle$$
$$= \langle AT^{\# n-1}u|T^{\# n}Tu \rangle$$
$$= \left\langle A^{\frac{1}{2}}T^{\# n-1}u \middle| A^{\frac{1}{2}}T^{\# n}Tu \right\rangle$$

$=\|T^{\# n-1}u\|_A \|T^{\# n}Tu\|_A$
$\leq \|T^{\# n-1}u\|_A \|T^{n+1}u\|_A$ remark(2,2)
The following discusses the conditions for product and sum of two (n,m) power-A-Quasihyponormal operators to be (n,m) power-A-Quasihyponormal.

Proposition 2.3 Let $T, S \in \mathcal{B}_A(\mathcal{H})$ be (n,n) power-A-quasi-hyponormal. Then the following properties hold

1. if $(T^{\#n}T)S = S(T^{\#n}T)$ and $(TS)^k = T^kS^k$ for $k \in \{n, n+1\}$ then TS is an (n,n) power-A- quasi-hyponormal.
2. if $(S^{\#n}S)T = T(S^{\#n}S)$ and $(TS)^k = T^kS^k$ for $k \in \{n, n+1\}$ then ST is an (n,n) power-A- quasi-hyponormal.

Proof **1**. Assume T and S are (n,n) power-A-quasi-hyponormal, we have for all $u \in \mathcal{H}$

$$\|(TS)^{\#n}(TS)u\|_A = \|S^{\#n}T^{\#n}TSu\|_A$$

$= \|S^{\#n}ST^{\#n}Tu\|_A$ (condition 1)

$$\leq \|S^{n+1}T^{\#n}Tu\|_A$$
$$= \|T^{\#n}TS^{n+1}u\|_A$$
$$\leq \|T^{n+1}S^{n+1}u\|_A$$
$$= \|(TS)^{n+1}u\|_A$$

Hence $(TS) \in [(n,n)QH]_A$

2. $\|(ST)^{\#n}(ST)u\|_A = \|T^{\#n}S^{\#n}STu\|_A$

$= \|T^{\#n}TS^{\#n}Su\|_A$ (condition 2)

$$\leq \|T^{n+1}S^{\#n}Su\|_A$$
$$= \|S^{\#n}ST^{n+1}u\|_A$$
$$\leq \|S^{n+1}T^{n+1}u\|_A$$
$$= \|(ST)^{n+1}u\|_A$$

Then $(ST) \in [(n,n)QH]_A$

Proposition 2.4 Let $T, S \in \mathcal{B}_A(\mathcal{H})$ be commuting (n,n) power-A-quasi-hyponormal. Then the following statements hold

1. If $TS^{\#} = S^{\#}T$ then TS is (n,n) power-A-quasi-hyponormal.

2. If $ST^{\#} = T^{\#}S$ then ST is (n,n) power-A-quasi-hyponormal

Proof **1**. Since T and S are (n,n) power-A-quasi-hyponormal and from condition (1) we have $TS^{\#} = S^{\#}T \Rightarrow T^{n+1}S^{\#n} = S^{\#n}T^{n+1}$

Then for all $u \in \mathcal{H}$ $\|(TS)^{\#n}(TS)u\|_A = \|T^{\#n}S^{\#n}TSu\|_A$

$$= \|T^{\#n}TS^{\#n}Su\|_A$$
$$\leq \|T^{n+1}S^{\#n}Su\|_A$$
$$= \|S^{\#n}ST^{n+1}u\|_A$$
$$\leq \|S^{n+1}T^{n+1}u\|_A$$
$$= \|(ST)^{n+1}u\|_A$$

Hence $(TS) \in [(n,n)QH]_A$

2. We omit the proof, since the techniques are similar to those of 1

The following proposition generalized proposition $(2,3)$

Proposition 2.5 Let $T, S \in \mathcal{B}_A(\mathcal{H})$ be (n,m) power- A -quasi-hyponormal. Then the following properties hold

1. if $T(S^{\#n}S) = (S^{\#n}S)T$ and $(TS)^k = T^kS^k$ for $k \in \{n, n+1\}$ then ST is an (n,m) power-A- quasi-hyponormal.

2. if $(T^{\#n}T)S = S(T^{\#n}T)$ and $(TS)^k = T^kS^k$ for $k \in \{n, n+1\}$ then TS is an (n,m) power-A- quasi-hyponormal.

Proof **1.** $\langle (ST)^{n+1}u \mid (ST)^{m+1}u\rangle_A = \langle S^{n+1}T^{n+1}u \mid S^{m+1}T^{m+1}u\rangle_A$

$\geq \langle S^{\#m}ST^{n+1}u \mid S^{\#n}ST^{m+1}u\rangle_A$ (Since $S \in [(n,m)QH]_A$)

$$= \langle T^{n+1}S^{\#m}Su \mid T^{m+1}S^{\#n}Su\rangle_A$$
$$\geq \langle T^{\#m}TS^{\#m}Su \mid T^{\#n}TS^{\#n}Su\rangle_A$$
$$= \langle T^{\#m}S^{\#m}STu \mid T^{\#n}S^{\#n}STu\rangle_A$$
$$= \langle (ST)^{\#m}STu \mid (ST)^{\#n}STu\rangle_A$$

Hence ST is an (n,m) power-A- quasi-hyponormal.

2. By same way hence TS is an (n,m) power-A- quasi-hyponormal.

Proposition 2.6 Let $T, S \in \mathcal{B}_A(\mathcal{H})$ be (n,m) power- A -quasi-hyponormal operators such that $TS^{\#} = S^{\#}T, ST^{\#} = T^{\#}S$ and $TS = ST = 0$. Then $(S+T)$ is an (n,m) power-A- quasihyponormal operator.

Proof Assume T and S are (n,m) power-A- quasi-hyponormal, uder assumption we have

$$(S+T)^{\#m+1}(S+T)^{n+1} = (S^{\#m+1} + T^{\#m+1})(S^{n+1} + T^{n+1})$$
$$= S^{\#m+1}S^{n+1} + S^{\#m+1}T^{n+1} + T^{\#m+1}S^{n+1} + T^{\#m+1}T^{n+1}$$

$\geq_A S^{\#}S^n.S^{\#m}S + S^{\#m+1}T^{n+1} + T^{\#m+1}S^{n+1} + T^{\#}T^n.T^{\#m}T$ (2.1)

Since $TS = ST = 0$, $TS^{\#} = S^{\#}T$ and $ST^{\#} = T^{\#}S$ we get $S^{\#}S^nS^{\#m}T = S^{\#}S^nT^{\#m}S = S^{\#}S^nT^{\#m}T = S^{\#}T^nS^{\#m}S = T^{\#}S^nS^{\#m}S = T^{\#}S^nT^{\#m}T = T^{\#}T^nS^{\#m}S = 0$

And other terms are null, a simple computation shows that

$(S+T)^{\#}(S+T)^n.(S+T)^{\#m}(S+T)^{\#} = S^{\#}S^nS^{\#m}S + S^{\#}T^nS^{\#m}T + T^{\#}S^nT^{\#m}S + T^{\#}T^nT^{\#m}T$

$= S^{\#}S^nS^{\#m}S + S^{\#m+1}T^{n+1} + T^{\#m+1}S^{n+1} + T^{\#}T^nT^{\#m}T$ (2.2)

Combinig (2.1) and (2.2) we obtain

$(S+T)^{\#m+1}(S+T)^{n+1} \geq_A (S+T)^{\#}(S+T)^n.(S+T)^{\#m}(S+T)^{\#}$

Then $(T+S) \in [(n,m)QH]_A$ and the proof is complete

Proposition 2.6 Let$T,S \in \mathcal{B}_A(\mathcal{H})$such that $\mathcal{N}(A)$ is invariant subspace of T.the following statements are equivalent

1. T is (n,m) power-A-quasi-hyponormal operator
2. T is (m,n) power-A-quasi-hyponormal operator

Proof. from the condition we have $P_{\overline{\mathcal{R}(A)}}T = TP_{\overline{\mathcal{R}(A)}}$,$P_{\overline{\mathcal{R}(A)}}T^\# = T^\# P_{\overline{\mathcal{R}(A)}}$ and $P_{\overline{\mathcal{R}(A)}}A = AP_{\overline{\mathcal{R}(A)}} = A$

(1) implies (2) . Assume that T is a (n,m) power- A -hyponormal operator it follow that

$$(T^\#)^{m+1}T^{n+1} \geq_A T^\# T^n T^{\#m} T \Rightarrow ((T^\#)^{m+1}T^{n+1})^\# \geq_A (T^\# T^n T^{\#m} T)^\#$$
$$\Rightarrow (T^\#)^{n+1}((T^\#)^\#)^{m+1} \geq_A T^\#((T^\#)^\#)^m (T^\#)^n (T^\#)^\#$$
$$\Rightarrow (T^\#)^{n+1}\left(P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}\right)^{m+1} \geq_A T^\#\left(P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}\right)^m (T^\#)^n \left(P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}\right)$$
$$\Rightarrow P_{\overline{\mathcal{R}(A)}}(T^\#)^{n+1}T^{m+1} \geq_A P_{\overline{\mathcal{R}(A)}}T^\# T^m (T^\#)^n T$$
$$\Rightarrow P_{\overline{\mathcal{R}(A)}}\left((T^\#)^{n+1}T^{m+1} - P_{\overline{\mathcal{R}(A)}}T^\# T^m (T^\#)^n T\right) \geq_A 0$$
$$\Rightarrow P_{\overline{\mathcal{R}(A)}}((T^\#)^{n+1}T^{m+1} - T^\# T^m T^{\#n} T) \geq_A 0$$
$$\Rightarrow (T^\#)^{n+1}T^{m+1} - T^\# T^m T^{\#n} T \geq_A 0$$

Then T is a (m,n) power-A-quasi-hyponormal operator

Eqt(2) implies eqt(1) and hence the result.

Example 2.3 Let us consider the operators S and A given in Example $(2,2)$. We have S is (3,2)power-A-quasi-hyponormal then S is (2,3) power-A-quasi-hyponormal

Proposition 2.8 Let$T \in \mathcal{B}_A(\mathcal{H})$such that $T \in ([(1,m)N]_A \cap [(n+1,m)QH]_A)$, if T is A-positive and commutes with $(T^{\#m+1}T^{n+2} - T^\# T^{n+1}.T^{\#m}T)$,then $T \in [(n+2,m)QH]_A$ for some positive integres n and m

*Proof.*Let$T \in ([(1,m)N]_A \cap [(n+1,m)QH]_A)$ then

$T^{\#m}T = TT^{\#m}$ and $(T^{\#m+1}T^{n+2} \geq_A T^\# T^{n+1}.T^{\#m}T)$.Since $T \geq_A 0$ and T commutes with $(T^{\#m+1}T^{n+2} - T^\# T^{n+1}.T^{\#m}T)$,then from lemma (1.3) we deduce that

$$(T^{\#m+1}T^{n+2} - T^\# T^{n+1}.T^{\#m}T)T \geq_A 0 \Rightarrow T^{\#m+1}T^{n+2}.T \geq_A T^\# T^{n+1}.T^{\#m}T.T \geq_A 0$$
$$\Rightarrow T^{\#m+1}T^{n+3} \geq_A T^\# T^{n+2}.T^{\#m}T$$

Hence $T \in [(n+2,m)QH]_A$.

The following examples shows that the classes $[(n,m)QH]_A$and $[(n+1,m)QH]_A$are not the same.

Example 2.4 Let , S be the operators given in Examples $(2,1)$and $(2,2)$ respectively . We have T as of class $(2,1)$power-A-quasi-hyponormal but is not of class $(3,1)$power-A-quasi-hyponormal. Moreover S is of class $(3,2)$power-A-quasi-hyponormal but is not of class $(2,2)$ power-A-quasi-hyponormal.

Proposition 2.9 *Let* $T \in \mathcal{B}_A(\mathcal{H})$be an (n,m) power-A-quasi-hyponormal such that$\mathcal{N}(A)$is a reducing subspace for T, *if* $S = UTU^\#$ where U is A-unitary operator ,then S is (n,m) power- A-quasi-hyponormal operator

*Proof.*Let T be an(n,m) power-A-quasihyponormal operator and $S = UTU^\#$,easily we obtain$S^n = UT^nU^\#$ and $S^\# = UT^\#U^\#$. Hence,

$$(S^\#)^{m+1}S^{n+1} = (U(T^\#)^{m+1}U^\#)(UT^{n+1}U^\#)$$
$$= UT^{\#m+1}U^\# UT^{n+1}U^\#$$
$$= UT^{\#m+1}P_{\overline{\mathcal{R}(A)}}T^{n+1}U^\#$$
$$= UT^{\#m+1}T^{n+1}U^\#$$
$$\geq_A UT^\# T^n.T^{\#m}TU^\# \qquad (Lemma\ (1.2))$$
$$= (UT^\#U^\#)(UT^nU^\#)(UT^{\#m}U^\#)(UTU^\#,)$$
$$= S^\# S^n.S^{\#m}S$$

Hence,$(S^\#)^{m+1}S^{n+1} \geq_A S^\# S^n.S^{\#m}S$ then $S \in [(n,m)QH]_A$

Proposition2.10*Let* $T \in \mathcal{B}_A(\mathcal{H})$,$X = T^{\#m}T - T^\# T^n$,$Y = T^{\#m}T + T^\# T^n$ and $Y = T^{\#m}T.T^\# T^n$ .If $T^\# T = TT^\#$ then the following statements hold

1. T is an(n,m) power-A-hyponormal operator if and only if $[X,Y] \geq_A 0$
2. If $T \in [(n,m)QH]_A$ such that $T^{\#m}T$ and $T^\# T^n$are A-positive and commutes with $(T^{\#m+1}T^{n+1} - T^\# T^n.T^{\#m}T)$ then ,$[X,Z] \geq_A 0$ and $[Z,Y] \geq_A 0$
3. T is an(n,m) power-A-quasihyponormal operator if and only if $[X,T^{\#m}T] \geq_A 0$
4. T is an(n,m) power-A-quasihyponormal operator if and only if $[T^\# T^n,Y] \geq_A 0$

Proof **1.** $[X,Y] \geq_A 0 \Leftrightarrow XY - YX \geq_A 0$

$$\Leftrightarrow (T^{\#m}T - T^\# T^n)(T^{\#m}T + T^\# T^n) - (T^{\#m}T + T^\# T^n)(T^{\#m}T - T^\# T^n) \geq_A 0$$
$$\Leftrightarrow 2T^{\#m}T.T^\# T^n - 2T^\# T^n.T^{\#m}T \geq_A 0$$
$$\Leftrightarrow T^{\#m+1}T^{n+1} - T^\# T^n.T^{\#m}T \geq_A 0$$
$$\Leftrightarrow T \in [(n,m)QH]_A$$

2. Assume $T \in [(n,m)QH]_A$.Under the assumption 2 and lemma (1.3) we get
$T^{\sharp m}T.T^{\sharp m+1}T^{n+1} \geq_A T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp m}T$ and$T^{\sharp m+1}T^{n+1}.T^{\sharp}T^n \geq_A T^{\sharp}T^n.T^{\sharp m}T.T^{\sharp}T^n$ then,$[X,Z] = XZ - ZX$

$$= (T^{\sharp m}T - T^{\sharp}T^n).T^{\sharp m}T.T^{\sharp}T^n - T^{\sharp m}T.T^{\sharp}T^n.(T^{\sharp m}T - T^{\sharp}T^n)$$
$$= T^{\sharp m}T.T^{\sharp m}T.T^{\sharp}T^n - T^{\sharp}T^n.T^{\sharp m}T.T^{\sharp}T^n - T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp m}T - T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp}T^n$$
$$= T^{\sharp m}T.T^{\sharp m+1}T^{n+1} - T^{\sharp}T^n.T^{\sharp m}T.T^{\sharp}T^n - T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp m}T - T^{\sharp m+1}T^{n+1}.T^{\sharp m}T$$
$$\geq_A T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp m}T - T^{\sharp}T^n.T^{\sharp m}T.T^{\sharp}T^n - T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp m}T - T^{\sharp m}T.T^{\sharp}T^n.T^{\sharp m}T$$
$$= 0$$

Simmilary we proved the statements3 and 4.

Tensor Product of (n,m)Power-A-quasi-Hyponormal Operators in Semi-Hilbertian Spaces

The operation of taking tensor products $T\otimes S$preserves many properties of $T,S \in \mathcal{B}(\mathcal{H})$ but by no means all of them.Whereas $T\otimes S$is normal if and only if $T$and$S$ are normal,there exist paranormal operators T and S such that $T\otimes S$is notparanormal.it is proved that for non-zero $T,S \in \mathcal{B}(\mathcal{H})$, $T\otimes S$is p-hyponormal if and only if T and S are p-hyponormal. This result was extended to P-quasi-hyponormaloperators, for more details see ([9],[10],[12],[14])
Recall that $(T\otimes S)^*(T\otimes S) = T^*T\otimes S^*S$and $(A\otimes B) = (A\otimes I)(I\otimes B) = (I\otimes B)(A\otimes I)$
The following elementary results on tensor products of operators will be used often
in the sequel.

Lemma 2.11([7])Let $T_K, S_K \in \mathcal{B}(\mathcal{H}), k = 1,2$ and let $A,B \in \mathcal{B}(\mathcal{H})^+$ such that $T_1 \geq_A T_2 \geq_A 0$ and $S_1 \geq_A S_2 \geq_A 0$, then$T_1 \otimes S_1 \geq_{A\otimes B} T_2 \otimes S_2 \geq_{A\otimes B} 0$

Proposition 2.12 ([7])Let $T_K, S_K \in \mathcal{B}(\mathcal{H}), k = 1,2$ and let $A,B \in \mathcal{B}(\mathcal{H})^+$ such that
$T_K$is A-positive and S_K is B-positive .If $T_1 \neq 0$ and$S_1 \neq 0$,then the following conditions are equivalents

1. $T_2 \otimes S_2 \geq_{A\otimes B} T_1 \otimes S_1$

2. There exist $d > 0$ such that $dT_2 \geq_A T_1$ and $d^{-1}S_2 \geq_A S_1$

In the following theorem we will prove the stability of the class of (n,m) power-A-quasihyponormal operators under the direct sum and tensor product

Theorem 2.13 Let$\mathrm{T}, S \in \mathcal{B}_A(\mathcal{H})$be (n,m) power-A-quasi-hyponormal operators such that$T^{\sharp}T^n.T^{\sharp m}T \geq_A 0$ and $S^{\sharp}S^n.S^{\sharp m}S \geq_A 0$,then $T\oplus S$ is (n,m) power-$(A\oplus A)$-quasi-hyponormal operator and $T\otimes S$is (n,m) power-$(A\otimes A)$-quasi-hyponormal operator

Proof . Assume T, S are (n,m) power-A-quasi-hyponormal operators,then we have
$T^{\sharp m+1}T^{n+1} \geq_A T^{\sharp}T^n.T^{\sharp m}T$ and $S^{\sharp m+1}S^{n+1} \geq_A S^{\sharp}S^n.S^{\sharp m}S$. So

$$(T\oplus S)^{\sharp m+1}(T\oplus S)^{n+1} = (T^{\sharp m+1}\oplus S^{\sharp m+1})(T^{n+1}\oplus S^{n+1})$$
$$= (T^{\sharp m+1}T^{n+1}\oplus S^{\sharp m+1}S^{n+1})$$
$$\geq_{A\oplus A} (T^{\sharp}T^n.T^{\sharp m}T\oplus S^{\sharp}S^n.S^{\sharp m}S)$$
$$= (T^{\sharp}T^n\oplus S^{\sharp}S^n)(T^{\sharp m}T\oplus S^{\sharp m}S)$$
$$= (T^{\sharp}\oplus S^{\sharp})(T^n\oplus S^n)(T^{\sharp m}\oplus S^{\sharp m})(T\oplus S)$$
$$= (T\oplus S)^{\sharp}(T\oplus S)^n(T\oplus S)^{\sharp m}(T\oplus S)$$

Then $(T\oplus S)$ is (n,m) power-$(A\oplus A)$-quasi-hyponormal operator.

$$(T\otimes S)^{\sharp m+1}(T\otimes S)^{n+1} = (T^{\sharp m+1}\otimes S^{\sharp m+1})(T^{n+1}\otimes S^{n+1})$$
$$= (T^{\sharp m+1}T^{n+1}\otimes S^{\sharp m+1}S^{n+1})$$
$$\geq_{A\otimes A} (T^{\sharp}T^n.T^{\sharp m}T\otimes S^{\sharp}S^n.S^{\sharp m}S)T$$
$$= (T^{\sharp}T^n\otimes S^{\sharp}S^n)(T^{\sharp m}T\otimes S^{\sharp m}S)$$
$$= (T^{\sharp}\otimes S^{\sharp})(T^n\otimes S^n)(T^{\sharp m}\otimes S^{\sharp m})(T\otimes S)$$
$$= (T\otimes S)^{\sharp}(T\otimes S)^n(T\otimes S)^{\sharp m}(T\otimes S)$$

Then $(T\otimes S)$ is (n,m) power-$(A\otimes A)$-quasi-hyponormal operator.
The following theorem gives a necessary and sufficient condition for $(T\otimes S)$to be (n,m) power-$(A\otimes B)$-quasi-hyponormal operator when T and $S$are nonzero operators.

Theorem 2.14 Let $A,B \in \mathcal{B}(\mathcal{H})^+$.If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_B(\mathcal{H})$ are nonzero operators such that $T^{\sharp}T^n.T^{\sharp m}T \geq_A 0$ and $S^{\sharp}S^n.S^{\sharp m}S \geq_A 0$, then $(T\otimes S)$ is (n,m)power-$(A\otimes B)$-quasi-hyponormal if and only if T is (n,m) power-A-quasihyponormal and $S$is (n,m)power-B-quasihyponormal.

.*Proof.* Assume that T is (n,m)power-A-quasihyponormal and S is (n,m)power-B-
quasihyponormal operators. Then

$$(T\otimes S)^{\sharp m+1}(T\otimes S)^{n+1} = (T^{\sharp m+1}\otimes S^{\sharp m+1})(T^{n+1}\otimes S^{n+1})$$
$$= (T^{\sharp m+1}T^{n+1}\otimes S^{\sharp m+1}S^{n+1})$$
$$\geq_{A\otimes B} (T^{\sharp}T^n.T^{\sharp m}T\otimes S^{\sharp}S^n.S^{\sharp m}S)(Lemma\ (2.11))$$
$$= (T^{\sharp}T^n\otimes S^{\sharp}S^n)(T^{\sharp m}T\otimes S^{\sharp m}S)$$
$$= (T\otimes S)^{\sharp}(T\otimes S)^n(T\otimes S)^{\sharp m}(T\otimes S)$$

Which implies that $(T\otimes S)$ is (n,m) power-$(A\otimes B)$-quasi-hyponormal operator.
Conversely, assume that $(T\otimes S)$ is (n,m) power $(A\otimes B)$ quasi-hyponormal operator. We aim to show that T is (n,m) power A-quasi-hyponormal and S is (n,m) power B-quasi-hyponormal.

Since $(T\otimes S) \in [(n,m)QH]_{A\otimes B}$ then

$$(T\otimes S)^{\sharp m+1}(T\otimes S)^{n+1} \geq_{A\otimes B} (T\otimes S)^{\sharp}(T\otimes S)^{n}(T\otimes S)^{\sharp m}(T\otimes S)$$

i.e. $$(T^{\sharp m+1}T^{n+1}\otimes S^{\sharp m+1}S^{n+1}) \geq_{A\otimes B} (T^{\sharp}T^{n}.T^{\sharp m}T\otimes S^{\sharp}S^{n}.S^{\sharp m}S)$$

By proposition (2.12)there exists $d>0$such that $\begin{cases} d\, T^{\sharp m+1}T^{n+1} \geq_A T^{\sharp}T^{n}.T^{\sharp m}T \\ and \\ d^{-1}S^{\sharp m+1}S^{n+1} \geq_B S^{\sharp}S^{n}.S^{\sharp m}S \end{cases}$

A simple computation shows that $d=1$ and hence

$T^{\sharp m+1}T^{n+1} \geq_A T^{\sharp}T^{n}.T^{\sharp m}T$ and $S^{\sharp m+1}S^{n+1} \geq_B S^{\sharp}S^{n}.S^{\sharp m}S$

Therefore, T is (n,m) power A-quasihyponormal and S is (n,m)power A-quasihyponormal.

CONCLUSION

In this article we have worked on the class of operators (n,m) power-A-quasi-hyponormal in Semi-Hilbertian Space, we have given the characterization of this class and found conditions so that the sum and the product of two (n,m) A-quasi-hyponormal operators to be $(n,m)A$-quasi-hyponormal and we have proved that this class is stable by the tensor product and the direct sum.

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