APPLICATION OF THE FINITE ELEMENT METHOD TO SOLVING THE DUFFING EQUATION OF GROUND MOTION

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ABSTRACT

In this paper, we applied the Galerkin Finite Element Method to solve a damped, externally forced, second order ordinary differential equation with cubic nonlinearity known as the Duffing Equation. The Galerkin method uses the functional minimization technique which sets the equation in systems of algebraic equations to be solved. Various simulation on the effect of change on some parametric values of the Duffing equation are shown.

KEYWORDS: Galerkin Finite Element Method, stiffness matrix, Duffing Equation, shape functions, basis functions, weight functions.
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INTRODUCTION

The Finite Element Method (FEM) is a numerical/computational analysis tool used in obtaining approximate solutions to boundary value problems which are governed by a differential equation and a set of boundary conditions. The main idea behind the finite element method is the representation of the domain with smaller subdomains called the finite elements. The distribution of the primary unknown quantity inside an element is then interpolated based on the values at the nodes, so far as nodal elements are used. The interpolation or shape functions must also be a complete set of polynomials. The accuracy of the solution depends, among other factors, on the order of these polynomials, whose order may be linear, quadratic, or higher. The numerical solution corresponds to the values of the primary unknown quantity at the nodes or the edges of the discretized domain. The solution is obtained after solving a system of linear equations. To form such a linear system of equations, the governing differential equation and associated boundary conditions must first be converted to an integro-differential form by using a weighted residual method as the Galerkin approach. This integro-differential formulation is applied to a single element and with the use of proper weight and interpolation functions called the shape functions, the respective element equations are obtained. The assembly of all elements results in a global matrix system that represents the entire domain of the BVP.

The FEM was first used by Clough [1 – 2] and it was invented when aircraft engineers wanted to solve problems relating to structures, wheels and to calculate stress in structures. Since then, the FEM has been used by many researchers in areas such as Electromagnetics [3 – 5], Computational Fluid Dynamics [6], Differential Equations [7], implant dentistry [8], heat transfer [9] and incompressible fluid [10].

THE DUFFING EQUATION

The Duffing Oscillator or equation is a damped, externally forced, second order nonlinear oscillator with constant coefficients which has gained wide recommendations as the simplest equation which is used to study and describe the chaotic behaviour of a system. It is a nonlinear differential equation which describes an oscillator with a cubic nonlinearity [11]. It was developed by a German Engineer named Georg Wilhem Christian Caspar Duffing in 1918 who aimed to tackle problems of nonlinear oscillators in a systematic way by starting with the linear oscillator and also examining the effects of quadratic and cubic stiffness nonlinearities. He then emphasized the differences between the linear and nonlinear oscillators for both the free and forced vibrations while considering their damping effects.

As a dynamical system that exhibits chaotic behaviour,

✓ It has a periodic long term behaviour as $t \rightarrow \infty$ making the system to have irregular pattern which does not oscillate nor repeat in a periodic manner.

✓ It is always vigilant in the change of its initial conditions as any change will alter the trajectory thereby giving a significant difference in its long term behaviour.

✓ The nonlinearity of the system makes it to exhibit irregular behaviour hence making it to be deterministic.

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The general Duffing equation is given as

$$\ddot{x} + \delta\dot{x} \pm \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{1}$$

where the unknown function $x = x(t)$ is the displacement at time t. The damping factor δ controls the size of the damping. The α controls the size of the stiffness and the β controls the amount of nonlinearity in the restoring force. If $\beta = 0$, the Duffing equation describes a damped and driven simple harmonic oscillator. The quantity γ controls the amplitude of the periodic driving force. If $\gamma = 0$, we have a system without driving force. The quantity ω controls the frequency of the periodic driving force. See [12 – 14]. When the Duffing equation has a negative linear stiffness, it is said to be a double-well potential. See [15 – 17]. The Duffing double-well potential equation is given as

$$\ddot{x} + \delta\dot{x} - \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{2}$$

THE FINITE ELEMENT METHOD
The Finite Element Method is an element wise application of the weighted residual or Galerkin method which involves the following steps:
a.   Discretization of the domain
b.   Formulation or derivation of element equations
c.   Assembly of the element equations
d.   Imposition of the boundary conditions
e.   Solution of the assembled equations. See [18, 19]

APPLICATION:
Consider the given Duffing oscillator,
$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos wt \tag{3}$$
Subject to $x(0) = 0, \dot{x}(1) = 1$
We seek for an approximate solution of the form

$$X(t) = \sum_{i=1}^{n} \alpha_i \xi_i \tag{4}$$

The error or residue E is given as
$$E = \ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 - \gamma\cos\omega t \tag{5}$$
and by Galerkin FEM,

$$\int_0^l Ew(t)\,dt = 0$$

Step 1: Discretization of the Domain
Let the function be divided into four elements in the domain [0,1]. Let $w = w(t)$ be the test function and let s_1 and s_2 be successive node points.
For each element $e_i, i = 1, 2, 3, 4$, we have two nodes and from (

$$\int_{s_1}^{s_2}(\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3)w\,dt - \int_{s_1}^{s_2}\gamma w\cos\omega dt = 0 \tag{6}$$

$$\Rightarrow \int_{s_1}^{s_2}(-\dot{x}\dot{w} + \delta\dot{x}w + \alpha x w + \beta x^3 w)dt = \int_{s_1}^{s_2}\gamma w\cos\omega t\,dt - w\dot{x}\Big|_{s_1}^{s_2}$$

Step 2: Derivation of element equations
Let

$$w(t) = \sum_{j=1}^{2}\gamma_j\psi_j(t) \text{ and } x(t) = \sum_{i=1}^{2}\alpha_i\psi_i(t) \tag{7}$$

where ψ_i and ψ_j are shape functions; γ_i and α_i are coefficients for the particular node which is 1 and 0 at other nodes.

$$\psi_i = \frac{s_{i+1} - t}{s_{i+1} - s_i}, \qquad \psi_{i+1} = \frac{t - s_i}{s_{i+1} - s_i} \tag{8}$$

But $s_1 = s_i, s_2 = s_{i+1}$, when $i = 1$ and $h = s_{i+1} - s_i$
Substituting (7,8) into (6), we obtain

$$\int_{s_i}^{s_{i+1}}\left(-\psi_1'\psi_j' + \delta\psi_1'\psi_j + \alpha\psi_i\psi_j + \beta\psi_i^3\psi_j\right)dt = \gamma\int_{s_1}^{s_2}\psi_j\cos\omega t\,dt - \psi_j\dot{x}\Big|_{s_1}^{s_2} \tag{9}$$

Equation (9) can be written as
$$k_{ij}^e = g_i^e \tag{10}$$

where k_{ij}^e is the stiffness matrix and it is given as

$$k_{ij}^e = \int_{s_i}^{s_{i+1}} \left(-\dot{\psi}_i \dot{\psi}_j + \delta \dot{\psi}_i \psi_j + \alpha \psi_i \psi_j + \beta \psi_i^3 \psi_j \right) dt \tag{11}$$

$$g_i^e = \gamma \int_{s_1}^{s_2} \psi_j \cos\omega t\, dt - \psi_j \dot{x} \Big|_{s_1}^{s_2} \tag{12}$$

Then, from (11)

$$k_{11} = \alpha \frac{h}{3} + \frac{\beta h}{5} - \frac{1}{h} - \frac{\delta}{2} = k_{22}$$

and

$$k_{12} = \left\{ \frac{1}{h} - \frac{\delta}{2} - \frac{\alpha}{2h}\left(s_2 + 2s_1\right)\left(s_2 + s_1\right) + \frac{\beta h}{20} \right\} = k_{21}$$

Hence, for a typical element, its stiffness matrix is

$$K^e = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Step 3: Assembling of element equations

For every element,

$$K^e = \begin{bmatrix} \dfrac{\alpha h}{3} + \dfrac{\beta h}{5} - \dfrac{1}{h} - \dfrac{\delta}{2} & \dfrac{1}{h} - \dfrac{\delta}{2} - \dfrac{\alpha}{2h}\left(s_2 + 2s_1\right)\left(s_2 + s_1\right) + \dfrac{\beta h}{20} \\ \dfrac{1}{h} - \dfrac{\delta}{2} - \dfrac{\alpha}{2h}\left(s_2 + 2s_1\right)\left(s_2 + s_1\right) + \dfrac{\beta h}{20} & \dfrac{\alpha h}{3} + \dfrac{\beta h}{5} - \dfrac{1}{h} - \dfrac{\delta}{2} \end{bmatrix}$$

The subscripts 1 and 2 are nodes 1 and 2 of a particular element. When we start assembling the matrices, the subscripts will be relabeled to take note of the element in question.

Then

$$K^1 = \begin{pmatrix} \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 & \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{\alpha}{8} + 4 \\ \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{\alpha}{8} + 4 & \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 \end{pmatrix}, \quad K^2 = \begin{pmatrix} \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 & \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{3}{2}\alpha + 4 \\ \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{3}{2}\alpha + 4 & \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 \end{pmatrix}$$

$$K^3 = \begin{pmatrix} \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 & \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{3}{2}\alpha + 4 \\ \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{3}{2}\alpha + 4 & \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 \end{pmatrix}, \quad K^4 = \begin{pmatrix} \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 & \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{35}{4}\alpha + 4 \\ \dfrac{\beta}{80} - \dfrac{\delta}{2} - \dfrac{35}{4}\alpha + 4 & \dfrac{\alpha}{12} + \dfrac{\beta}{20} - \dfrac{\delta}{2} - 4 \end{pmatrix}$$

The global system of matrices equals:

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & 0 \\ 0 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & 0 \\ 0 & 0 & k_{21}^3 & k_{22}^3 + k_{11}^4 & k_{12}^4 \\ 0 & 0 & 0 & k_{21}^4 & k_{22}^4 \end{pmatrix} \tag{13}$$

which gives to

$$K = \begin{pmatrix} \frac{\alpha}{12}+\frac{\beta}{20}-\frac{\delta}{2}-4 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{\alpha}{8}+4 & 0 & 0 & 0 \\ \frac{\beta}{80}-\frac{\delta}{2}-\frac{\alpha}{8}+4 & \frac{\alpha}{6}+\frac{\beta}{10}-\delta-2 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{3}{2}\alpha+4 & 0 & 0 \\ 0 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{3}{2}\alpha+4 & \frac{\alpha}{6}+\frac{\beta}{10}-\delta-2 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{8}\alpha+4 & 0 \\ 0 & 0 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{8}\alpha+4 & \frac{\alpha}{6}+\frac{\beta}{10}-\delta-2 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{4}\alpha+4 \\ 0 & 0 & 0 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{4}\alpha+4 & \frac{\alpha}{12}+\frac{\beta}{20}-\frac{\delta}{2}-4 \end{pmatrix} \quad (14)$$

The load vectors are given as (12)
Then each of the load vectors gives:

$$G_i = \begin{bmatrix} \frac{\gamma}{\omega h}\left[s_1 \sin \omega h - \frac{\cos \omega h}{\omega} \right] \\ \frac{\gamma}{h}\left[(t-s_1)\frac{\sin \omega t}{\omega} + \frac{\cos \omega t}{w^2} \right]\Big|_{s_1}^{s_2} \end{bmatrix} \quad (15)$$

The global load vectors are summarized as thus:

$$G = \begin{pmatrix} g_1^1 - x(0) \\ g_2^1 + g_1^2 \\ g_2^2 + g_1^3 \\ g_2^3 + g_1^4 \\ g_2^4 + \dot{x}(1) \end{pmatrix} \quad (16)$$

which gives

$$G = \begin{pmatrix} -\frac{4\gamma}{\omega}\cos\frac{\omega}{4} - x(0) \\ \frac{2\gamma}{\omega}\sin\frac{\omega}{4} \\ \frac{2\gamma}{\omega}\sin\frac{\omega}{4} \\ \frac{2\gamma}{\omega}\sin\frac{\omega}{4} \\ \frac{4\gamma}{\omega}\left(\cos\frac{\omega}{4} - \frac{1}{2}\sin\frac{\omega}{4}\right) + \dot{x}(1) \end{pmatrix} \quad (17)$$

Then combining the equations together, we have

$$\begin{pmatrix} \frac{\alpha}{12}+\frac{\beta}{20}-\frac{\delta}{2}-4 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{\alpha}{8}+4 & 0 & 0 & 0 \\ \frac{\beta}{80}-\frac{\delta}{2}-\frac{\alpha}{8}+4 & \frac{\alpha}{6}+\frac{\beta}{10}-\delta-2 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{3}{2}\alpha+4 & 0 & 0 \\ 0 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{3}{2}\alpha+4 & \frac{\alpha}{6}+\frac{\beta}{10}-\delta-2 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{8}\alpha+4 & 0 \\ 0 & 0 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{8}\alpha+4 & \frac{\alpha}{6}+\frac{\beta}{10}-\delta-2 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{4}\alpha+4 \\ 0 & 0 & 0 & \frac{\beta}{80}-\frac{\delta}{2}-\frac{35}{4}\alpha+4 & \frac{\alpha}{12}+\frac{\beta}{20}-\frac{\delta}{2}-4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} -\frac{4\gamma}{\omega}\cos\frac{\omega}{4} - x(0) \\ \frac{2\gamma}{\omega}\sin\frac{\omega}{4} \\ \frac{2\gamma}{\omega}\sin\frac{\omega}{4} \\ \frac{2\gamma}{\omega}\sin\frac{\omega}{4} \\ \frac{4\gamma}{\omega}\left(\cos\frac{\omega}{4}-\frac{1}{2}\sin\frac{\omega}{4}\right)+\dot{x}(1) \end{pmatrix}$$

The first and last rows have been covered by the boundary conditions and hence

$$
\begin{pmatrix}
\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2 & \dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4 & 0 \\[2mm]
\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4 & \dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2 & \dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4 \\[2mm]
0 & \dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4 & \dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2
\end{pmatrix}
\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix}
=
\begin{pmatrix}
\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4} \\[2mm]
\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4} \\[2mm]
\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}
\end{pmatrix}
$$

Then we solve for u_2, u_3, u_4 which gives $\alpha_2, \alpha_3, \alpha_4$.

$$
\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix}
=
\begin{pmatrix}
\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2 & \dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4 & 0 \\[2mm]
\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4 & \dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2 & \dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4 \\[2mm]
0 & \dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4 & \dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2
\end{pmatrix}^{-1}
\begin{pmatrix}
\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4} \\[2mm]
\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4} \\[2mm]
\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}
\end{pmatrix}
$$

Therefore,

$$
u_2=\frac{\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2+\dfrac{23}{8}\alpha\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)\right]}{\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)^2\right]}=\alpha_2
$$

$$
u_3=\frac{\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}\left[\dfrac{145}{24}\alpha+\dfrac{3}{40}\beta-10\right]}{\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)^2\right]}=\alpha_3
$$

$$
u_4=\frac{\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+5\right)\right]}{\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)^2\right]}=\alpha_4
$$

The general weak solution of the Duffing equation using FEM is given as:

$$
X(t)=\xi_1+\frac{\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2+\dfrac{23}{8}\alpha\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)\right]}{\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)^2\right]}\xi_2
$$

$$
+\frac{\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}\left[\dfrac{145}{24}\alpha+\dfrac{3}{40}\beta-10\right]}{\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)^2\right]}\xi_3
$$

$$
+\frac{\dfrac{2\gamma}{\omega}\sin\dfrac{\omega}{4}\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+5\right)\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)\right]}{\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)\left[\left(\dfrac{\alpha}{6}+\dfrac{\beta}{10}-\delta-2\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{3}{2}\alpha+4\right)^2-\left(\dfrac{\beta}{80}-\dfrac{\delta}{2}-\dfrac{35}{8}\alpha+4\right)^2\right]}\xi_4 + \xi_5
$$

(18)

where

$$
\xi_1=x(0)=0, \qquad\qquad \xi_5=\dot{x}(1)=l
$$

are the boundary conditions.

Equation (18) is the finite element method solution of the Duffing equation. To obtain the numerical strings of approximate solutions, we give numerical values to the constants α, β, γ, δ, and ω.

NUMERICAL EXPERIMENTS

In this section, we would give some numerical values to the parameters of the Duffing equation whose solution was derived in (18) above. These approximating values will make the solution be in form of a polynomial. We take two examples.

a. Let $\alpha = 1$, $\beta = 5$, $\delta = 0.02$, $\gamma = 8$ and $\omega = 0.5$ then, the solution (18) becomes

$$X(t) = \xi_1 + 0.28079\xi_2 - 0.03861\xi_3 - 0.03812\xi_4 + \xi_5 \qquad (19)$$

b. Let $\alpha = -2$, $\delta = 0.1$, $\beta = 2$, $\gamma = 3$ and $\omega = 1.2$, then, the solution (18) becomes

$$X(t) = \xi_1 - 0.00389\xi_2 - 0.00125\xi_3 - 0.00832\xi_4 + \xi_5 \qquad (20)$$

SOME NUMERICAL SIMULATIONS AND INTERPRETATIONS

We show the numerical simulations of the behaviour of the damped Duffing oscillator which was obtained by the use of the Galerkin finite element techniques. Using MATLAB ODE 45 package, we show in figures below, the effects of the change in damping factor δ, and the amount of nonlinearity β even at steady state and variation in size of stiffness α.
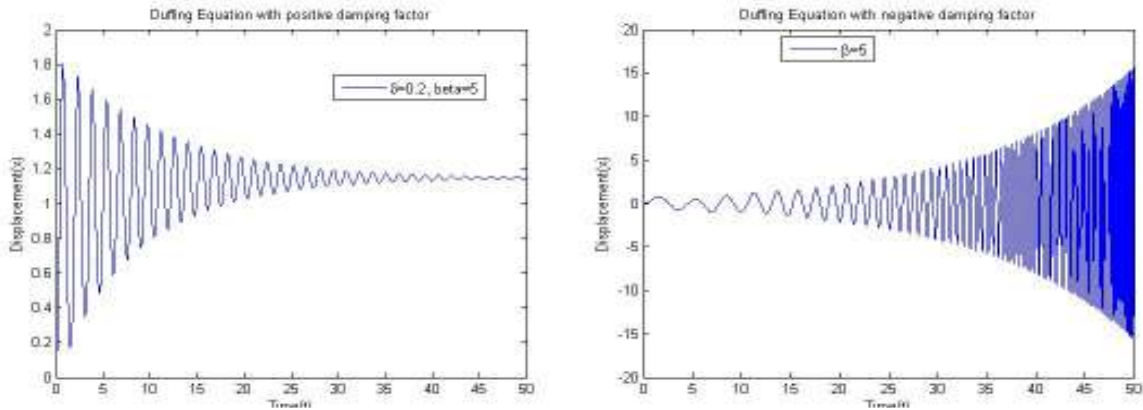

Fig. A and B: Duffing Oscillator with positive and negative damping factor. The damping has effects on earthquake occurrence as the positivity helps to reduce the tremor. Negative damping factor increases its chaotic behaviour.
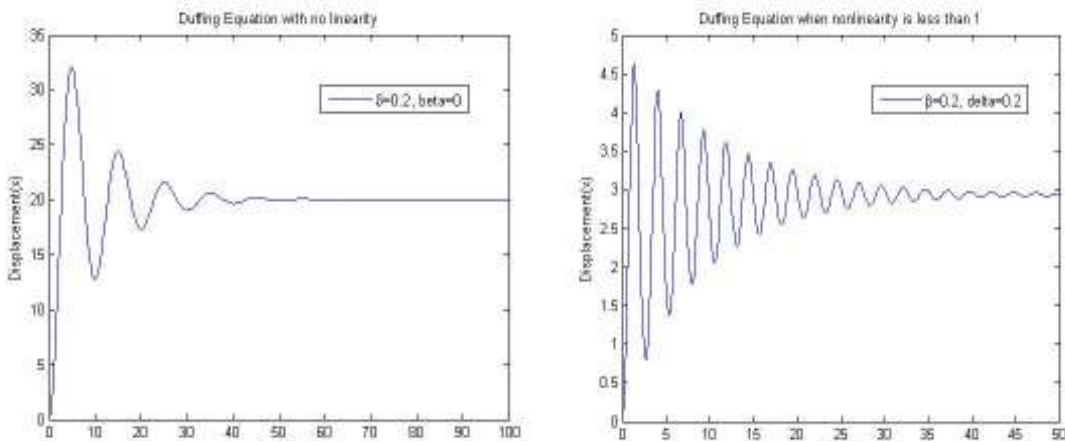

Fig. C and D: Duffing Oscillator with no linear term and when linearity is less than 1.
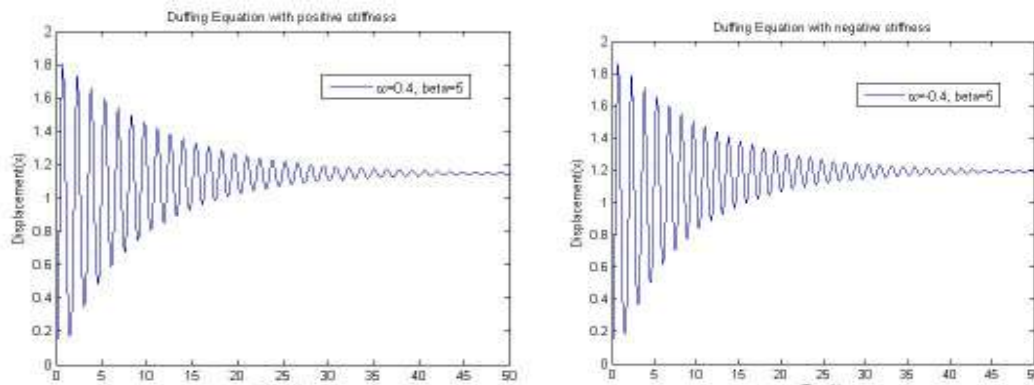
Fig. E and F: Duffing Oscillator with positive and negative stiffness gives no change.

CONCLUSION

In this research, we have applied the use of the Finite Element Method to solving the Duffing Equation of ground motion. The FEM has been seen as a powerful tool for solving nonlinear dynamic problems which have no definite analytical solution. This method gave us a promising iterative result that such equation can be reduced to a polynomial.

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