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NUMERICAL SOLUTION OF INITIAL VALUE PROBLEMS BY RATIONAL INTERPOLATION METHOD USING CHEBYSHEV POLYNOMIALS

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ABSTRACT

In this research, a modified rational interpolation method for the numerical solution of initial value problem is presented. The proposed method is obtained by fitting the classical rational interpolation formula in Chebyshev polynomials leading to a new stability function and new scheme. Three numerical test problems are presented in other to test the efficiency of the proposed method. The numerical result for each test problem is compared with the exact solution. The approximate solutions are show competitiveness with the exact solutions of the ODEs throughout the solution interval.

KEYWORDS AND PHRASES: Chebyshev polynomial, Rational Interpolation, Minimaxpolynomial, Initial Value Problems and Ordinary Differential Equations (ODEs).

INTRODUCTION

Many of the differential equations encountered in practice cannot be solved analytically and recourse must necessarily be made to numerical methods. Fortunately, there is a wide range of methods developed by researchers that can be efficiently implemented with the computer and has become widely applied in engineering, sciences and many fields. Recent research on efficient methods to solve differential equations is the motivation for this work. The need to develop direct methods for solving higher order ordinary differential equation cannot be over emphasized in the theory of initial value problems (Pandey, 2012).

In recent years, researchers have applied non-standard finite difference method and obtained competitive results to those obtained with other methods. Our aim is to improve the classical implicit difference methods for systems of first order initial value problems. Though implicit methods are in general more expensive, but they have advantage in terms of stability and convergence (Horner, 1977).

The classical methods for solving first order ordinary differential equations include the Runge-Kutta methods and multistep methods (Lambert, 1974, Gear and Qsterby, 1984). One limitation of these methods is that they may be inefficient when they are used to solve problems with singularities. For this reason, it becomes imperative to find alternative methods that take into account the effect of singularities (Gear and Qsterby, 1984; Luke *et al.,* 1975; Fatunla, 1990; Otunta and Ikhile, 2004.

One popular method is the rational interpolation method which is based on the inverse polynomial functions and where points of singularities of the functions are made to coincide with that of the solution. The method is applicable to higher order equations since they can always be converted to an equivalent system of first order equations. Rational functions have the advantage of automatically picking up the singularities of a given function to the zeros of the denominator. The need to have an integrator that can efficiently cope with either singularity or stiffness or both is enough reason for the search for new integrator schemes or methods.

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In this work, we will fit the classical integrator formulae proposed by Aashikelokhia (1991) in to Chebyshev polynomials to obtain a new scheme; this is possible since any polynomial of degree n is uniquely expressible as linear combination of the Chebyshev polynomials with the objective to guarantee a special specified accuracy (Conte, 1965).

Some existing methods for integration schemes are the linear multi step methods (LMM), Exponential Based methods (EBM) and the Rational Interpolant based methods.

Different methods for enhancing the performance of numerical integration formulae design for approximation of theoretical solutions of first order IVP in ODEs of the form

$$y' = f(x, y); y(x_0) = y_0 \quad (1)$$

are in Fatunla (1976) given by;

$$F_k(x) = \frac{A}{1 + \sum_{r=1}^{k} a_r x^r}$$

where A and the polynomial coefficients a_r are real parameters, other methods suggested are in Okosun and Ademuluyi (2007), Aashikelokhia (1991,1997), Enright and Pryce (1978) Lambert and Shaw (1965).

Most of these methods have low order of discontinuities when applied to IVPs.
However, there is a need to have a method that can efficiently cope with different classes of IVPs. In other to achieve the set objective, we shall be concerned with addressing the problem of improving the accuracy of the rational interpolation method by modifying the existing method of Anetor et al., (2014), (Nwachukwu 2005), (Okosun and Ademuluyi 2007) and others by the introduction of Chebyshev polynomials.

DERIVATION OF THE NEW SCHEME

The derivation of the scheme is similar to Aashikelokhia (1991).
We consider Aashikepelokhia (1991) class of rational integrator formulas given by;

$$y_{n+1} = \frac{\sum_{i=0}^{k-1} p_i x^i{}_{n+1}}{1+\sum_{i=1}^{k} q_i x^i{}_{n+1}} \quad (2)$$

where,

$$p_i = \frac{(2k-1-i)!\binom{k-1}{i}\tilde{h}^i}{(2k-1)!x^i{}_{n+1}} \quad ; \qquad i = 0(1)k-1 \quad (3)$$

$$q_i = \frac{(-1)^i(2k-1-i)!\binom{k-1}{i}\tilde{h}^i}{(2k-1)!x^i{}_{n+1}} \quad ; \qquad i = 1(1)k \quad (4)$$

$$p_j = \frac{\sum_{i=1}^{j} h^{(j+1-i)} y_n^{(j+1-i)} q_{i-1}}{\sum_{i=1}^{j} (j+1-i) x_{n+1}^{(j+1-i)}} + y_n q_j; j = 1(1)k \quad (5)$$

Setting k =5 in(2); we obtain

$$y_{n+1} = \frac{\sum_{i=0}^{4} p_i x^i{}_{n+1}}{1+\sum_{i=1}^{5} q_i x^i{}_{n+1}} = \frac{p_0+p_1x_{n+1}+p_2x^2{}_{n+1}+p_3x^3{}_{n+1}+p_4x^4{}_{n+1}}{1+q_1x_{n+1}+q_2x^2{}_{n+1}+q_3x^3{}_{n+1}+q_4x^4{}_{n+1}+q_5x^5{}_{n+1}} \quad (6)$$

where $p_0, p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4$ and q_5 are real undetermined coefficients. Let y_{n+1} be the numerical estimate to the theoretical solution (2) andset $y_n = y(x_n)$with mesh pointsx_n, x_{n+1} and stepsize$h = (x_{n+1} - x_n)$ to be sufficiently small enough so that x_{n+1} and x_n are very close.

DETERMINATION OF THE COEFFICIENTS OF THE SCHEME

We impose the following constraints, (i) and (ii) respectively on the interpolating rational function (6) in order to obtain the undetermined coefficients

(i).The interpolating function must coincide with the theoretical solution at the x_n and x_{n+1}.

(ii). Points of singularities or discontinuities of the inverse polynomial functions are made to coincide with that of the solution.

Consider equation (**5**)

$$p_j = \frac{\sum_{i=1}^{j} h^{(j+1-i)} y_n^{(j+1-i)} q_{i-1}}{\sum_{i=1}^{j} (j+1-i) x_{n+1}^{(j+1-i)}} + y_n q_j; \quad j = 1(1)k$$

Put j = 0, i = 1 in (**5**) implies $p_0 = y_n q_0$, but $q_0 = 1$

Hence, $p_0 = y_n$. **(7)**

j = 0, i = 1in (**5**) we have

$$\frac{h y_n^{(1)}}{1!} = x_{n+1}[-p_0 q_1 + p_1] \quad \textbf{(8)}$$

Similarly, for j = $2,3,4,5,6,..10, and$ i= 1are obtained from the relation:

$$\frac{h^m y_n^{(m)}}{m!} = x_{n+1}^m \left[-p_0 q_i - \frac{h^{m-v} y_n^{(m-v)} q_{i-(m-1)}}{(m-v)!\, x_{n+1}^{(m-v)}} - \frac{h^{m-v+1} y_n^{(m-v+1)} q_{i-(m-2)}}{(m-v+1)!\, x_{n+1}^{(m-v+1)}} - \frac{h^{m-v+2} y_n^{(m-v+2)} q_{i-(m-3)}}{(m-v+2)!\, x_{n+1}^{(m-v+2)}} - \cdots - \frac{h^{m-v+r} y_n^{(m-v+r)} q_{i-(m-r)}}{(m-v+r)!\, x_{n+1}^{(m-v+r)}} + p_i \right] \quad \textbf{(9)}$$

where (**9**) is the general derivation formula, with $p_0 = y_n; v = 1; m, i, r = 1,2,3...$

We obtainq_1, q_2, q_3, q_4 and q_5 by combiningequations containing our undetermined coefficientsgenerated from our general derivation formula (**9**).

$$\frac{h^9 y_n^{(9)} q_1}{9!x_{n+1}^9} + \frac{h^8 y_n^{(8)} q_2}{8!x_{n+1}^8} + \frac{h^7 y_n^{(7)} q_3}{7!x_{n+1}^7} + \frac{h^6 y_n^{(6)} q_4}{6!x_{n+1}^6} + \frac{h^5 y_n^{(5)} q_5}{5!x_{n+1}^5} = -\frac{h^{10} y_n^{(10)}}{10!x_{n+1}^{10}} \quad \textbf{(10)}$$

$$\frac{h^8 y_n^{(8)} q_1}{8!x_{n+1}^8} + \frac{h^7 y_n^{(7)} q_2}{7!x_{n+1}^7} + \frac{h^6 y_n^{(6)} q_3}{6!x_{n+1}^6} + \frac{h^5 y_n^{(5)} q_4}{5!x_{n+1}^5} + \frac{h^4 y_n^{(4)} q_5}{4!x_{n+1}^4} = -\frac{h^9 y_n^{(9)}}{9!x_{n+1}^9} \quad \textbf{(11)}$$

$$\frac{h^7 y_n^{(7)} q_1}{7!x_{n+1}^7} + \frac{h^6 y_n^{(6)} q_2}{6!x_{n+1}^6} + \frac{h^5 y_n^{(5)} q_3}{5!x_{n+1}^5} + \frac{h^4 y_n^{(4)} q_4}{4!x_{n+1}^4} + \frac{h^3 y_n^{(3)} q_5}{3!x_{n+1}^3} = -\frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} \quad \textbf{(12)}$$

$$\frac{h^6 y_n^{(6)} q_1}{6!x_{n+1}^6} + \frac{h^5 y_n^{(5)} q_2}{5!x_{n+1}^5} + \frac{h^4 y_n^{(4)} q_3}{4!x_{n+1}^4} + \frac{h^3 y_n^{(3)} q_4}{3!x_{n+1}^3} + \frac{h^2 y_n^{(2)} q_5}{2!x_{n+1}^2} = -\frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} \quad \textbf{(13)}$$

$$\frac{h^5 y_n^{(5)} q_1}{5!x_{n+1}^5} + \frac{h^4 y_n^{(4)} q_2}{4!x_{n+1}^4} + \frac{h^3 y_n^{(3)} q_3}{3!x_{n+1}^3} + \frac{h^2 y_n^{(2)} q_4}{2!x_{n+1}^2} + \frac{h y_n^{(1)} q_5}{1!x_{n+1}} = -\frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} \quad \textbf{(14)}$$

Equations (**10**) - (**14**) written in the form $AX = b$ as:

$$\begin{bmatrix} \frac{h^9 y_n^{(9)}}{9!x_{n+1}^9} & \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} & \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} \\ \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} & \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} \\ \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} \\ \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} & \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} \\ \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} & \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} & \frac{h y_n^{(1)}}{1!x_{n+1}} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} \frac{-h^{10} y_n^{(10)}}{10!x_{n+1}^{10}} \\ \frac{-h^9 y_n^{(9)}}{9!x_{n+1}^9} \\ \frac{-h^8 y_n^{(8)}}{8!x_{n+1}^8} \\ \frac{-h^7 y_n^{(7)}}{7!x_{n+1}^7} \\ \frac{-h^6 y_n^{(6)}}{6!x_{n+1}^6} \end{bmatrix} \quad \textbf{(15)}$$

Re-writing (15) as an upper triangular matrix, first the test equation $y' = \lambda y$ and the stability function$\tilde{h}$ =λh wasappliedand dividing each row by y_n.

$$\begin{bmatrix} 1 & \frac{9x_{n+1}}{\tilde{h}} & \frac{72x_{n+1}^2}{\tilde{h}^2} & \frac{504x_{n+1}^3}{\tilde{h}^3} & \frac{3024x_{n+1}^4}{\tilde{h}^4} \\ 0 & 1 & \frac{16x_{n+1}}{\tilde{h}} & \frac{168x_{n+1}^2}{\tilde{h}^2} & \frac{1344x_{n+1}^3}{\tilde{h}^3} \\ 0 & 0 & 1 & \frac{21x_{n+1}}{\tilde{h}} & \frac{252x_{n+1}^2}{\tilde{h}^2} \\ 0 & 0 & 0 & 1 & \frac{24x_{n+1}}{\tilde{h}} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} \frac{-9!\tilde{h}^{10}}{10!x_{n+1}^{10}} \\ \frac{8!\tilde{h}^9}{10!x_{n+1}^9} \\ \frac{-7!\tilde{h}^8}{10!x_{n+1}^8} \\ \frac{6!\tilde{h}^7}{10!x_{n+1}^7} \\ \frac{-5!\tilde{h}^6}{10!x_{n+1}^6} \end{bmatrix} \qquad \textbf{(16)}$$

Back substitution by solving (**16**) yields,

$$\begin{bmatrix} q_1(\tilde{h}) \\ q_2(\tilde{h}) \\ q_3(\tilde{h}) \\ q_4(\tilde{h}) \\ q_5(\tilde{h}) \end{bmatrix} = \begin{bmatrix} \frac{-1814400 \quad \tilde{h}}{10!x_{n+1}} \\ \frac{403200 \quad \tilde{h}^2}{10!x_{n+1}^2} \\ \frac{-50400 \quad \tilde{h}^3}{10!x_{n+1}^3} \\ \frac{3600 \quad \tilde{h}^4}{10!x_{n+1}^4} \\ \frac{-120 \quad \tilde{h}^5}{10!x_{n+1}^5} \end{bmatrix} \qquad \textbf{(17)}$$

Similarly, the undetermined coefficientsp_0, p_1, p_2, p_3 and p_4 can be generated by rearranging the equations obtained from (**9**) as follows;

$$p_0 = y_n \qquad \textbf{(18)}$$

$$p_1 = \frac{hy'_n}{1!x_{n+1}} + y_n q_1 \qquad \textbf{(19)}$$

$$p_2 = \frac{h^2 y''_n}{2!x_{n+1}^2} + \frac{hy'_n}{1!x_{n+1}} q_1 + y_n q_2 \qquad \textbf{(20)}$$

$$p_3 = \frac{h^3 y'''_n}{3!x_{n+1}^3} + \frac{h^2 y''_n}{2!x_{n+1}^2} q_1 + \frac{hy'_n}{1!x_{n+1}} q_2 + y_n q_3 \qquad \textbf{(21)}$$

$$p_4 = \frac{h^4 y_n^{(4)}}{3!x_{n+1}^4} + \frac{h^3 y'''_n}{3!x_{n+1}^3} q_1 + \frac{h^2 y''_n}{2!x_{n+1}^2} q_2 \frac{hy'_n}{1!x_{n+1}} q_3 + y_n q_4 \qquad \textbf{(22)}$$

Applying the test equation $y' = \lambda y$ and $\tilde{h} = \lambda h$ to the equations (**18**)-(**22**) and substituting (**16**) we have;

$$\begin{bmatrix} p_1(\tilde{h}) \\ p_2(\tilde{h}) \\ p_3(\tilde{h}) \\ p_4(\tilde{h}) \end{bmatrix} = \begin{bmatrix} \dfrac{1814400\tilde{h}}{10!x_{n+1}} \\ \dfrac{403200\tilde{h}^2}{10!x_{n+1}^2} \\ \dfrac{50400\tilde{h}^3}{10!x_{n+1}^3} \\ \dfrac{3600\tilde{h}^4}{10!x_{n+1}^4} \end{bmatrix} \quad (\mathbf{23})$$

Recall, $y_n = p_0 = 1$ from (7)

Applying the results (**17**) and (**23**) to the integrator (**6**), and the fact that $y' = \lambda y$, we obtain the stability function given by;

$$y_{n+1} = S(\tilde{h}) = \frac{3628800+1814400\tilde{h}+403200\tilde{h}^2+50400\tilde{h}^3+3600\tilde{h}^4}{3628800-1814400\tilde{h}+403200\tilde{h}^2-50400\tilde{h}^3+3600\tilde{h}^4-120\tilde{h}^5} \quad (\mathbf{24})$$

2.2 Application of the Chebyshev Polynomials to the Method

In order to find a rational interpolation function, which spread the error evenly over the whole interval of interest with the same accuracy, we approximate the numerator and the denominator of (**6**) by Minimax polynomials, this is with the aim of modifying the method(**6**) and stability function (**24**) by introducing the Chebyshev Polynomial in the numerator and denominator of (**6**), of course any polynomial of degree n is uniquely expressible as a linear combination of the Chebyshev Polynomial, i.e. constants c_k exist such that

$$P_n(x) = c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + \cdots + c_nT_n(x)$$

(Conte, 1965).

For application purpose, we express each x^k in (6) in terms of the Chebyshev polynomials as expressed in Column II below

Table 1:Chebyshev polynomial for various T_n

Column I	Column II
$T_0(x) = 1$	$1 = T_0$
$T_1(x) = x$	$x = T_1$
$T_2(x) = 2x^2 - 1$	$x^2 = 2^{-1}(T_0 + T_2)$
$T_3(x) = 4x^3 - 3x$	$x^3 = 2^{-2}(3T_1 + T_3)$
$T_4(x) = 8x^4 - 8x^2 + 1$	$x^4 = 2^{-3}(3T_0 + 4T_2 + T_4)$
$T_5(x) = 16x^5 - 20x^3 + 5x$	$x^5 = 2^{-4}(10T_1 + 5T_3 + T_5)$
$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$	$x^6 = 2^{-5}(10T_0 + 15T_2 + 6T_4 + T_6)$
.	.
.	.
.	.
$T_n(x) = 2^{n-1}x^n \dots$	$x^n = 2^{-n+1}(\dots)$

MINIMAX POLYNOMIAL APPROXIMATION

If $P_n(x)$ is any polynomial of degree n with leading coefficients a_n, then its Minimax polynomial approximation of degree $\leq n-1$ on $[-1, 1]$ is
$M_{n-1}(x) = P_n(x) - a_n 2^{n-1} T_n(x)$. (Samelson, 1972).

In (**6**), set $P_n(x)$ as $P_n(x) = a_0 + a_1 x + a_1 x^2 + \cdots + a_n x^n$ (**25**)
Column II in Table I expresses $x^k (k = 0,1, \dots n-1)$ in terms of Chebyshev polynomials so that
$P_n(x) = c_0 T_0 + c_1 T_1(x) + c_2 T_2(x) + \dots + c_n T_n(x)$ (**26**)
Since the term x^n appears only in $T_n(x)$, we must have $c_n = a_n 2^{n-1}$ and therefore from (**26**)$M_{n-1}(x) = c_0 T_0 + c_1 T_1 + c_2 T_2 + \dots + c_{n-1} T_{n-1}$
(27)
If we retain only the terms through $k < n$ in (**26**) and if we use the fact that $|T_n(x)| \leq 1$ for all n, we observed that the error committed is bounded by;
$\max_{-1 \leq x \leq 1} |P_n(x) - (c_0 T_0 + c_1 T_1 + c_2 T_2 + \dots + c_n T_n)| \leq |c_{n+1}| + |c_{n+2}| + \cdots$ (**28**)
Hence, to obtain the Minimax polynomial approximation to the polynomial (**6**), we just express $P_n(x)$ as the series (**27**) of Chebyshev polynomials and then drop the last term in (**26**).
We summarize the procedure for obtaining an economized rational approximation to (**6**)

$$y_{n+1} = \frac{p_0 + p_1 x_{n+1} + p_2 x^2{}_{n+1} + p_3 x^3{}_{n+1} + p_4 x^4{}_{n+1}}{1 + q_1 x_{n+1} + q_2 x^2{}_{n+1} + q_3 x^3{}_{n+1} + q_4 x^4{}_{n+1} + q_5 x^5{}_{n+1}}$$

Consider,

$$y(x) = \frac{3628800 + 1814400x + 403200x^2 + 50400x^3 + 3600x^4}{3628800 - 1814400x + 403200x^2 - 50400x^3 + 3600x^4 - 120x^5} = \frac{P_4(x)}{Q_5(x)} \quad \textbf{(29)}$$

We then find the Minimax polynomials for $P_4(x)$and $Q_5(x)$. Which shall be denoted by $N_3$and N_4respectively; then

$$y_{n+1} = \frac{N_3(\tilde{h})}{N_4(\tilde{h})}$$

Next, we substitute the value x^k as given in column II of **Table 1** in (**29**)

$$y(x) = \frac{3628800T_0 + 1814400T_1 + 403200[2^{-1}(T_0 + T_2)] + 50400[2^{-2}(3T_1 + T_3)] + 3600[2^{-3}(3T_0 + 4T_2 + T_4)]}{3628800T_0 + 1814400T_1 + 403200[2^{-1}(T_0 + T_2)] + 50400[2^{-2}(3T_1 + T_3)] + 3600[2^{-3}(3T_0 + 4T_2 + T_4)] - 120[2^{-4}(10T_1 + 5T_3 + T_5)]}$$

$$y(x) = \frac{3831750T_0 + 1852200T_1 + 203400T_2 + 12600T_3 + 450T_4}{3831750T_0 - 1852275T_1 + 201600T_2 - 12637.5T_3 + 450T_4 - 7.5T_5} \quad (\mathbf{30})$$

Next, we substitute the values of $T_n(x)$ as given in Column I of Table 1 into (**30**) and simplify to get a new numerical integrator.

$$y(x) = \frac{3628350 + 1814400x + 406800x^2 + 50400x^3}{3628800 - 1814362.5x + 403200x^2 - 505500x^3 + 3600x^4} \quad (\mathbf{31})$$

Therefore,

$$y_{n+1} = \frac{3628350 + 1814400x_{n+1} + 406800x^2{}_{n+1} + 50400x^3{}_{n+1}}{36208800 - 1814362.5x_{n+1} + 403200x^2{}_{n+1} - 505500x^3{}_{n+1} + 3600x^4{}_{n+1}} \quad (\mathbf{32})$$

Hence, **(32)** is the new scheme for solution of the first order differential equations.

NUMERICAL IMPLEMENTATION OF THE SCHEME IN (32)

This section considers the numerical implementation of the scheme on three initial value problems using C++ programming language and run on a digital computer. Three test problems considered are found in Mathews 2005, Otunta and Nwachuckwu (2005) and Ayinde and Ibijola (2015)

Problem 1: Consider the linear system. $\begin{cases} x_1' = -4x_1, \\ x_2' = 2x_2 \end{cases}$

with the initial condition$X(0) = \binom{1}{1}$ and exact solution $X = \binom{e^{-4t}}{e^{2t}}$ in the interval $0 \le t \le 1$, where $X = \binom{x_1}{x_2}$. Mathews 2005. The application of the numerical integrator (**32**) with uniform mesh size $h = 0.1$ onproblem 1 yields the results below:

Table 2: Numerical Result for Problem 1

t_n	h	x	Exact Solution	Proposed Scheme (32)	Error
0.1	0.1	x_1	0.670320046	0.670328981	8.93E-06
		x_2	1.221402758	1.221305387	9.74E-05
0.2	0.1	x_1	0.449328964	0.449406382	7.74E-05
		x_2	1.491824698	1.491832363	7.66E-06
0.3	0.1	x_1	0.301194212	0.300822854	0.000371358
		x_2	1.8221188	1.822257552	0.000138752
0.4	0.1	x_1	0.201896518	0.200173972	0.001722546
		x_2	2.225540928	2.225688904	0.000147976
0.5	0.1	x_1	0.135335283	0.131108807	0.004226476
		x_2	2.718281828	2.718009847	0.000271981
0.6	0.1	x_1	0.090717953	0.082725008	0.007992946
		x_2	3.320116923	3.318421596	-0.001695327
0.7	0.1	x_1	0.060810063	0.047796279	0.013013784
		x_2	4.055199967	4.05005194	0.005148027
0.8	0.1	x_1	0.040762204	0.021567312	0.019194892
		x_2	4.953032424	4.940602539	0.012429885
0.9	0.1	x_1	0.027323722	0.000936046	0.026387677
		x_2	6.049647464	6.022969098	0.026678366
1.0	0.1	x_1	0.018315639	0.016099655	0.002215984
		x_2	7.389056099	7.335702861	0.053353238

Problem 2: Using the scheme (**32**)to solve the initial value problem $\acute{y} = -y, y(0) = 1$ in the interval $0 \leq t \leq 1$, With uniform mesh size $h = 0.1$The exact solution is given as $y(t) = e^{-t}$. (Otunta and Nwachuckwu2005)

Table 3: Numerical Solution for problem 2

t_n	h	Exact Solution	Proposed Scheme	Error
0.1	0.1	0.904837418	0.90472964	0.000107778
0.2	0.1	0.818730753	0.818654277	7.65E-05
0.3	0.1	0.740818221	0.74078276	3.55E-05
0.4	0.1	0.670320046	0.670328981	8.93E-06
0.5	0.1	0.60653066	0.606580414	4.98E-05
0.6	0.1	0.548811636	0.548891337	7.97E-05
0.7	0.1	0.496585304	0.496676671	9.14E-05
0.8	0.1	0.449328964	0.449406382	7.74E-05
0.9	0.1	0.40656966	0.406600396	3.07E-05
1	0.1	0.367879441	0.367823981	5.55E-05

Problem 3: Using the scheme (**32**) to solve the initial value problem $x' = 2tx, x(0) = 1$in the interval $0 \leq t \leq 1$ with uniform mesh size $h = 0.1$.Exact solution is given as: $x(t) = e^{t^2}$. (Ayinde and Ibijola, 2015)

Table 4: Numerical Result for Problem 3

t_n	h	Exact Solution	Proposed Scheme (32)	Error
0.1	0.1	1.105170918	1.105049707	0.000121211
0.2	0.1	1.221402758	1.221305387	9.74E-05
0.3	0.1	1.349858808	1.349805733	5.31E-05
0.4	0.1	1.491824698	1.491832363	7.66E-06
0.5	0.1	1.648721271	1.648797635	7.64E-05
0.6	0.1	1.8221188	1.822257552	0.000138752
0.7	0.1	2.013752707	2.013925791	0.000173083
0.8	0.1	2.225540928	2.225688904	0.000147976
0.9	0.1	2.459603111	2.459622711	1.96E-05
1	0.1	2.718281828	2.718009847	0.000271981

DISCUSSION OF NUMERICAL RESULTS

Problem 1 is a linear system with exact exponential solutions. The results obtained from the application of the proposed method and the exact solutions of the differential equation are sufficiently comparable for the fixed step-size h as shown in Table 2.Problem 2 is an autonomous first order ordinary differential equation. The proposed method shows competitive results with small error for each mesh size as can be seen in Table 3. In problem 3, the trend is the same and the results are comparable.

SUMMARY AND CONCLUSION

In this work, we proposed a numerical method by using the Minimax polynomials as a rational integrator in which the numerator is a polynomial of degree 4 and the denominator a polynomial of degree 5. This was possible since any polynomial of degreen can be approximated by a polynomial of degree $\leq n-1$ for which the absolute value of$|P_n(x) - M_{n-1}(x)|$ on $[-1, 1]$ is as small as possible.

The proposed method can be used for higher order equations, since they can always be converted to an equivalent system of first order. The approximate solutions compare favourably with the exact solution of the ODEs. This shows that the integrator formula is consistent and stable. The implementation of the modified rational interpolation was carried out on C++ programming language and runon a digital computer.

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